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On semi-symmetric recurrent-metric conjugate connection

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Abstract In this paper, we study a semi-symmetric recurrent-metric connection and we establish its conjugate connection, generalized conjugate connection and semi-conjugate connection, respectively.

1 Introduction

Friedmann and Schouten [11] were the first ones who studied the notion of a semi-symmetric linear connection on a Riemannian manifold (M, g). Recall that a linear connection $\widetilde{\nabla}$ is called semi-symmetric connection if its torsion $\widetilde{T}$ has the following form:

$$\widetilde{T}(X, Y) = u(Y)X - u(X)Y, \tag{1.1}$$

where u is a 1-form associated with the vector field U on (M, g) by

$$u(X) = g(X, U). \tag{1.2}$$

Firsly, the notion of semi-symmetric metric connection on a Riemannian manifold (M, g) was introduced by Hayden [12] and was later studied by several other mathematicians [6, 13, 20, 21]. Let recall that a semi-symmetric connection $\widetilde{\nabla}$ is said to be a semi-symmetric metric connection if

$$\widetilde{\nabla} g = 0. \tag{1.3}$$

Afterwards, the notion of semi-symmetric non-metric connection on a Riemannian manifold (M, g) was introduced by Agashe and Chafle [1]. We recall that a semi-symmetric connection $\widetilde{\nabla}$ is said to be a semi-symmetric non-metric connection if

$$\widetilde{\nabla} g \neq 0. \tag{1.4}$$

Different model of semi-symmetric non-metric connections have been studied [2, 17]. Recently, the authors [8], studied a type of semi-symmetric non-metric connection and etablish its conjugate connection, generalized conjugate connection and semi-conjugate connection. Warped products endowed with semi-symmetric non-metric connections are studied in [16].

In contrast, the Weyl connection $\widetilde{\nabla}$ defined from of a 1-form ω and its corresponding vector field B in a Riemannian manifold (M, g) is a symmetric non-metric connection. In fact, the Riemannian metric g of the manifold M is recurrent with respect to the Weyl connection with the recurrence 1-form ω, that is $\widetilde{\nabla} g = \omega \otimes g$ [18]. Inspired by [8], we develop a type of semi-symmetric recurrent-metric on (M, g) and we establish its conjugate connection, generalized conjugate connection and the semi-conjugate connection, respectively.

The paper is organised as follows. In section 2, we supply the semi-symmetric recurrent-metric connection which generalize the semi-symmetric non-metric connection studied in [8].

In section 3, we give the mathematical expression of the conjugate connection of the semi-symmetric recurrent-metric connection. The expressions of the generalized conjugate connection and the semi-conjugate connection of the semi-symmetric recurrent-metric connection are presented in section 4 and section 5, respectively. It's worth noting that the generalized conjugate connection and the semi-conjugate conjugate are both generalizations of the conjugate connection.

2 On a semi-symmetric recurrent-metric connection

In this section, we establish a type of semi-symmetric recurrent-metric connection on a Riemannian manifold.

Theorem 2.1. *Let (M,g) be a Riemannian manifold. Let f be a function on (M,g) and let u, u_1 are 1-forms associated with the vector fields U, U_1 on (M,g) defined by*

$$u(X) = g(U,X), \quad u_1(X) = g\big(U_1, X\big). \tag{2.1}$$

Then there exists a unique linear connection $\widetilde{\nabla}$ on (M,g) given by

$$\begin{aligned} \widetilde{\nabla}_X Y &= \mathring{\nabla}_X Y + u(Y)X - g(X,Y)U \\ &\quad -f\Big[u_1(X)Y + u_1(Y)X - g(X,Y)U_1\Big], \end{aligned} \tag{2.2}$$

where $\mathring{\nabla}$ is the Levi Civita connection of (M,g). This unique linear connection $\widetilde{\nabla}$ on M satisfies

$$\widetilde{T}(X,Y) = u(Y)X - u(X)Y, \tag{2.3}$$

and

$$\left(\widetilde{\nabla}_X g\right)(Y,Z) = 2fu_1(X)g(Y,Z), \tag{2.4}$$

where $\widetilde{T}$ is the torsion of $\widetilde{\nabla}$.

