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A performance comparison of two nonconvex separable programming techniques

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Abstract Convex separable programming is a nonlinear optimization method that yields the global optimum efficiently. In contrast to that, its nonconvex counterpart is challenging and requires further computational techniques to reach the global optimum. We compare two such nonconvex separable programming techniques: the restricted basis entry rule and the mixed integer programming reformulation, by implementing them and running on several test problems. Our experimentation indicates that the restricted basis entry rule is much more powerful when the problem deviates significantly from convexity.

1 Introduction

The method of separable programming is of particular importance as a technique used in nonlinear programming. It plays a key role in solving industry–level optimization problems, as it allows the approximation of nonlinear functions by piecewise linear functions, thus converting the nonlinear problem into a format close to a linear program. Several nonlinear real problems including agricultural planning [1], linear complementarity problem [2], Newsboy problem [3, 4] and demand allocation [5] have been addressed effectively, with the aid of this solution technique.

Separable programming was first introduced by Charnes and Lemke in [6] as a technique that particularly aimed at solving constrained optimization of nonlinear convex functions, whenever these functions are *separable*; that is, if they are expressible as sums of functions of a single variable. The major reason had been the observation that whenever the objective function and the constraints satisfy certain *convexity conditions*, the problem can be readily transformed into a linear program, guaranteeing its efficient solvability. In contrast to this, *nonconvex* optimization problems cannot be restated as linear programs straightaway, as the imposition of an extra nonlinear condition becomes inevitable. Despite this barrier, several researchers have attempted to extend the technique of separable programming beyond the realm of convex optimization [7, 8]. Consequently, several methods were introduced to overcome the challenges and solve separable programs effectively [8, 9]. From these methods, *restricted basis entry rule* (RBER) and *mixed integer programming reformulation* (MIPR) have long served as the two standard methods, and they have been used frequently for solving nonconvex separable problems in literature. Considering the real–world applications, the RBER has been helpful for circuit analysis [10], transportation problems [11], structural engineering [12, 13] and petroleum engineering [14]. On the other hand, MIPR has been used in solving real–world problems such as resource allocation and scheduling [15, 16, 17, 18, 19].

In this context, it is important to explore the computational capacities of these separable programming techniques. Recall that these techniques are required due to the nonconvexity of the original nonlinear optimization problem. It is particularly important to explore how the two techniques converge to the optimum when the problem deviates from convexity. This is the major problem we investigate in this work. Accordingly, we describe the separable programming method and the two techniques by providing examples in section 2. In order to measure the convexity of a separable problem, we use an existing convexity index and modify it to fit into our scope in section 3. We run the algorithms on several test problems and present a comparison of the results in section 4. Our concluding remarks can be found in section 5.

2 Preliminaries

2.1 Separable programming

A function $f : X \to R$, $X \subset R^n$ is said to be *separable* if it can be expressed as a sum of single–variables functions as follows.

$$f(x) = \sum_{j=1}^{n} f_j(x_j) \tag{2.1}$$

If the objective function and the constraints of a nonlinear optimization problem are separable, the program is called a *separable nonlinear program* [20], and it can be stated as follows.

$$Maximize/Minimize \qquad f(x) = \sum_{j=1}^{n} f_j(x_j) \tag{2.2}$$

subject to,

$$\sum_{j=1}^{n} g_{ij}(x_j) \le b_i, \qquad i = 1, 2, \ldots, m \tag{2.3a}$$

$$l_j \le x_j \le u_j, \qquad j = 1, 2, ..., n \tag{2.3b}$$

where all f_j's and g_{ij}'s are separable.

Due to separability, the optimization problem can be restated as a problem format close to a linear program by replacing each nonlinear function with piecewise linear approximations. The idea of piecewise approximation of a function is graphically represented in Figure 1. In order to do this, consider the continuous single variabled function $f(x)$ on the interval $[a, b]$. We subdivide the interval using grid points such that,

$$a = x_1 \le x_2 \le ... \le x_k = b. \tag{2.4}$$

Let x be the convex combination of x_t and x_{t+1}. Then there exists $\lambda \in [0, 1]$ such that

$$x = \lambda x_t + (1 - \lambda)x_{t+1}. \tag{2.5}$$

Accordingly,

$$\hat{f}(x) = \lambda f(x_t) + (1 - \lambda)f(x_{t+1}) \tag{2.6}$$

The approximation of the function f can be done over the interval $[a, b]$ using grid points $x_1, x_2, \ldots, x_k$ by considering the following piecewise linear function $\hat{f}$.

$$\hat{f}(x) = \sum_{t=1}^{k} \lambda_t f(x_t) \tag{2.7}$$

where,

$$\sum_{t=1}^{k} \lambda_t = 1. \tag{2.8}$$

The non–negativity of λ_t must be specified by,

$$\lambda_t \ge 0 \qquad t = 1, 2, ..., k \tag{2.9}$$

Hence,

$$x = \sum_{t=1}^{k} \lambda_t x_t \tag{2.10}$$

where at most two λ_t's are positive and they must be adjacent.

