Palestine Journal of Mathematics

Vol 13(4)(2024) , 848–858

Common Fixed Point Results for Generalized L^*-Contraction in b-Rectangular Metric Spaces

A.N. Sakelo, K.K. Tola, M.A. Mamud and M.T. A/mecha

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 20M99, 13F10; Secondary 13A15, 13M05.

Keywords and phrases: Fixed points, Common fixed points, b-rectangular metric spaces, Generalized L^*-contraction mappings.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Corresponding Author: K.K. Tola

Abstract In this paper, we introduce generalized L^*-contraction mappings for a pair of self-mappings, establish common fixed point theorems and prove the existence and uniqueness of common fixed points for the mappings introduced in the setting of b-rectangular metric spaces. Our results extend and generalize several related fixed point results in the existing literature, particularly that of Saleh et al. [23] from rectangular metric space of single mappings to b-rectangular metric space of a pair of mappings. Also, we provide an example in support of our main finding.

1 Introduction and Preliminaries

Fixed point theory is an important tool in the study of nonlinear analysis. It is considered to be the key connection between pure and applied mathematics. It is also widely applied in different fields of study such as Economics, Chemistry, Physics and almost all engineering fields. In 1922, the Polish mathematician Banach [5] established a remarkable fixed point theorem known as the Banach contraction principle (BCP) which is one of the most important results of nonlinear analysis and considered as the main source of metric fixed point theory. It confirms the existence and uniqueness of fixed point of self-maps of metric spaces and provides a constructive method to find fixed points. Banach contraction principle has been extended and generalized in different directions by many researchers (e.g., see [9], [10], [17], [18]). Bakhtin [4] and Czerwik [11] introduced the concept of b-metric space as a generalization of the nation of metric space and proved some fixed point theorems for some contraction mappings in b-metric spaces which generalized Banach contraction principle in metric spaces.

In 2000, Branciari [7] initiated the concept of a generalized metric by replacing the natural triangle inequality of a metric with a relatively more general inequality termed as rectangular (or quadrilateral) inequality which involves four points instead of three and proved some fixed point results in rectangular metric space. Many authors extended and generalized the works of Branciari for different mappings and contractions (e.g., see [3, 6, 8, 13, 14, 15, 19, 23] and references therein.

In 2015, George et al. [12] announced the notion of b-rectangular metric space as a generalization of metric, b-metric space and rectangular metric space. On the other hand, in 2020, Öztürk [26] introduced the existence of common fixed point theorem in b-rectangular metric space.

Also, in 2020, Saleh et al. [23] introduced the notions of generalized L^*-contractions mappings and studied existence and uniqueness of fixed point results in the setting of rectangular metric space.

Inspired and motivated by the work of Saleh et al. [23] and related works aforementioned above the main purpose of this paper is to introduce generalized L^*-contraction for a pair of mappings, establish common fixed point theorem and prove the existence and uniqueness of common fixed points for the mappings introduced in the setting of b-rectangular metric spaces.

In what follows we recall basic definitions and results on the topics which we use in the sequel.

In this work we denote, R^+, R and $\mathbb{N}$ by the set of non-negative real numbers, the set of real numbers and the set of all natural numbers respectively.

Definition 1.1. [11] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric if for all $x, y, z \in X$, the following conditions are satisfied:
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (X, d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, since a metric is a b-metric with $s = 1$.

Definition 1.2. [7] Let X be a nonempty set and $d : X \times X \to R^+$ be a function satisfying the following conditions:
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all $x, y \in X$ and all distinct point $u, v \in X\backslash\{x, y\}$.
Then d is called a rectangular metric on X and the pair (X, d) is called a rectangular metric space.

Definition 1.3. [12] Let X be a nonempty set $s \geq 1$ be a given real number and $d : X \times X \to R^+$ be a function satisfying the following conditions:
(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq s[d(x, u)+d(u, v)+d(v, y)]$ for all $x, y \in X$ and all distinct point $u, v \in X\backslash\{x, y\}$.
Then d is called a b-rectangular metric on X and the pair (X, d) is called a rectangular b-metric space.
Note: Every metric space is a rectangular metric space and every rectangular metric space is a b-rectangular metric space with coefficient $s = 1$, but the converse is not true in general.

Definition 1.4. [12] Let (X, d) be a b-rectangular metric space, $\{x_n\}$ be sequence in X and $x \in X$.
We say that:

(i) $\{x_n\}$ is convergent to x if $d(x_n, x) \to 0$ as $n \to \infty$, and we denote this by $x_n \to x$.

(ii) $\{x_n\}$ is a b-rectangular Cauchy sequence if for each $\varepsilon > 0$, there exists a natural number N such that $d(x_n, x_m) < \varepsilon$ for all $m, n > N$.

(iii) X is a b-rectangular complete if every b-rectangular Cauchy sequence in X is convergent in X.

The following is an example to show that not every b-rectangular metric space is a rectangular metric space.