Proof. Assume that $\widetilde{\nabla}$ is a linear connection on (M,g) given by

$$\widetilde{\nabla}_X Y = \mathring{\nabla}_X Y + \theta(X,Y), \tag{2.5}$$

where $\mathring{\nabla}$ represents the Levi-Civita connection of (M,g) and θ is a tensor type $(1,2)$ defined on (M,g) which ensures $\widetilde{\nabla}$ satisfies (2.3) and (2.4). From (2.5), we have:

$$\widetilde{T}(X,Y) = \theta(X,Y) - \theta(Y,X). \tag{2.6}$$

Let's pose:

$$G(X,Y,Z) = \left(\widetilde{\nabla}_X g\right)(Y,Z). \tag{2.7}$$

From (2.5) and (2.7), we obtain:

$$\begin{aligned} \widetilde{\nabla}_X\Big(g(Y,Z)\Big) &= \Big(\widetilde{\nabla}_X g\Big)(Y,Z) + g(\widetilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}_X Z) \\ &= G(X,Y,Z) + g(\mathring{\nabla}_X Y, Z) + g(\theta(X,Y),Z) \\ &\quad + g(Y, \mathring{\nabla}_X Z) + g(Y, \theta(X,Z)), \end{aligned}$$

that means:

$$g(\theta(X,Y),Z) + g(Y,\theta(X,Z)) = -G(X,Y,Z). \tag{2.8}$$

Using (2.6) and (2.8), we get:

$$\begin{aligned} g(\widetilde{T}(X,Y),Z) &+ g(\widetilde{T}(Z,X),Y) + g(\widetilde{T}(Z,Y),X) \\ &= 2g(\theta(X,Y),Z) + G(X,Y,Z) \\ &\quad + G(Y,X,Z) - G(Z,X,Y). \end{aligned} \tag{2.9}$$

Putting (2.7) in (2.9), we get:

$$\begin{aligned} & g(\widetilde{T}(X,Y),Z) + g(\widetilde{T}(Z,X),Y) + g(\widetilde{T}(Z,Y),X) \\ & = 2g(\theta(X,Y),Z) + (\widetilde{\nabla}_X g)(Y,Z) + (\widetilde{\nabla}_Y g)(X,Z) \\ & -(\widetilde{\nabla}_Z g)(X,Y). \end{aligned} \tag{2.10}$$

From (2.4), the relation (2.10) become:

$$\begin{aligned} & g(\widetilde{T}(X,Y),Z) + g(\widetilde{T}(Z,X),Y) + g(\widetilde{T}(Z,Y),X) \\ & = 2g(\theta(X,Y),Z) + 2fu_1(X)g(Y,Z) \\ & +2fu_1(Y)g(X,Z) - 2fu_1(Z)g(X,Y), \end{aligned} \tag{2.11}$$

which implies that

$$\begin{aligned} \theta(X,Y) &= \frac{1}{2}\Big(\widetilde{T}(X,Y) + \widetilde{T}'(X,Y) + \widetilde{T}'(Y,X)\Big) \\ & -fu_1(X)g(Y,Z) - fu_1(Y)g(X,Z) \\ & +fu_1(Z)g(X,Y), \end{aligned} \tag{2.12}$$

whereas

$$g(\widetilde{T}'(X,Y),Z) = g(\widetilde{T}(Z,X),Y). \tag{2.13}$$

By making use of (2.1) and (2.3) in (2.10), we find:

$$g(\widetilde{T}'(X,Y),Z) = u(X)g(Y,Z) - g(X,Y)g(U,Z), \tag{2.14}$$

leading to :

$$\widetilde{T}'(X,Y) = u(X)Y - g(X,Y)U. \tag{2.15}$$

By making use of (2.3) and (2.15) in (2.12), we obtain:

$$\begin{aligned} \theta(X,Y) &= u(X)Y - g(X,Y)U \\ & -f\Big[u_1(X)Y + u_1(Y)X - g(X,Y)U_1\Big]. \end{aligned} \tag{2.16}$$