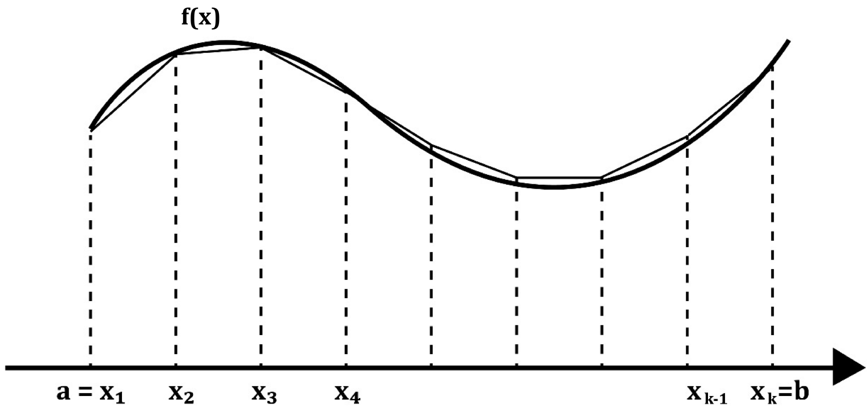

Figure 1. Piecewise linear approximation of a function

2.2 The approximating problem

Consider the nonlinear program given by Equations (2.2, 2.3a–2.3b) and suppose all the f_j and g_{ij} are continuous in the interval $[a_j, b_j]$. There exist $j \in R$ such that f_j and all g_{ij} where $i = 1, 2, ..., m$ are linear. Then we define the set,

$$L=\{j : f_j \text{ and } g_{ij} \text{ , } i = 1, 2, ..., m \text{ are linear}\}.$$

For $j \notin L$, consider the functions on the interval $[a_j, b_j]$, where $a_j, b_j \geq 0$. The grid points x_{tj}, where $t = 1, ..., k_j$, of the j^{th} variable x_j can be defined as,

$$a_j = x_{1j} \leq x_{2j} \leq ... \leq x_{kj} = b_j$$

The grid points may or may not be equidistant. When the number of grid points increases, the accuracy of the approximation improves [20]. For each $j \notin L$, the functions f_j and g_{ij} where $i = 1, 2, ..., m$ can be approximated as follows.

$$\hat{f}_j(x_j) = \sum_{t=1}^{k_j} \lambda_{tj} f_j(x_{tj}) \tag{2.11}$$

$$\hat{g}_{ij}(x_j) = \sum_{t=1}^{k_j} \lambda_{tj} g_{ij}(x_{tj}), \qquad i = 1, 2, ..., m \tag{2.12}$$

$$\sum_{t=1}^{k_j} \lambda_{tj} = 1 \tag{2.13}$$

Finally, the non–negativity of λ_{ij} must be specified by,

$$\lambda_{tj} \geq 0, \qquad t = 1, 2, ..., k_j. \tag{2.14}$$

Here, λ_{tj}'s are the grid variables of the j^{th} variable x_j. Then the approximating problem for the separable problem given by Equations (2.2, 2.3a-2.3b) is,

$$Minimize \qquad \hat{f}(x) = \sum_{j \in L} f_j(x_j) + \sum_{j \notin L} \sum_{t=1}^{k_j} \lambda_{tj} f_j(x_{tj}) \tag{2.15}$$

subject to,

$$\hat{g}_i(x) = \sum_{j \in L} g_{ij}(x_j) + \sum_{j \notin L} \sum_{t=1}^{k_j} \lambda_{tj} g_{ij}(x_{tj}) \le b_i, \quad i = 1, 2, \ldots, m \tag{2.16a}$$

$$\sum_{t=1}^{k_j} \lambda_{tj} = 1, \qquad j \notin L \tag{2.16b}$$

$$\lambda_{tj} \ge 0, \qquad t = 1, 2, ..., k_j, j \notin L \tag{2.16c}$$

$$x_j \ge 0, \qquad j \in L \tag{2.16d}$$

For each $j \notin L$ at most two λ_{tj} values are positive and they must be adjacent. This additional restriction is called as the *adjacency criterion*.

When considering the separable programming problem, in the minimization case, the following theorem [21] is of particular importance.

Theorem 2.1. *Suppose that for each $j \notin L$, $f_j(x_j)$ are strictly convex and $g_{ij}(x_j)$ are convex for $i = 1, 2, ...m$. Further, suppose that x_j^0 $(j \notin L)$ and λ_{tj} $(t = 1, 2, ..., k_j; j \notin L)$ solve the approximating program given by Equations (2.15, 2.16a–2.16d) without the additional restriction. Then*

(i) The vector $\hat{x}$, whose components are given by

- $\hat{x}_j$=x_j^0, *for* $j \in L$
- $\hat{x}_j = \sum_{t=1}^{k_j} \lambda_{tj}^0 x_{tj}$, *for* $j \notin L$, *is feasible to the original problem.*

(ii) For each $j \notin L$, at most two λ_{tj}^0's are positive and they must be adjacent.