Example 1.5. [12] Let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ and B is the set of all positive integers. Define $d = X \times X \to R^+$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$ and

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 2\alpha, & \text{if } x, y \in A \\ \frac{\alpha}{2n}, & \text{if } x \in A \text{ and } y \in \{2, 3\}; \\ \alpha, & \text{otherwise}, \end{cases}$$

where $\alpha > 0$ is a constant. Then (X, d) is a b-rectangular metric space with coefficient $s = 2 > 1$. However we have the following:

(i) (X, d) is not a rectangular metric space, as

$$d(\tfrac{1}{2}, \tfrac{1}{3}) = 2\alpha > \tfrac{17\alpha}{12} = d(\tfrac{1}{2}, 2) + d(2, 3) + d(3, \tfrac{1}{3}).$$

(ii) The sequence $\{\frac{1}{n}\}$ converges to 2 and 3 in b-rectangular metric space and so limit of a convergent sequence is not unique. Also $d(\frac{1}{n}, \frac{1}{n+p}) = 2\alpha \nrightarrow 0$ as $n \to \infty$. Therefore, $\{\frac{1}{n}\}$ is not a b-rectangular Cauchy sequence in b-rectangular metric space.

Lemma 1.6. *[22] Let X be a b-rectangular metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is not a b-rectangular Cauchy sequence, then there exists $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, such that $d(x_{m_k}, x_{n_k}) \geq \varepsilon$ and with n_k is the smallest index with $n_k > m_k > k$ for which $d(x_{m_k}, x_{n_{k-1}}) < \varepsilon$. Then the following hold:*

(i) $\varepsilon \leq \liminf_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq s\varepsilon.$

(ii) $\varepsilon \leq \liminf_{k\to\infty} d(x_{m_{k+1}}, x_{n_k}) \leq \limsup_{k\to\infty} d(x_{m_{k+1}}, x_{n_k}) \leq s\varepsilon.$

(iii) $\varepsilon \leq \liminf_{k\to\infty} d(x_{m_k}, x_{n_{k+1}}) \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_{k+1}}) \leq s\varepsilon.$

(iv) $\varepsilon \leq \liminf_{k\to\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) \leq \limsup_{k\to\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) \leq s^2\varepsilon.$

Lemma 1.7. *[25] Let (X, d) be a b-rectangular metric spaces with $s \geq 1$ and $\{x_n\}$ be a Cauchy sequence in X such that $x_n \neq x_m$ whenever $n \neq m$. Then $\{x_n\}$ can converge to at most one point.*

Lemma 1.8. *[25] Let (X, d) be a b-rectangular metric spaces with $s \geq 1$.*
(a) Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, with $x \neq y$ and $x_n \neq x$,$y_n \neq y$ for $n \in \mathbb{N}$. Then we have,

$$\frac{1}{s}d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq sd(x, y).$$

(b) If $y \in X$ and $\{x_n\}$ is a b-rectangular Cauchy sequence in X with $x_n \neq x_m$, for infinitely many $m, n \in \mathbb{N}$, $n \neq m$, converging to $x \neq y$, then

$$\frac{1}{s}d(x, y) \leq \liminf_{n\to\infty} d(x_n, y) \leq \limsup_{n\to\infty} d(x_n, y) \leq sd(x, y), \ for\ all\ x \in X.$$

Definition 1.9. [1] Let T and S be self-mappings of a set X. If $y = Tx = Sx$ for some x in X, then x is called a **coincidence point** of T and S and y is called a **point of coincidence** of T and S.

Definition 1.10. [24] Let (X, d) be a metric space. The mappings $T, S : X \to X$ are said to be Compatible if

$$\lim_{n\to\infty} d(TSx_n, STx_n) = 0$$

whenever $\{x_n\}$ is sequence in X such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$$

for some $t \in X$.

Definition 1.11. [20] Let (X, d) be a metric space. The mappings $T, S\colon X \to X$ are said to be weakly compatible if the mappings commute at their coincidence points, that is $Tx = Sx$, for some $x \in X$ implies $T(Sx) = S(Tx)$.

Lemma 1.12. *[21] Let f and g be weakly compatible self-mappings of a set X. If f and g have a unique point of coincidence say ω, then ω is the unique common fixed point of f and g, that is, $\omega = fx = gx$, then ω is the unique common fixed point of f and g.*

In 2014, Jleli and Samet, [19] defined a Θ-contraction mappings and studied fixed point results for rthe mappings defined in the setting of rectangular metric spaces.

Definition 1.13. [19] Let (X, d) be a rectangular metric space. A mapping $T : X \to X$ is said to be a Θ-contraction if there exist $\Theta \in \Omega_{1,2,3}$ and $k \in (0,1)$ such that (for all $x, y \in X$)

$$d(Tx,Ty) > 0 \Rightarrow \Theta(d(Tx,Ty)) \leq [\Theta(d(x,y))]^k,$$

where $\Omega_{1,2,3}$ is the family of all functions: $\Theta : (0,\infty) \to (1,\infty)$ which satisfies the following conditions:
(Θ_1) Θ is nondecreasing;
(Θ_2) For each sequence $\{\alpha_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1 \Leftrightarrow \lim_{n\to\infty} \alpha_n = 0^+$;
(Θ_3) There exists $r \in (0,1)$ and $l \in (0,\infty)$ such that $\lim_{n\to\infty} \frac{\Theta(\alpha_n)}{\alpha^r} = l$.