Substituting (2.16) in (2.5), we get:

$$\begin{aligned} \widetilde{\nabla}_X Y &= \mathring{\nabla}_X Y + u(Y)X - g(X,Y)U \\ & -f\Big[u_1(X)Y + u_1(Y)X - g(X,Y)U_1\Big]. \end{aligned}$$

As a consequence, a linear connection given by (2.2) satisfies the conditions (2.3) and (2.4). □

In particular, when $f = 1$, then we obtain the following semi-symmetric reccurent-metric connection $\widetilde{\nabla}$ given by :

$$\begin{aligned} \widetilde{\nabla}_X Y &= \mathring{\nabla}_X Y + u(Y)X - g(X,Y)U \\ & -u_1(X)Y - u_1(Y)X + g(X,Y)U_1. \end{aligned}$$

This connection satisfies $\widetilde{\nabla} g = 2u_1 \otimes g$. See [4] and [14] for more details. If $f = 1$ and $u_1 = u$, then, we obtain the following semi-symmetric recurrent-metric connection given by

$$\widetilde{\nabla}_X Y = \mathring{\nabla}_X Y - u(X)Y$$

This above connection verifes $\widetilde{\nabla} g = 2u \otimes g$.

3 Conjugate connection of the semi symmetric recurrent metric connection

Let (M,g) be a Riemannian manifold and ∇ an affine connection on M. A connection ∇^* is called conjugate connection of ∇ with respect to the metric g if

$$X \cdot g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z), \tag{3.1}$$

for arbitrary X, Y and Z vectors fields on M. The triple of a Riemannian metric and a pair of conjugate connections (g, ∇, ∇^*) satisfying (3.1) is called a dualistic structure on M.

According to [3], dualistic structures are essential notions of information geometry, particularly in the examination of the natural differential geometric structure presented by families of probability distributions. Information geometry is presently used in a wide range of subjects and settings, including the theory of data, stochastic methods, systems with dynamical properties and time series, statistical physics, quantum systems, and the mathematical theory of neural networks [5, 10]. For more information about dualistic structures, we refer to [7, 9, 19] and references therein.

In the following proposition, we obtain the conjugate connection of the semi-symmetric recurrent-metric connection given in Theorem 2.1.

Proposition 3.1. *The conjugate connection of the semi symmetric recurrrent-metric connection (2.2) is given by*

$$\begin{aligned} \widetilde{\nabla}^*_X Y &= \mathring{\nabla}_X Y + u(Y)X - g(X,Y)U \\ &\quad + f\Big[u_1(X)Y - u_1(Y)X + g(X,Y)U_1\Big]. \end{aligned} \tag{3.2}$$

Proof. By making use of (2.2) and (3.1), we obtain:

$$\begin{aligned} g(Z, \widetilde{\nabla}^*_X Y) &= Xg(Y,Z) - g(\widetilde{\nabla}_X Z, Y) \\ &= g(\mathring{\nabla}_X Z, Y) + g(Z, \mathring{\nabla}_X Y) - g(\mathring{\nabla}_X Z, Y) \\ &\quad - u(Z)g(X,Y) + u(Y)g(X,Z) \\ &\quad + f\Big[u_1(X)g(Z,Y) - u_1(Y)g(X,Z) + g(X,Y)g(U_1,Z)\Big]. \end{aligned}$$

Hence

$$\begin{aligned} \widetilde{\nabla}^*_X Y &= \mathring{\nabla}_X Y + u(Y)X - g(X,Y)U \\ &\quad + f\Big[u_1(X)Y - u_1(Y)X + g(X,Y)U_1\Big]. \end{aligned}$$

□

We additionally made the following observation:

Proposition 3.2. *The semi-symmetric recurrent-metric connection (2.2) and its conjugate connection (3.2) satisfies*

$$\widetilde{\nabla}^*_X Y - \widetilde{\nabla}_X Y = 2fu_1(X)Y, \tag{3.3}$$

and

$$\widetilde{T}^*(X,Y) - \widetilde{T}(X,Y) = 2f\Big(u_1(X)Y - u_1(Y)X\Big), \tag{3.4}$$

where $\widetilde{T}^*, \widetilde{T}$ are the torsion tensors of $\widetilde{\nabla}^*$ and $\widetilde{\nabla}$ respectively.

Proof. The relation (3.3) is obtained by applying (2.2) and (3.2). The relation (3.4) is obtained through a simple calculation based on the following:

$$\widetilde{T}^*(X,Y) = \widetilde{\nabla}^*_X Y - \widetilde{\nabla}^*_Y X - [X,Y].$$

□

Corollary 3.3. *The conjugate connection (3.2) of the semi-symmetric recurrent-metric connection (2.2) satisfies*

$$\left(\widetilde{\nabla}^*_X g\right)(Y,Z) = -2fu_1(X)g(Y,Z), \tag{3.5}$$

that is, its is also a semi-symmetric recurrent-metric connection.

This connection $\widetilde{\nabla}^*$ will be called the semi-symmetric recurrent-metric conjugate connection of $\widetilde{\nabla}$ with respect to g. And we have:

$$Xg(Y,Z) = g(\widetilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}^*_X Z). \tag{3.6}$$

4 Generalized conjugate connection of the semi-symmetric recurrent-metric connection

Let (M,g) be a Riemannian manifold and $\mathcal{C}$ be a conformal structure on (M,g) determined by the metric g and a torsion-free affine connection ∇. The pair $(\nabla,\mathcal{C})$ is called a Weyl structure on (M,g) if there exists a 1-form u such that

$$(\nabla_X g)(Y,Z) = -u(X)g(Y,Z).$$

Thus the triplet (M,∇,g) is called a Weyl manifold and the connection ∇ a Weyl connection. A Weyl structure or a Weyl manifold is said to be locally trivial if the 1-form u is closed [15].

Definition 4.1. [8] Let (M,g) be a Riemannian manifold. Let ∇ an affine connection on M and u a 1-form on M. The generalized conjugate connection $\overline{\nabla}^*$ of ∇ with respect to g by u is defined by

$$g(Z,\overline{\nabla}^*_X Y) = X\cdot g(Z,Y) - g(\nabla_X Z, Y) + u(X)g(Z,Y). \tag{4.1}$$

Note that the generalized conjugate connection is a generalization of conjugate connection introduced in Weyl geometry to characterize Weyl connections.

The next proposition provides, the generalized conjugate connection of the semi-symmetric recurrent-metric connection given in Theorem 2.1.

Proposition 4.2. *The generalized conjugate connection $\overline{\widetilde{\nabla}}^*$ of the semi symmetric recurrent-metric connection (2.2) is given by*

$$\begin{aligned}\overline{\widetilde{\nabla}}^*_X Y &= \mathring{\nabla}_X Y + u(X)Y - u(Y)X + g(X,Y)U \\ &\quad + f\Big[u_1(X)Y - u_1(Y)X + g(X,Y)U_1\Big].\end{aligned} \tag{4.2}$$

Proof. We obtain by using relations (2.2) and (4.1), the following:

$$\begin{aligned}g(Z,\overline{\widetilde{\nabla}}^*_X Y) &= X\cdot g(Z,Y) - g(\widetilde{\nabla}_X Z, Y) + u(X)g(Z,Y) \\ &= g(\mathring{\nabla}_X Y, Z) + u(X)g(Y,Z) \\ &\quad + u(Z)g(X,Y) - u(Y)g(X,Z) \\ &\quad + f\Big[u_1(X)g(Y,Z) + u_1(Z)g(X,Y) - u_1(Y)g(X,Z)\Big]\end{aligned}$$

Hence, we have relation (4.2). □

We will call this connection $\overline{\widetilde{\nabla}}^*$ the semi-symmetric recurrent-metric generalized conjugate connection of $\widetilde{\nabla}$ with respect to g.