According to this theorem, under the convexity conditions mentioned in the theorem, the adjacency criterion is automatically satisfied. Therefore the simplex algorithm can be used to solve the nonlinear problem without modifications [22]. On the other hand, when it does not satisfy the convexity conditions, that is, when the objective function and the constraints deviate from convexity, techniques such as RBER and MIPR must be used.

2.3 Simplex method with restricted basic entry rule

The simplex method can be applied to solve the approximated problem given by Equations (2.15, 2.16a–2.16d) with RBER if the introduction into the basis of non–basis variable improves the value of the objective function and if the new basis satisfies the additional restriction of no more than two λ_{tj}'s can be positive only if they are adjacent. This process is repeated until the optimality criterion is satisfied or until it is impossible to introduce a new λ_{tj} without violating the restricted basis entry rule. The last simplex tableau gives the approximate optimal solution to the problem given by Equations (2.2, 2.3a–2.3b).

Numerical example:

Consider the following nonconvex nonlinear separable programming problem,

$$Minimize \qquad x_1^2 - 6x_1 + 2x_2^2 - 8x_2 + \frac{1}{2}x_3 \tag{2.17}$$

subject to,

$$x_1 + x_2 + x_3 \le 5 \tag{2.18a}$$

$$x_1^3 - x_2 \le 3 \tag{2.18b}$$

$$x_j \ge 0, \quad j = 1, 2, 3 \tag{2.18c}$$

Since the variable x_3 is linear in the whole problem, then $L = \{3\}$. Therefore, the is no need to transform to a piecewise linear approximation. Considering these two constraints, clearly both x_1 and x_2 lie in the interval [0,5]. Here, we use the grid points [0,2,4,5] for variable x_1 and x_2.

$$x_{11} = 0, x_{21} = 2, x_{31} = 4, x_{41} = 5$$
$$x_{12} = 0, x_{22} = 2, x_{32} = 4, x_{42} = 5$$

Original variables in terms of grid variables and grid points,

$$x_1 = 0\lambda_{11} + 2\lambda_{21} + 4\lambda_{31} + 5\lambda_{41} \tag{2.19}$$

$$x_2 = 0\lambda_{12} + 2\lambda_{22} + 4\lambda_{32} + 5\lambda_{42} \tag{2.20}$$

Then the approximation problem,

$$\begin{aligned} \text{Minimize} \quad & 0\lambda_{11} - 8\lambda_{21} - 8\lambda_{31} - 5\lambda_{41} + 0\lambda_{12} - 8\lambda_{22} - 0\lambda_{32} + 10\lambda_{42} + \frac{1}{2}x_3 \\ \text{subject to,} \quad & 0\lambda_{11} + 2\lambda_{21} + 4\lambda_{31} + 5\lambda_{41} + 0\lambda_{12} + 2\lambda_{22} + 4\lambda_{32} + 5\lambda_{42} + x_3 + s_1 = 5 \\ & 0\lambda_{11} + 8\lambda_{21} + 64\lambda_{31} + 125\lambda_{41} + 0\lambda_{12} - 2\lambda_{22} - 4\lambda_{32} - 5\lambda_{42} + s_2 = 3 \\ & \lambda_{11} + \lambda_{21} + \lambda_{31} + \lambda_{41} = 1 \\ & \lambda_{12} + \lambda_{22} + \lambda_{32} + \lambda_{42} = 1 \\ & \lambda_{tj} \geq 0, \quad t = 1,2,3,4 \quad and \quad j = 1,2 \\ & x_3 \geq 0 \end{aligned}$$

At most two $\lambda_{1j}, \lambda_{2j}, \lambda_{3j}$ and λ_{4j} are positive and they must be adjacent for each $j = 1, 2$.

Then we solve the linear problem by the simplex method with restricted basic entry rule.
Initial tableau:

λ_{11}	λ_{21}	λ_{31}	λ_{41}	λ_{12}	λ_{22}	λ_{32}	λ_{42}	x_3	s_1	s_2	RHS
0	2	4	5	0	2	4	5	1	1	0	5
0	8	64	125	0	-2	-4	-5	0	0	1	3
1	1	1	1	0	0	0	0	0	0	0	1
0	0	0	0	1	1	1	1	0	0	0	1
0	-8	-8	-5	0	-8	0	10	$\frac{1}{2}$	0	0	0

Pivot step 1(second tableau):