Theorem 1.14. *[19] Let (X, d) be a complete rectangular metric space and $T : X \to X$ a Θ-contraction mapping. Then T has a unique fixed point.*

Imdad et al. [16] observed that this theorem can be proved without the condition (Θ_1). Also, Ahmad et al. [2] replaced the condition (Θ_3) by the following one:
(Θ_4) is continuous.

Remark 1.15. It is known that every Θ-contraction mapping is continuous. In the sequel, we adopt the following notations:

(i) $\Omega_{1,2,3}$ is the class of all functions Θ which satisfy $(\Theta_1) - (\Theta_3)$;

(ii) $\Omega_{1,2,4}$ is the class of all functions Θ which satisfy (Θ_1), (Θ_2), and (Θ_4);

(iii) $\Omega_{1,2,3,4}$ is the class of all functions Θ which satisfy $(\Theta_1) - (\Theta_4)$.

Definition 1.16. [8] Let (X, d) be a rectangular metric space. A mapping $T : X \to X$ is said to be an L-contraction with respect to $\zeta \in L$ if there exists $\Theta \in \Omega_{1,2,4}$ such that (for all $x, y \in X$)

$$d(Tx,Ty) > 0 \Rightarrow \zeta[\Theta(d(Tx,Ty)), \Theta(d(x,y))] \geq 1,$$

where L is the class of all functions ζ: $[1,\infty) \times [1,\infty) \to R$ which satisfies the following conditions (ζ^*):

(i) (ζ_1^*) $\zeta(1,1) = 1$;

(ii) (ζ_2^*) $\zeta(t,s) < (s/t)$, for all $t, s > 1$;

(iii) (ζ_3^*) If $\{t_n\}$ and $\{s_n\}$ are two sequences in $(1,\infty)$ with $t_n < s_n$, such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 1$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 1$.

Example 1.17. [8] Let ζ_k, ζ_ψ: $[1,\infty) \times [1,\infty) \to R$ be two functions defined as under:

(i) $\zeta_k(t,s) = (s^k/t)$ for all $t, s \geq 1$, where $k \in (0,1)$.

(ii) $\zeta_\psi(t,s) = (s/t\psi(s))$ for all $t, s \geq 1$, where ψ: $[1,\infty) \to [1,\infty)$ is a lower semi-continuous and nondecreasing function with $\psi^{-1}(1) = 1$.

Then, $\zeta_k, \zeta_\psi \in L$.

Definition 1.18. [23] Let (X, d) be a rectangular metric space and $T : X \to X$. Then T is said to be an L^*-contraction with respect to $\zeta \in L$ if there exists $\Theta \in \Omega_{1,2,4}$ such that (for all $x, y \in X$)

$$d(Tx,Ty) > 0 \Rightarrow \zeta[\Theta(d(Tx,Ty)), \Theta(M(x,y))] \geq 1,$$
$$where\ M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Theorem 1.19. *[23] Let (X, d) be a complete rectangular metric space and $T : X \to X$ an L^*-contraction with respect to $\zeta \in L$. Then T has a unique fixed point.*

2 Main Results

In this section, we define generalized L^*-contraction mappings in the setting of b-rectangular metric spaces and establish a common fixed point theorem for the mappings defined.
Before presenting our main result of this section, we give the following definition.

Definition 2.1. Let (X, d) be a b-rectangular metric space with coefficient $s \geq 1$ and $f, g : X \to X$ be self-mappings. Then f and g are said to be generalized L^*-contraction mappings with respect to $\zeta \in L$ if there exists $\Theta \in \Omega_{1,2,4}$ such that for all $x, y \in X$

$$d(fx, fy) > 0 \Rightarrow \zeta[\Theta(s^2 d(fx, fy)), \Theta(M(gx, gy))] \geq 1, \tag{2.1}$$

where

$$M(gx, gy) = max\{d(gx, gy), d(gx, fx), d(gy, fy)\}.$$

Now, we state and prove the following fixed point theorem.

Theorem 2.2. *Let (X, d) be a b-rectangular metric space with coefficient $s \geq 1$, $f, g{:}X \to X$ be self-maps of X, with $f(X) \subseteq g(X)$. Let f and g be a generalized L^*-contraction mappings with respect to $\zeta \in L$. If (2.1) holds and either $g(X)$ or $f(X)$ is complete, then f and g have a unique point of coincidence.*
Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. We first prove that the point of coincidence of f and g is unique if it exists. Let v_1 and v_2 be points of coincidence of f and g. Thus, there exists some $x, y \in X$ such that $v_1 = fx = gx$ and $v_2 = fy = gy$. By (2.1), we derive that

$$\begin{aligned} d(v_1, v_2) &= d(fx, fy) > 0 \Rightarrow \zeta[\Theta(s^2 d(fx, fy)), \Theta(M(gx, gy))] \geq 1. \\ where & \\ M(gx, gy) &= max\{d(gx, gy), d(gx, fx), d(gy, fy)\} \\ &= max\{d(v_1, v_2), d(v_1, v_1), d(v_2, v_2)\} \\ &= max\{d(v_1, v_2), 0, 0\} \\ &= d(v_1, v_2), \end{aligned}$$