Definition 4.3. Let (M,g) be a Riemannian manifold.

(i) Two affine connections ∇^1 and ∇^2 on (M,g) are said to be projectively equivalent if there exists a 1-form u on (M,g) such that

$$\nabla^1_X Y = \nabla^2_X Y + u(Y)X + u(X)Y.$$

(ii) Two affine connections ∇^1 and ∇^2 on (M,g) are said to be dual-projectively equivalent if there exists a 1-form u on M such that

$$\nabla^1_X Y = \nabla^2_X Y - g(X,Y)U.$$

We have the following observations:

Proposition 4.4. *Let $\widetilde{\nabla}$ be a semi-symmetric recurrent-metric connection on a Riemannian manifold (M,g) and let $\widetilde{\nabla}^*$ its conjugate connection with respect to g. Assume that an affine connection ∇' is projectively equivalent to $\widetilde{\nabla}$ by u. Then the generalized conjugate connection $\overline{\nabla'}^*$ of ∇' by u is dual-projectively equivalent to $\widetilde{\nabla}^*$ by u with respect to g.*

Proof. By using (3.1), we obtain:

$$X \cdot g(Y,Z) = g(\widetilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}^*_X Z). \tag{4.3}$$

Making use of (4.1), implies that :

$$X \cdot g(Y,Z) = g(\nabla'_X Y, Z) + g(Y, \overline{\nabla'}^*_X Z) - u(X) g(Y,Z). \tag{4.4}$$

Since the affine connections ∇' and $\widetilde{\nabla}$ are projectively equivalent, then there exists a 1-form u on M such that:

$$\nabla'_X Y = \widetilde{\nabla}_X Y + u(Y)X + u(X)Y. \tag{4.5}$$

By applying (4.4) and (4.5), we obtain:

$$X \cdot g(Y,Z) = g(\widetilde{\nabla}_X Y, Z) + g(Y, \overline{\nabla'}^*_X Z) + u(Y) g(X,Z). \tag{4.6}$$

Using (4.3) and (4.6), we observe that :

$$g(Y, \widetilde{\nabla}^*_X Z) = g(Y, \overline{\nabla'}^*_X Z) + u(Y) g(X,Z). \tag{4.7}$$

From (4.7), we have:

$$\overline{\nabla'}^*_X Z = \widetilde{\nabla}^*_X Z - g(X,Z)U. \tag{4.8}$$

This indicates that $\overline{\nabla'}^*$ is dual-projectively equivalent to $\widetilde{\nabla}^*$ by u with respect to g. □

Proposition 4.5. *Let $\widetilde{\nabla}$ be a semi-symmetric recurrent-metric connection on a Riemannian manifold (M,g) and let $\widetilde{\nabla}^*$ its conjugate connection with respect to g. Assume that an affine connection ∇' is dual-projectively equivalent to $\widetilde{\nabla}$ by u. Then the generalized conjugate connection $\overline{\nabla'}^*$ of ∇' by u is projectively equivalent to $\widetilde{\nabla}^*$ by u with respect to g.*

Proof. By using (3.1), we have:

$$X \cdot g(Y,Z) = g(\widetilde{\nabla}_X Y, Z) + g(Y, \widetilde{\nabla}^*_X Z). \tag{4.9}$$

From (4.1), one as:

$$X \cdot g(Y,Z) = g(\nabla'_X Y, Z) + g(Y, \overline{\nabla'}^*_X Z) - u(X) g(Y,Z). \tag{4.10}$$