λ_{11}	λ_{21}	λ_{31}	λ_{41}	λ_{12}	λ_{22}	λ_{32}	λ_{42}	x_3	s_1	s_2	RHS
0	0	-12	$-\frac{105}{4}$	0	$\frac{5}{2}$	5	$\frac{25}{4}$	1	1	$-\frac{1}{4}$	$\frac{17}{4}$
0	1	8	$\frac{125}{8}$	0	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{5}{8}$	0	0	$\frac{1}{8}$	$\frac{3}{8}$
1	0	-7	$-\frac{117}{8}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{8}$	0	0	$\frac{1}{8}$	$\frac{5}{8}$
0	0	0	0	1	1	1	1	0	0	0	1
0	0	56	120	0	-10	-4	5	$-\frac{1}{2}$	0	0	3

Pivot step 2(Final tableau):

λ_{11}	λ_{21}	λ_{31}	λ_{41}	λ_{12}	λ_{22}	λ_{32}	λ_{42}	x_3	s_1	s_2	RHS
0	0	-12	$-\frac{105}{4}$	$-\frac{5}{2}$	0	$\frac{5}{2}$	$\frac{15}{4}$	1	1	$-\frac{1}{4}$	$\frac{7}{4}$
0	1	8	$\frac{125}{8}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{3}{8}$	0	0	$\frac{1}{8}$	$\frac{5}{8}$
1	0	-7	$-\frac{117}{8}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{3}{8}$	0	0	$-\frac{1}{8}$	$\frac{3}{8}$
0	0	0	0	1	1	1	1	0	0	0	1
0	0	56	120	10	0	6	15	$\frac{1}{2}$	0	1	13

In the initial tableau, one of λ_{21} or λ_{22} variables could be introduced into the basis as they satisfy the restricted basic entry rule. Hence arbitrary we choose λ_{21} as the basis. Note that λ_{31} should be neglected as it violated the rule.

In the second tableau, since λ_{22} satisfies the restricted basic entry rule, it could be introduced into the basis of the next step.

In the final tableau the optimal solution is obtained.

$$(\lambda_{11}^*, \lambda_{21}^*, \lambda_{31}^*, \lambda_{41}^*, \lambda_{12}^*, \lambda_{22}^*, \lambda_{32}^*, \lambda_{42}^*, x_3^*) = (\tfrac{3}{8}, \tfrac{5}{8}, 0, 0, 0, 1, 0, 0, 0)$$

Therefore the optimal solution to the approximation problem could be defined as follows.

$$x^* = (x_1^*, x_2^*, x_3^*) = (\tfrac{5}{4}, 2, 0)$$

In the above approximated problem, corresponding value of the objective function is,

$$\hat{f}_{min} = \hat{f}(x^*) = -13$$

However, when considering the original separable problem given by Equations (2.17, 2.18a–2.18c) at this point x^*, the value of the objective function shows a different figure of,

$$f_{min} = f(x^*) = -13.9375$$

Therefore to reduce the gap between these two different objective values, we can increase the number of grid points with a sufficiently decreased grid length. This will increase the accuracy of the approximation [23].

2.4 Mixed integer programming reformulation

In MIPR, the adjacency criterion is replaced by a set of constraints with Boolean variables [16]. The approximated problem is defined as follows.

$$Minimize \qquad \hat{f}(x) = \sum_{j \in L} f_j(x_j) + \sum_{j \notin L} \sum_{t=1}^{k_j} \lambda_{tj} f_j(x_{tj}) \tag{2.21}$$

subject to,

$$\hat{g}_i(x) = \sum_{j\in L} g_{ij}(x_j) + \sum_{j\notin L}\sum_{t=1}^{k_j} \lambda_{tj} g_{ij}(x_{tj}) \le b_i, \qquad i = 1,2,\dots,m \tag{2.22a}$$

$$0 \le \lambda_{1j} \le \delta_{1j}, \qquad j \notin L \tag{2.22b}$$

$$0 \le \lambda_{tj} \le \delta_{t-1,j} + \delta_{tj}, \quad t = 1,2,...,k_j - 1; \qquad j \notin L \tag{2.22c}$$

$$0 \le \lambda_{k_j,j} \le \delta_{k_j-1,j}, \qquad j \notin L \tag{2.22d}$$

$$\sum_{t=1}^{k_j-1} \delta_t = 1, \qquad j \notin L \tag{2.22e}$$

$$\sum_{t=1}^{k_j} \lambda_t = 1, \qquad j \notin L \tag{2.22f}$$

$$\delta_{tj} = 0 \text{ or } 1, \quad t = 1,2,...,k_j - 1, \qquad j \notin L \tag{2.22g}$$

$$\lambda_{tj} \ge 0, \quad t = 1,2,...,k_j, \qquad j \notin L \tag{2.22h}$$

$$x_j \ge 0, \qquad j \in L \tag{2.22i}$$

Numerical example:

Consider the nonconvex separable programming problem given by Equations (2.17, 2.18a–2.18c). The approximated problem given by Equations (2.21, 2.22a -2.22i) can be defined as follows.