which implies that

$$\zeta[\Theta(s^2 d(v_1, v_2)), \Theta(d(v_1, v_2))] \geq 1. \tag{2.2}$$

By (ζ_2^*), we have

$$1 < \frac{\Theta(d(v_1, v_2))}{\Theta(s^2 d(v_1, v_2))} \quad \Rightarrow \quad \Theta(s^2 d(v_1, v_2)) < \Theta(d(v_1, v_2)),$$

in view of (Θ_1), we get

$$s^2 d(v_1, v_2) < d(v_1, v_2),$$

which is a contradiction. So, we conclude that $d(v_1, v_2) = 0$. That is, $v_1 = v_2$. Hence, v_1 is the unique point of coincidence of f and g.
Now, we prove the existence of a point of coincidence of f and g. Let x_0 be an arbitrary point of X. Since $f(X) \subseteq g(X)$, we define two iterative sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$y_n = fx_n = gx_{n+1}, \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

If $y_n = y_{n+1}$, that is, $d(y_n, y_{n+1}) = 0$, then $gx_{n+1} = y_n = y_{n+1} = fx_{n+1}$ and f and g have a point of coincidence. And this complete the proof. Now, assume that $d(y_n, y_{n+1}) > 0$, for all $n \in \mathbb{N} \cup \{0\}$.
Using (2.1)) and (ζ_2^*), we have

$$\begin{aligned} 1 &\leq \zeta[\Theta(s^2 d(fx_n, fx_{n+1})), \Theta(M(gx_n, gx_{n+1}))] \\ &= \zeta[\Theta(s^2 d(y_n, y_{n+1})), \Theta(M(gx_n, gx_{n+1}))] \\ &< \frac{\Theta(M(d(gx_n, gx_{n+1})))}{\Theta(s^2 d(y_n, y_{n+1}))}, \end{aligned}$$

which give

$$\Theta(s^2 d(y_n, y_{n+1})) < \Theta(M(d(gx_n, gx_{n+1}))), \tag{2.3}$$

where

$$\begin{aligned} M(gx_n, gx_{n+1}) &= max\{d(gx_n, gx_{n+1}), d(gx_n, fx_n), d(gx_{n+1}, fx_{n+1})\} \\ &= max\{d(gx_n, gx_{n+1}), d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})\} \\ &= max\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1})\} \\ &= max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}. \end{aligned}$$

If $M(gx_n, gx_{n+1}) = d(y_n, y_{n+1})$, then (2.3) becomes

$$\Theta(s^2 d(y_n, y_{n+1})) < \Theta(d(y_n, y_{n+1})), for\ all\ n \in \mathbb{N},$$

which is a contradiction. Hence, we must have

$$M(gx_n, gx_{n+1}) = d(y_{n-1}, y_n),\ \ for\ all\ n \in \mathbb{N}.$$

Therefore, (2.3) becomes

$$\Theta(s^2 d(y_n, y_{n+1})) < \Theta(d(y_{n-1}, y_n)).$$

Since, $s \geq 1$ for all $n \in \mathbb{N}$, we have

$$\Theta(d(y_n, y_{n+1}))) \leq \Theta(s^2 d(y_n, y_{n+1})) < \Theta(d(y_{n-1}, y_n)), \tag{2.4}$$

which implies (in view of (Θ_1)), that

$$d(y_n, y_{n+1}) < d(y_{n-1}, y_n),\ \ for\ all\ n \in \mathbb{N}.$$

Thus, the sequence $\{d(y_n, y_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. Hence, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(y_n, y_{n+1}) = r$. Suppose that $r \neq 0$, then it follows from (Θ_2), that

$$\lim_{n\to\infty} \Theta(d(y_n, y_{n+1})) > 1. \tag{2.5}$$

Taking $t_n = \Theta(d(y_n, y_{n+1}))$ and $s_n = \Theta(d(y_n, y_{n+1}))$, for all $n \in \mathbb{N}$.
It is clear that from (2.4), (2.5) and (Θ_4) that $t_n < s_n$, for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n > 1$.
Using (ζ_3^*), we get

$$1 \leq \limsup_{n\to\infty} \zeta(t_n, s_n) < 1,$$

which is a contradiction. Therefore, $r = 0$, i.e., we have

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \text{ for all } n \in \mathbb{N}. \tag{2.6}$$

Now, let us assume that $y_m = y_n$, for some $m > n$. Then we have $y_{m+1} = y_{n+1}$. Using (2.4), we get

$$\begin{aligned} \Theta(d(x_m, x_{m+1})) &< \Theta(d(x_{m-1}, x_m)) \\ &< \Theta(d(x_{m-2}, x_{m-1})) \\ &\ldots \\ &\ldots \\ &\ldots \\ &< \Theta(d(x_n, x_{n+1})) \\ &= \Theta(d(x_m, x_{m+1})), \end{aligned}$$

which is a contradiction. This concludes that $y_m \neq y_n$, for all $n \neq m$.
Next, we prove that the sequence $\{y_n\}$ is a b-rectangular Cauchy sequence in (X, d). On the contrary, assume that it is not a b-rectangular Cauchy, then there exists an $\varepsilon > 0$ for which we can find two subsequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $n_k > m_k > k$, for all $k \in \mathbb{N}$ and