Since the affine connections ∇' and $\widetilde{\nabla}$ are dual-projectively equivalent, then there exists a 1-form u on M such that

$$\nabla'_X Y = \widetilde{\nabla}_X Y - g(X,Y)U. \tag{4.11}$$

From (4.10) and (4.11), we observe that :

$$\begin{aligned} X \cdot g(Y,Z) &= g(\widetilde{\nabla}_X Y, Z) + g(Y, \overline{\nabla'}^*_X Z) \\ &\quad - u(X) g(Y,Z) - g(U,Z) g(X,Y). \end{aligned} \tag{4.12}$$

By using (4.9) and (4.12), we remark that :

$$g(Y, \widetilde{\nabla}^*_X Z) = g(Y, \bar{\nabla}^*_X Z) - u(X) g(Y,Z) - u(Z) g(X,Y). \tag{4.13}$$

Hence

$$\overline{\nabla'}^*_X Z = \widetilde{\nabla}^*_X Z + u(Z)X + u(X)Z.$$

This implies that $\overline{\nabla'}^*$ is projectively equivalent to $\widetilde{\nabla}^*$ with respect to u. □

5 Semi-conjugate connection of the semi-symmetric recurrent-metric connection

Let (M, g) be a Riemannian manifold and $\mathcal{C}$ be a conformal structure on (M, g) determined by the metric g and a torsion-free affine connection ∇. The pair $(\nabla, \mathcal{C})$ is called a semi-Weyl structure on (M, g) if there exists a 1-form u such that

$$(\nabla_X g)(Y, Z) + u(X)g(Y, Z) = (\nabla_Y g)(X, Z) + u(Y)g(X, Z).$$

Thus the triplet (M, ∇, g) is called a semi-Weyl manifold and the affine connection ∇ is called semi-Weyl compatible with g by u [15]. If $u = 0$, then (M, ∇, g) is a statistical manifold. We introduce a $(0, 3)$ - tensor C by

$$C(X, Y, Z) := (\nabla_X g)(Y, Z) + u(X)g(Y, Z).$$

If the $(0, 3)$-tensor C vanishes everywhere, then (M, ∇, g) is a Weyl manifold.

Definition 5.1. [8] Let (M, g) be a Riemannian manifold. Let ∇ an affine connection on M and u a 1-form on M. The semi-conjugate connection $\widehat{\nabla}^*$ of ∇ with respect to g by u is defined as

$$g\left(Z, \widehat{\nabla}^*_X Y\right) = X \cdot g(Z, Y) - g(\nabla_X Z, Y) - u(Y)g(X, Z) \tag{5.1}$$

From the relation (5.1), we see that, the semi-conjugate connection is a generalization of the conjugate connection.

Next, we calculate the semi-conjugate connection associated with the semi-symmetric recurrent-metric connection given in Theorem 2.1.

Proposition 5.2. *The semi-conjugate connection $\widehat{\widetilde{\nabla}}^*$ of the semi symmetric recurrent-metric connection $\widetilde{\nabla}$ (2.2) is given by*

$$\begin{aligned} \widehat{\widetilde{\nabla}}^*_X Y &= \mathring{\nabla}_X Y - g(X, Y)U \\ &\quad + f\left[u_1(X)Y - u_1(Y)X + g(X, Y)U_1\right]. \end{aligned} \tag{5.2}$$

Proof. From relationss (2.2) and (5.1), we obtain:

$$\begin{aligned} g(Z, \widehat{\widetilde{\nabla}}^*_X Y) &= g(Z, \mathring{\nabla}_X Y) - g(X, Y)g(Z.U) \\ &\quad + f\left[u_1(X)g(Z, Y) - u_1(Y)g(Z, X) + g(Z, U_1)g(X, Y)\right]. \end{aligned}$$

As a result, we have Equation (5.2). □

We have the following observations:

Proposition 5.3. *Let $\widetilde{\nabla}$ be a semi symmetric recurrent-metric connection on a Riemannian manifold (M, g) and let $\widetilde{\nabla}^*$ its conjugate connection with respect to g. Assume that an affine connection ∇' is projectively equivalent to $\widetilde{\nabla}$ by u. Then the semi-conjugate connection $\widehat{\nabla'}^*$ of ∇' by u is given by*

$$\widehat{\nabla'}^*_X Y = \widetilde{\nabla}^*_X Y - u(Y)X - u(X)Z - g(X, Y)U. \tag{5.3}$$

Proof. By (3.1), we observe that :

$$X \cdot g(Y, Z) = g\left(\widetilde{\nabla}_X Y, Z\right) + g\left(Y, \widetilde{\nabla}^*_X Z\right) \tag{5.4}$$

By using (5.1), we have

$$X \cdot g(Y, Z) = g(\nabla'_X Y, Z) + g\left(Y, \widehat{\nabla'}^*_X Z\right) + u(Z)g(X, Y). \tag{5.5}$$

Recall that affine connections ∇' and $\tilde{\nabla}$ are said to be projectively equivalent if there exists a 1-form u on M such that

$$\nabla'_X Y = \tilde{\nabla}_X Y + u(Y)X + u(X)Y. \tag{5.6}$$

Making use of (5.5) and (5.6), we have :

$$\begin{aligned} X \cdot g(Y,Z) &= g\left(\tilde{\nabla}_X Y, Z\right) + g\left(Y, \widehat{\nabla'}^*_X Z\right) + u(Z)g(X,Y) \\ &\quad + u(Y)g(X,Z) + u(X)g(Y,Z). \end{aligned} \tag{5.7}$$

using (5.4) and (5.7), we observe that:

$$g\left(Y, \tilde{\nabla}^*_X Z\right) = g\left(Y, \widehat{\nabla'}^*_X Z\right) + u(Z)g(X,Y) + u(Y)g(X,Z) + u(X)g(Y,Z).$$

Hence

$$\widehat{\nabla'}^*_X Z = \tilde{\nabla}^*_X Z - u(Z)X - u(X)Y - g(X,Z)U.$$

□

Proposition 5.4. *Let $\tilde{\nabla}$ be a semi-symmetric recurrent-metric connection on a Riemannian manifold (M,g) and let $\tilde{\nabla}^*$ its conjugate connection with respect to g. Assume that an affine connection ∇' is dual-projectively equivalent to $\tilde{\nabla}$ by u. Then the semi-conjugate connection $\widehat{\nabla'}^*$ of ∇' by u is projectively equivalent to $\tilde{\nabla}^*$ by u with respect to g.*

Proof. By using (3.1), we obtain:

$$X \cdot g(Y,Z) = g\left(\tilde{\nabla}_X Y, Z\right) + g\left(Y, \tilde{\nabla}^*_X Z\right). \tag{5.8}$$

using (5.1), we get

$$X \cdot g(Y,Z) = g\left(\nabla'_X Y, Z\right) + g\left(Y, \widehat{\nabla'}^*_X Z\right) + u(Z)g(X,Y). \tag{5.9}$$

Recall that affine connections ∇' and $\tilde{\nabla}$ are said to be dual-projectively equivalent if there exists a 1-form u on M such that

$$\nabla'_X Y = \tilde{\nabla}_X Y - g(X,Y)U \tag{5.10}$$

From equations (5.9) and (5.10), we obtain:

$$\begin{aligned} X \cdot g(Y,Z) &= g\left(\tilde{\nabla}_X Y, Z\right) + g\left(Y, \widehat{\nabla'}^*_X Z\right) \\ &\quad + u(Z)g(X,Y) - g(X,Y)g(U,Z). \end{aligned} \tag{5.11}$$

From equations (5.8) and (5.11), we obtain

$$g\left(Y, \tilde{\nabla}^*_X Z\right) = g\left(Y, \widehat{\nabla'}^*_X Z\right) + u(Z)g(X,Y) - g(X,Y)g(U,Z).$$

Hence

$$\widehat{\nabla'}^*_X Z = \tilde{\nabla}^*_X Z.$$

□

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