$$Minimize \qquad 0\lambda_{11} - 8\lambda_{21} - 8\lambda_{31} - 5\lambda_{41} + 0\lambda_{12} - 8\lambda_{22} - 0\lambda_{32} + 10\lambda_{42} + \frac{1}{2}x_3$$

subject to,

$$\begin{aligned}
0\lambda_{11} + 2\lambda_{21} + 4\lambda_{31} + 5\lambda_{41} + 0\lambda_{12} + 2\lambda_{22} + 4\lambda_{32} + 5\lambda_{42} + x_3 + s_1 &= 5\\
0\lambda_{11} + 8\lambda_{21} + 64\lambda_{31} + 125\lambda_{41} + 0\lambda_{12} - 2\lambda_{22} - 4\lambda_{32} - 5\lambda_{42} + s_2 &= 3\\
\lambda_{11} + \lambda_{21} + \lambda_{31} + \lambda_{41} &= 1\\
\lambda_{12} + \lambda_{22} + \lambda_{32} + \lambda_{42} &= 1\\
0 \le \lambda_{11} &\le \delta_{11}\\
0 \le \lambda_{21} &\le \delta_{11} + \delta_{21}\\
0 \le \lambda_{31} &\le \delta_{21} + \delta_{31}\\
0 \le \lambda_{41} &\le \delta_{31}\\
0 \le \lambda_{12} &\le \delta_{12}\\
0 \le \lambda_{22} &\le \delta_{12} + \delta_{22}\\
0 \le \lambda_{32} &\le \delta_{22} + \delta_{32}\\
0 \le \lambda_{42} &\le \delta_{32}\\
\delta_{11} + \delta_{21} + \delta_{31} &= 1\\
\delta_{12} + \delta_{22} + \delta_{32} &= 1\\
\delta_{tj} = 0 \; or \; 1, \quad t = 1,2,3 \quad and \quad j &= 1,2\\
\lambda_{tj} \ge 0, \quad t = 1,2,3 \quad and \quad j &= 1,2\\
x_3 &\ge 0
\end{aligned}$$

Then the optimal solution of this approximated problem is,

$$(\lambda^*_{11}, \lambda^*_{21}, \lambda^*_{31}, \lambda^*_{41}, \lambda^*_{12}, \lambda^*_{22}, \lambda^*_{32}, \lambda^*_{42}, x^*_3, \delta_{11}, \delta_{21}, \delta_{31}, \delta_{12}, \delta_{22}, \delta_{32}) = (\tfrac{3}{8}, \tfrac{5}{8}, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0)$$

Therefore the optimal solution to the approximation problem could be defined as follows.

$$x^* = (x_1^*, x_2^*, x_3^*) = (\tfrac{5}{4}, 2, 0)$$

Then the objective value for approximated problem is $\hat{f}_{min} = \hat{f}(x^*) = -13$. However, when considering the original separable problem given by Equations (2.17, 2.18a–2.18c) at this point x^*, the value of the objective function shows a different figure of,

$$f_{min} = f(x^*) = -13.9375$$

Here too, in order to increase the accuracy of the approximation, we can increase the number of grid points with sufficiently decreased grid length [23]. As a result, the number of constraints and the number of binary variables will increase.

3 Measuring nonconvexity

Convexity is commonly required by continuous optimization algorithms, and it has been discussed in detail by several researchers [25, 26, 27, 28]. Recall that the linearly approximated program provides the global solution to the optimization problem when the convexity conditions are satisfied. The capacity of nonconvex separable programming can be fairly assessed by considering its convergence rate and accuracy when the problem deviates from convexity. In particular, a comparison of two nonconvex separable programming techniques must consider their relative performance when the problem deviates more from convexity. For this to be a feasible task, it is important to quantify the convexity of the problem in an appropriate way.

In this regard we use the convexity measure introduced by Davydov et al in [24]. The definition is as follows: If $h(x)$ be a real–valued and twice continuously differentiable function on $[a, b]$, then the global convexity index of $h(x)$ is given by

$$C(h, (a, b)) = \frac{\int_a^b \lambda^+(x)\, dx}{\int_a^b |\lambda(x)|\, dx}, \tag{3.1}$$

where, $\lambda^+(x) = max\{h''(x), 0\}$, $\lambda^-(x) = max\{-h''(x), 0\}$ and $|\lambda(x)| = \lambda^+(x) + \lambda^-(x)$. It is clear that this convexity index should always be in $[0, 1]$. That is, if the function h is not convex at any point $x \in (a, b)$ then $C = 0$; and, if h is convex at any point $x \in (a, b)$ then $C = 1$. Thus, the index naturally leads towards a measure of nonconvexity, or, the deviation from convexity. We call this the nonconvexity index, and it is readily derived from Equation (3.1) as follows.

$$D(h, (a, b)) = 1 - C(h, (a, b)). \tag{3.2}$$

Considering our scope, where the convexity of the *individual* functions in the objective function and the constraints set plays a key role, the total deviation from convexity of $\sum_j f_j$ is given by $\sum_j |D(f_j, (a, b))|$. A similar definition applies to g_{ij} as well.