$$d(y_{m_k}, y_{n_k}) \geq \varepsilon. \tag{2.7}$$

Suppose that n_k is the least integer exceeding m_k satisfying (2.7). Then, we have

$$d(y_{m_k}, y_{n_{k-1}}) < \varepsilon. \tag{2.8}$$

Using (2.7), (2.8) and by applying b-rectangular inequality, we get

$$\begin{aligned} \varepsilon \leq d(y_{m_k}, y_{n_k}) &\leq sd(y_{m_k}, y_{n_{k-2}}) + sd(y_{n_{k-2}}, y_{n_{k-1}}) + sd(y_{n_{k-1}}, y_{n_k}). \\ &< s\varepsilon + d(y_{n_{k-2}}, y_{n_{k-1}}) + sd(y_{n_{k-1}}, y_{n_k}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.6), we get

$$\varepsilon \leq \liminf_{n\to\infty} d(y_{m_k}, y_{n_k}) \leq \limsup_{n\to\infty} d(y_{m_k}, y_{n_k}) \leq s\varepsilon. \tag{2.9}$$

Also,

$$\varepsilon \leq d(y_{m_k}, y_{n_k}) \leq s[d(y_{m_k}, y_{m_{k+1}}) + d(y_{m_{k+1}}, y_{n_{k+1}}) + d(y_{n_{k+1}}, y_{n_k})].$$

Taking $k \to \infty$, in the above inequality, using (2.6) and Lemma 1.6, we obtain

$$\frac{\varepsilon}{s} \leq \liminf_{n\to\infty} d(y_{m_{k+1}}, y_{n_{k+1}}) \leq \limsup_{n\to\infty} d(y_{m_{k+1}}, y_{n_{k+1}}) \leq s^2\varepsilon. \tag{2.10}$$

Now, we substitute $x = x_{m_{k+1}}$ and $y = x_{n_{k+1}}$ in (2.1), we obtain

$$\zeta[\Theta(s^2 d(fx_{m_{k+1}}, fx_{n_{k+1}})), \Theta(M(gx_{m_{k+1}}, gx_{n_{k+1}}))] \geq 1.$$

It gives

$$\zeta[\Theta(s^2 d(y_{m_{k+1}}, y_{n_{k+1}})), \Theta(M(gx_{m_{k+1}}, gx_{n_{k+1}}))] \geq 1, \tag{2.11}$$

where

$$\begin{aligned} M(gx_{m_{k+1}}, gx_{n_{k+1}}) &= max\{d(gx_{m_{k+1}}, gx_{n_{k+1}}), d(gx_{m_{k+1}}, fx_{m_{k+1}}), d(gx_{n_{k+1}}, fx_{n_{k+1}})\} \\ &= max\{d(y_{m_k}, y_{n_k}), d(y_{m_k}, y_{m_{k+1}}), d(y_{n_k}, y_{n_{k+1}})\}. \end{aligned}$$

Letting $k \to \infty$ in the above inequality, using (2.6) and (2.9), we get

$$\limsup_{n\to\infty} M(gx_{m_{k+1}}, gx_{n_{k+1}}) = \limsup_{n\to\infty} d(y_{m_k}, y_{n_k}) \leq s\varepsilon. \tag{2.12}$$

Using (ζ_2^*), (2.11) becomes

$$\begin{aligned} 1 &\leq \zeta[\Theta(s^2 d(y_{m_{k+1}}, y_{n_{k+1}})), \Theta(M(y_{m_{k+1}}, y_{n_{k+1}}))] \\ &< \frac{\Theta(M(y_{m_{k+1}}, y_{n_{k+1}}))}{\Theta(s^2 d(y_{m_{k+1}}, y_{n_{k+1}}))}, \end{aligned}$$

which ie equivalent to

$$\Theta(s^2 d(y_{m_{k+1}}, y_{n_{k+1}})) < \Theta(M(y_{m_{k+1}}, y_{n_{k+1}})).$$

By taking the upper limits $k \to \infty$ in the above inequality and using (2.9), (2.10), (2.12) and the continuity Θ, we obtain that

$$\begin{aligned} \Theta(\frac{\varepsilon}{s} s^2) = \Theta(\varepsilon s) &\leq \limsup_{n\to\infty} \Theta(s^2 d(y_{m_{k+1}}, y_{n_{k+1}})) \\ &< \limsup_{n\to\infty} \Theta(d(y_{m_k}, y_{n_k})) \\ &\leq \Theta(s\varepsilon). \end{aligned} \tag{2.13}$$

So, $\Theta(\varepsilon s) < \Theta(s\varepsilon)$, which is a contradiction. Therefore, $\{y_n\}$ is b-rectangular Cauchy sequence in X.
Suppose that $g(X)$ is complete. Then there exists some $r \in X$ such that $gx_n \to gr = v \in g(X)$ and $y_n \to v$ as $n \to \infty$. Now to prove that $fr = gr$. Arguing by contradiction, we assume that $fr \neq gr$. Then, we have

$$d(gr, fr) > 0.$$

Hence, we can apply b-rectangular inequality to obtain,

$$\begin{aligned}\frac{1}{s}d(gr, fr) &\leq d(gr, gx_n) + d(gx_n, fx_n) + d(fx_n, fr)\\ &= d(gr, y_{n-1}) + d(y_{n-1}, y_n) + d(fx_n, fr).\end{aligned}$$

If $d(fx_n, fr) = 0$ for some n, then

$$\frac{1}{s}d(gr, fr) \leq \lim_{n\to\infty}\{d(gr, y_{n-1}) + d(y_{n-1}, y_n) + d(fx_n, fr)\} = 0.$$

Which gives $gr = fr$.
Now, we assume $d(fx_n, fr) > 0$.

$$d(fx_n, fr) > 0 \Rightarrow \zeta[\Theta(s^2 d(fx_n, fr)), \Theta(M(gx_n, gr))] \geq 1. \tag{2.14}$$

By (ζ_2^*), we have

$$1 < \frac{\Theta(M(gx_n, gr))}{\Theta(s^2 d(fx_n, fr))} \Rightarrow \Theta(s^2 d(fx_n, fr)) < \Theta(M(gx_n, gr)). \tag{2.15}$$

In view of (Θ_1), we get

$$s^2 d(fx_n, fr) < M(gx_n, gr) \Rightarrow d(fx_n, fr) < \frac{1}{s^2}M(gx_n, gr). \tag{2.16}$$

From (2.14) and (2.16), we get

$$\begin{aligned}\frac{1}{s}d(gr, fr) &\leq d(gr, gx_n) + d(gx_n, fx_n) + \frac{1}{s^2}M(gx_n, gr)\\ &\leq d(gr, gx_n) + d(gx_n, fx_n) + \frac{1}{s^2}max\{d(gx_n, gr), d(gx_n, fx_n), d(gr, fr)\}.\end{aligned}$$

Taking limit as $n \to \infty$ in (2.16), we obtain

$$\frac{1}{s}d(gr, fr) < \frac{1}{s^2}d(gr, fr), \tag{2.17}$$

which is a contradiction, it follows that $d(gr, fr) = 0$. That is, $gr = fr$. Therefore, $gr = fr = v$, v is a point of coincidence of f and g. Since f and g are weakly compatible, $fgr = gfr = fv = gv = w$. Thus, by Lemma 1.12, w is a unique common fixed point of f and g. Also, the proof is similar when $f(X)$ is complete. □

In the following we give an example in support of Theorem 2.2.

Example 2.3. Let $X = A \cup B$, where $A = \left\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\right\}$ and B = [1,2]. Define $d : X \times X \to R^+$ by:
$d(x, y) = d(y, x)$ for all $x, y \in X$ and

(i) $d(0, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{4}) = d(\frac{1}{5}, \frac{1}{6}) = 0.06$

(ii) $d(0, \frac{1}{3}) = d(\frac{1}{2}, \frac{1}{5}) = d(\frac{1}{4}, \frac{1}{5}) = 0.04$

(iii) $d(0, \frac{1}{4}) = d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{6}) = 0.08$

(iv) $d(0, \frac{1}{5}) = d(\frac{1}{2}, \frac{1}{6}) = d(\frac{1}{3}, \frac{1}{6}) = 0.18$

(v) $d(0, \frac{1}{6}) = d(\frac{1}{2}, \frac{1}{4}) = d(\frac{1}{3}, \frac{1}{5}) = 0.36$

(vi) $d(x, y) = |x - y|^2$, otherwise.

Clearly, (X, d) is a complete b-rectangular metric space with $s = 3.1$. But (X, d) is neither a metric nor a rectangular metric space. Because,

$$0.36 = d(\frac{1}{2}, \frac{1}{4}) > d(\frac{1}{2}, \frac{1}{5}) + d(\frac{1}{5}, \frac{1}{4}) = 0.04 + 0.04 = 0.08 \text{ and}$$

$$0.36 = d(\frac{1}{3}, \frac{1}{5}) > d(\frac{1}{3}, 0) + d(0, \frac{1}{2}) + d(\frac{1}{2}, \frac{1}{5}) = 0.04 + 0.06 + 0.04 = 0.14.$$

Let $f, g : X \to X$ be defined by:

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in A \\ \frac{1}{5}, & \text{if } x \in B. \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}\} \\ \frac{1}{5}, & \text{if } x = \frac{1}{3} \\ 2, & \text{if } x \in B. \end{cases}$$

Obviously, $f(X) \subseteq g(X)$ and f and g are an L^*-contraction mappings with respect to $\zeta \colon [1, \infty) \times [1, \infty) \to R$, where

$\zeta_k(t, s) = \frac{s^k}{t}$, for all $t, s \in [1, \infty)$, for any $k \in [0.12, 1)$ and $\Theta \colon (0, \infty) \to (1, \infty)$ such that $\Theta(t) = e^t$, for all $t \in (0, \infty)$.