3.1 Example

Consider the function $h(x, y) = -cos(2x) - 3sin(y)$ on the square $S_{0,0}(4)$. This is separable, and expressible as the summation of two functions $h_1(x) = -cos(2x)$ and $h_2(y) = -3sin(y)$. For $h_1(x)$,

$$\lambda^+(x) = max\{h_1''(x), 0\} = max\{4cos(2x), 0\}$$
$$\lambda^-(x) = max\{-h_1''(x), 0\} = max\{-4cos(2x), 0\}$$
$$|\lambda(x)| = \lambda^+(x) + \lambda^-(x)$$

For $h_2(y)$,

$$\lambda^+(y) = max\{h_2''(y), 0\} = max\{3sin(y), 0\}$$
$$\lambda^-(y) = max\{-h_2''(y), 0\} = max\{-3sin(y), 0\}$$
$$|\lambda(y)| = \lambda^+(y) + \lambda^-(y)$$

Then the value of global convexity index of h_1 and h_2 is given by $C(h_1,(-4,4)) = 0.598723$ and $C(h_2,(-4,4)) = 0.500000$. Hence, the total deviation from convexity of the function $h(x,y)$ on $[-4,4]$ can be calculated as follows.

$$\begin{aligned} D &= |D_1| + |D_2| \\ &= (1 - C(h_1)) + (1 - C(h_2)) \\ &= 0.901277 \end{aligned}$$

4 Experimental Results

We developed Python codes for both the RBER and the MIPR and implemented them for several test problems. The computer on which this experiment was run was equipped with an Intel i5 10^{th} generation processor, 4GB of RAM, and the Windows 10 operating system. The performance metrics were chosen as the execution time to reach an accurate solution, the size of the file that contains the problem, and the number of variables in the problem. The comparison took place in two phases. In the first phase, we made a comparison on the same optimization problem using RBER and MIPR algorithms. In the second phase, we considered a collection of test problems and measured their deviation from convexity (3.2). Then we executed the algorithms and compared the values obtained for performance metrics against the deviation from convexity.

4.1 Comparison-Phase I

We demonstrate this phase of our experimentation using the example problem given by Equations (2.17, 2.18a–2.18c). A comparison of the performance metrics are given in Table 1. The other test problems, too, showed somewhat similar results in this regard. The RBER required less computational time, approximately one–tenth of the time for MIPR for problems of similar size. However, it required a significantly large space. Further, the number of variables is relatively low in RBER, as the MIPR requires several artificial variables in its formulation.

Table 1. Performance metrics

Method	Computation time(sec)	File size on Disk(KB)	Number of variables
RBER	0.010793447494506836	11.7	9
MIPR	0.1007375717163086	1.86	15

4.2 Comparison-Phase II

This experimentation phase was carried out to calculate the deviation from convexity and make a comparison of performance metrics in both approaches, MIPR and RBER. To carry out this experimentation, we considered the following collection of example problems (P1–P12) which were obtained by changing the objective function. Here the grid points for variables x_1 and x_2 be $[-4,-2,0,2,4]$.

(P1)

$$\begin{aligned} \text{Minimize} \quad & x_1^3 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \leq 6 \\ & -4 \leq x_1, x_2 \leq 4 \end{aligned}$$

(P2)

$$\begin{aligned} \text{Minimize} \quad & x_1^4 + x_1^3 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \leq 6 \\ & -4 \leq x_1, x_2 \leq 4 \end{aligned}$$

(P3)

$$\begin{aligned} \text{Minimize} \quad & x_1^5 + x_1^4 + x_1^3 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \leq 6 \\ & -4 \leq x_1, x_2 \leq 4 \end{aligned}$$

(P4)

$$\begin{aligned} \text{Minimize} \quad & 6x_1^5 + x_1^4 + 2x_1^3 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \leq 6 \\ & -4 \leq x_1, x_2 \leq 4 \end{aligned}$$

(P5)

$$\begin{aligned} \text{Minimize} \quad & 3x_1^5 + x_1^4 + 2x_1^3 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \leq 6 \\ & -4 \leq x_1, x_2 \leq 4 \end{aligned}$$

(P6)

$$\begin{aligned} \text{Minimize} \quad & 3x_1^5 + 2x_1^4 + 2x_1^3 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \leq 6 \\ & -4 \leq x_1, x_2 \leq 4 \end{aligned}$$

(P7)

$$\begin{aligned} \text{Minimize} \quad & 5x_1^5 + 2x_1^4 + 2x_1^3 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \leq 6 \\ & -4 \leq x_1, x_2 \leq 4 \end{aligned}$$

(P8)

$$\begin{aligned} \text{Minimize} \quad & 6x_1^5 + 2x_1^4 + 2x_1^3 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \leq 6 \\ & -4 \leq x_1, x_2 \leq 4 \end{aligned}$$