Indeed, for $x \in A$ and $y \in B$, we have

$$d(fx, fy) = d(\frac{1}{2}, \frac{1}{5}) = 0.04 > 0.$$

And

$$\zeta[\Theta(s^2 d(fx, fy)), \Theta(M(gx, gy))] = \frac{[\Theta(M(gx, gy))]^k}{\Theta(s^2 d(fx, fy))}, \tag{2.18}$$

where

$$M(gx, gy) = max\{d(gx, gy), d(gx, fx), d(gy, fy)\}.$$

But,

$$d(gx, gy) = \begin{cases} d(\frac{1}{2}, \frac{1}{5}) = 0.04 & \text{if } x \in \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}\} \text{ and } y = \frac{1}{3} \\ d(\frac{1}{2}, 2) = 2.25 & \text{if } x \in \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}\} \text{ and } y \in B \\ d(\frac{1}{5}, 2) = 3.24 & \text{if } x = \frac{1}{3} \text{ and } y \in B, \end{cases}$$

$$d(gx, fx) = \begin{cases} d(\frac{1}{2}, \frac{1}{2}) = 0 & \text{if } x \in A \\ d(\frac{1}{5}, \frac{1}{2}) = 0.04 & \text{if } x \in A \\ d(2, \frac{1}{5}) = 3.24 & \text{if } x \in B. \end{cases}$$

$$d(gx, fy) = \begin{cases} d(\frac{1}{2}, \frac{1}{2}) = 0 & \text{if } y \in A \\ d(\frac{1}{5}, \frac{1}{2}) = 0.04 & \text{if } y \in A \\ d(2, \frac{1}{5}) = 3.24 & \text{if } y \in B. \end{cases}$$

Hence, $M(gx, gy) = 3.24$.
Now (2.18), becomes

$$\frac{[\Theta(M(gx, gy))]^k}{\Theta(s^2 d(fx, fy))} \geq \frac{e^{k(3.24)}}{e^{(3.1)^2 \times 0.04}} = \frac{e^{3.24k}}{e^{0.3844}} = e^{3.24k - 0.3844} \geq 1, \ for\ any\ k \in [0.12, 1).$$

Therefore, all conditions of Theorem 2.2 are satisfied, f and g have a common fixed point $v = \dfrac{1}{2}$ which is unique.

In the following we deduce corollaries to Theorem 2.2.

Corollary 2.4. *Let (X, d) be a b-rectangular metric space with coefficient $s \geq 1$ and $f, g: X \to X$ be self-maps with $f(X) \subseteq g(X)$. Suppose that there exists $\Theta \in \Omega_{1,2,4}$ and $k \in (0, 1)$ such that (for all $x, y \in X$)*

$$d(fx, fy) > 0 \Rightarrow \Theta(s^2 d(fx, fy)) \leq [\Theta(M(gx, gy))]^k,$$
$$where\ M(gx, gy) = max\{d(gx, gy), d(gx, fx), d(gy, fy)\}.$$

If either $g(X)$ or $f(X)$ is complete, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Observe that f and g are generalized L^*-contraction with respect to $\zeta_k(t, s) = (s^k/t)$. Then the result follows immediately from Theorem 2.2. □

Corollary 2.5. *Let (X, d) be a b-rectangular metric space with coefficient $s \geq 1$ and $f, g: X \to X$ be two mappings with $f(X) \subseteq g(X)$. Suppose that there exists $\Theta \in \Omega_{1,2,4}$ and $k \in (0, 1)$ such that (for all $x, y \in X$)*

$$d(fx, fy) > 0 \Rightarrow s^2 d(fx, fy) \leq M(gx, gy) - \varphi(M(gx, gy)), \tag{2.19}$$

$$where\ \ M(gx, gy) = max\{d(gx, gy), d(gx, fx), d(gy, fy)\}\ and\ \varphi\colon R^+ \to R^+$$

is non-decreasing and lower semi-continuous such that $\varphi^{-1}\{(0)\} = 0$.
If either $g(X)$ or $f(X)$ is complete, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $\Theta(t) = e^t$, for all $t > 0$. From (2.19), we have

$$\Theta(s^2 d(fx, fy)) = e^{s^2 d(fx,fy)} \leq e^{M(gx,gy) - \varphi(M(gx,gy))} = \frac{\Theta(M(gx, gy))}{e^{\varphi(M(gx,gy))}}, \tag{2.20}$$

for all $x, y \in X$ with $d(fx, fy) > 0$.
Now, define $\varphi(t) = ln(\psi(\Theta(t)))$, for all $t > 0$, where $\psi\colon [1, \infty) \to [1, \infty)$ is non-decreasing and lower semi-continuous such that $\psi^{-1}(\{1\}) = 1$.
From (2.20), we have

$$\Theta(s^2 d(fx, fy)) \leq \frac{\Theta(M(gx, gy))}{\psi(\Theta(M(gx, gy)))}. \tag{2.21}$$

Taking $\zeta(t, s) = (s/t\psi(s))$ and using (2.21), we have

$$\begin{aligned} 1 &\leq \frac{\Theta(M(gx, gy))}{\Theta(s^2 d(fx, fy))\psi(\Theta(M(gx, gy)))} \\ &= \zeta[\Theta(s^2 d(fx, fy)), \Theta(M(gx, gy))]. \end{aligned}$$

Therefore, all the requirements of Theorem 2.2 are satisfied and hence f and g have a unique common fixed point in X. □

References

[1] M. Abbas and G. Jungck, *Common fixed point results for non-commuting mappings without continuity in cone metric spaces*, Journal of Mathematical Analysis and Applications, **341(1)**, 416-420, (2008).