(P9)

$$\begin{aligned} \text{Minimize} \quad & 7x_1^5 + 2x_1^4 + 2x_1^3 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \leq 6 \\ & -4 \leq x_1, x_2 \leq 4 \end{aligned}$$

(P10)

$$\begin{aligned} \text{Minimize} \quad & 8x_1^5 + 2x_1^4 + 2x_1^3 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \leq 6 \\ & -4 \leq x_1, x_2 \leq 4 \end{aligned}$$

(P11)

$$\begin{aligned} \text{Minimize} \quad & 10x_1^5 + 2x_1^4 + 2x_1^3 + x_2^3 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \leq 6 \\ & -4 \leq x_1, x_2 \leq 4 \end{aligned}$$

(P12)

$$\begin{aligned} \text{Minimize} \quad & 10x_1^5 + 2x_1^4 + 2x_1^3 + 2x_2^3 + x_2^2 \\ \text{subject to} \quad & x_1 + x_2 \leq 6 \\ & -4 \leq x_1, x_2 \leq 4 \end{aligned}$$

First, we calculated the total deviation from convexity for each problem separately, as given in Table 2. After that, performance metrics for each problem were measured using both approaches. Following that, performance metrics were illustrated separately against the increments of total deviation as shown in Figures 2,3 and 4.

According to Figure 2, until the nonconvexity index of the P4 problem, the deviation shows small increments, and the required computational time for those was not much differ from each other. However, in both approaches after P4, a high increment in computation time for P11 was highlighted in Figure 2 due to the high deviation of 0.89739 in P11 problem. As in phase I, the RBER required less computational time than MILP for each problem. Furthermore, both required relatively high computational time for the problems that deviated largely from convexity.

As obtained in phase I, MIPR required less computation space than RBER. Furthermore, for each nonconvexity index, MIPR required almost the same size of space. However, as shown in Figure 3, even though the RBER required almost the same space until the deviation in P4, the required space was increased in a relatively higher manner as in P11 when the deviation was much further from the convexity.

For each problem, MIPR required more variables than RBER. Furthermore, as shown in Figure 4, the required number of variables remained consistent for every nonconvexity index in both approaches, as we used an equal number of variables and grid points for each problem.

Table 2. Total deviations from convexity

Problem No	Total deviation from convexity
1	0.033841133
2	0.000488
3	0.403615
4	0.483539
5	0.46748
6	0.434959
7	0.460591
8	0.467078
9	0.471732
10	0.475232
11	0.89739
12	0.938555

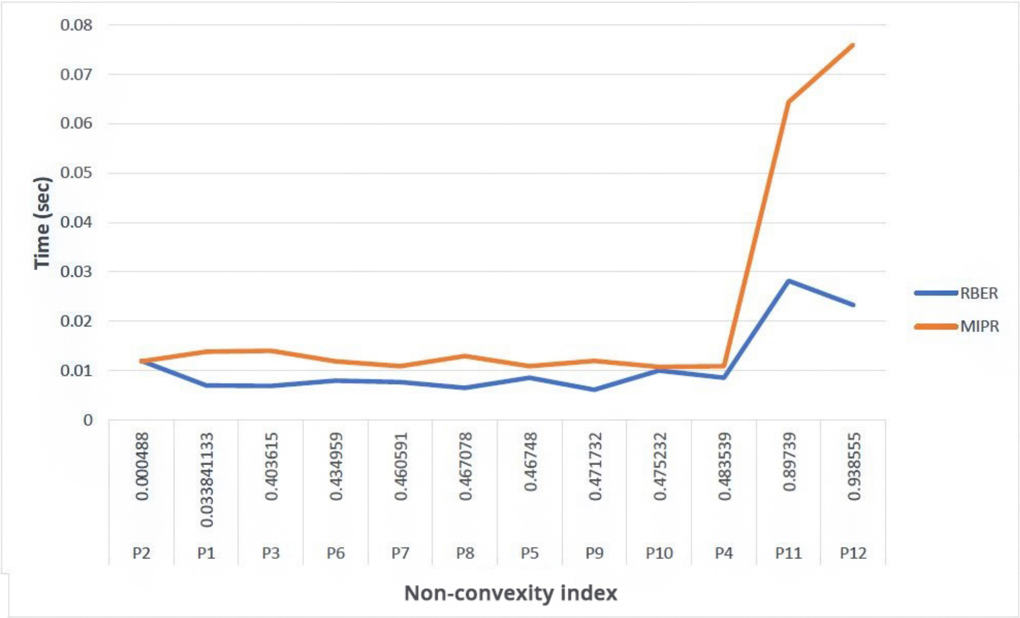

Figure 2. Computational time vs non–convexity index

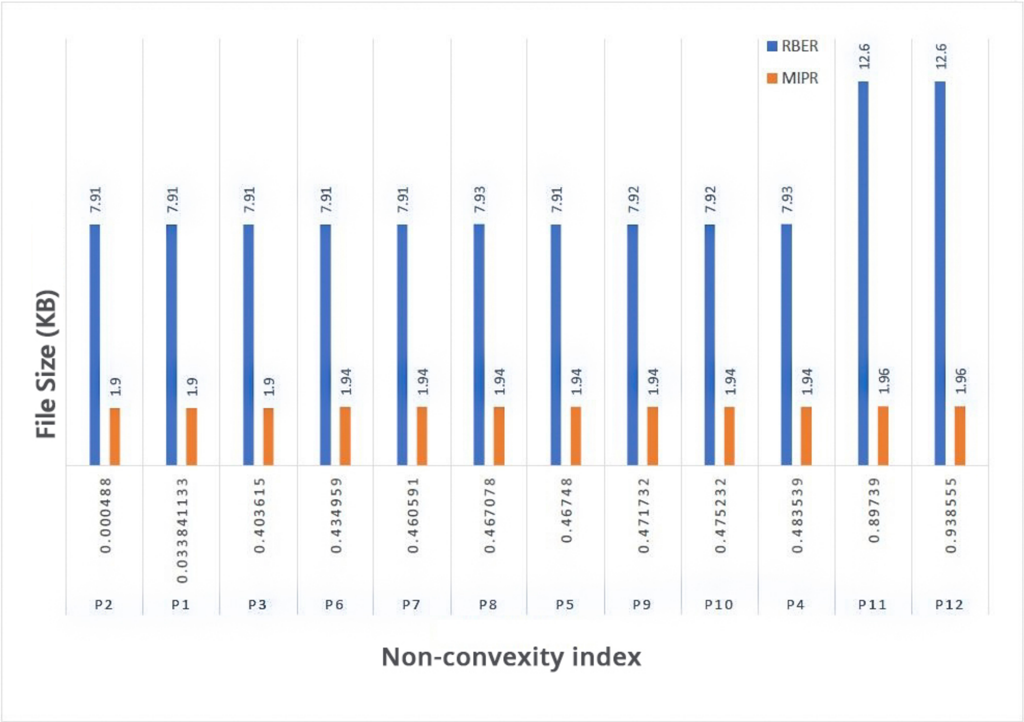

Figure 3. File size vs non–convexity index

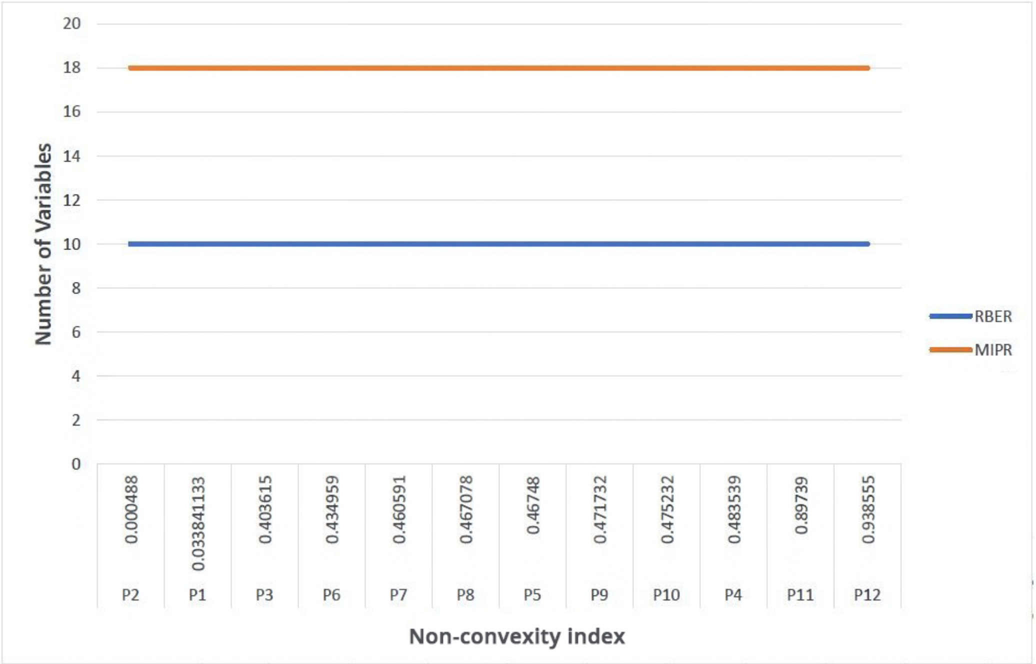

Figure 4. Number of variables vs non–convexity index

5 Concluding Remarks

We implemented the two nonconvex separable programming techniques, namely, RBER and MIPR, on several test problems. Our experimentation shows that RBER outperforms MIPR in particular on computational time when the problem deviates from convexity. Further, it was seen that the RBER requires more space on a computer, though MIPR generates a large number of artificial variables. It would be an interesting future task to find exact theoretical bounds for the two techniques and also to explore new techniques to solve nonconvex separable programs.

6 Declaration

The data used to support the findings of this study are included within the article.

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