[2] J. Ahmad, A.E. Al-Mazrooei, Y.J. Cho and Y.O. Yang, *Fixed point results for generalized theta-contractions*, Journal of Nonlinear Sciences and Applications, **10(5)**, 2350-2358, (2017).

[3] M. Arshad, J. Ahmad, E. Karapinar, *Some common fixed point results in rectangular metric spaces*, International Journal of analysis, 1-7, (2013).

[4] I. Bakhtin, *The contraction mapping principle in quasimetric spaces*, Functional analysis, **30**, 26-37, (1989).

[5] S. Banach, *Sur les opérationsdansles ensembles abstraitsetleurapplication aux equations integrals*, Fund.math., **3(1)**, 133-181, (1922).

[6] C.D. Bari, P. Vetro, *Common fixed points in generalized metric spaces*, Applied Mathematics and Computation, **218(13)**, 7322-7325, (2012).

[7] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publicationes Mathematicae Debrecen, **57(1)**, 31-37, (2000).

[8] S.H. Cho, *Fixed point theorems for-contractions in generalized metric spaces*, In Abstract and Applied Analysis 1-6, (2018).

[9] M.U. Rahman, M. Sarwar, *Coupled fixed point theorem for rational contraction conditions in dislocated quasi-metric space*, Palestine Journal of Mathematics, **5(2)**, 6-11, (2016).

[10] Haque, S. *New types of Meir-Keeler contractions in metric like spaces*, Palestine Journal of Mathematics, **12(2)**, (2023).

[11] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta mathematica etinformatica universita tisostraviensis **1(1)**, 5-11, (1993).

[12] R. George, S. Radenovic, K.P. Reshma and S. Shukla, *Rectangular b-metric space and contraction principles*, J. Nonlinear Sci. Appl., **8(6)**, 1005-1013, (2015).

[13] N.S. Hayel, I. Mohammad, A. Thabet and A. Mohammad, *New Contractive Mappings and Their Fixed Points in Branciari Metric Spaces,* Journal of Function Spaces, 11 pages, (2020).

[14] H. Guan, J. Li, and Y. Hao, *Common fixed point theorems for weakly contractions in rectangular b-metric spaces with supportive applications,* Journal of Function Spaces, 1-16, (2022).

[15] H.S. Ding, V. Ozturk and S. Radenovic, *On some new fixed point results in b-rectangular metric spaces*, J. Nonlinear Sci. Appl., **8**, 378-386, (2015).

[16] M. Imdad, W.M. Alfaqih and I.A. Khan, *Weak θ-contractions and some fixed point results with applications to fractal theory*, Advances in Difference Equations, **1**, 1-18, (2018).

[17] M.U. Rahman and M. Sarwar, *Some new fixed point theorems in dislocated quasi-metric spaces*, Palest. J. Math., **5**, 171-176, (2016).

[18] H. Kerim, W. Shatanawi and A. Tallafha, *COMMON FIXED POINT THEOREMS FOR SET-VALUED MAPS ON MODULAR B-GAUGE SPACES*, Palestine Journal of Mathematics, **11(3)**, (2022).

[19] M. Jleli and B. Samet, *A new generalization of the Banach contraction principle*, Journal of inequalities and applications, **1**, 1-8, (2014).

[20] G. Jungck, *Commuting mappings and fixed points,* Amer. Math. Monthly, **83**, 261-263, (1976).

[21] G. Jungck and B.E. Rhoades, *Fixed point for set valued functions without continuity*, Indian J.Pure Appl. Math., **29(3)**, 227-238, (1998).

[22] A. Kari, M. Rossafi, E.M. Marhrani and M. Aamri, *New fixed point theorems for θ, φ-contraction on rectangular b-metric space*, In Abstract and Applied Analysis, 1-12, (2020).

[23] H.N. Saleh, M. Imdad, T. Abdeljawad and M. Arif, *New contractive mappings and their fixed points in Branciari metric spaces*, Journal of Function Spaces, (2020).

[24] R.P. Pant, *Common fixed points of Lipschitz type mapping pairs*, Journal of Mathematical Analysis and Applications, **240(1)**, 280-283, (1999).

[25] J. R. Roshan, V. Parvaneh and Z. Kadelburg, *New fixed point results in b-rectangular metric spaces*, Nonlinear Analysis: Modelling and Control, **21(5)**, 614-634, (2016).

[26] V. Öztürk, *Fixed Point Theorems in b-Rectangular Metric Spaces*, Universal Journal of Mathematics and Applications, **3(1)**, 28-32, (2020).

Author information

A.N. Sakelo, Department of Mathematics, Jimma University, Ethiopia.
E-mail: `alemunegasa342@gmail.com`

K.K. Tola, Department of Mathematics, Jimma University, Ethiopia.
E-mail: `kidanekoyas@yahoo.com`

M.A. Mamud, Department of Mathematics, Jimma University, Ethiopia.
E-mail: `musteabduletif@gmail.com`

M.T. A/mecha, Department of Mathematics, Jimma University, Ethiopia.
E-mail: `temammohamedamin@gmail.com`

Received: 2023-08-23
Accepted: 2024-01-24