Palestine Journal of Mathematics

Vol 13(4)(2024) , 859–864

GENERALIZED DERIVATIONS AND COMMUTATIVITY OF RINGS WITH PRIME IDEAL

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Communicated by Muzibur Rahman Mozumder

MSC 2010 Classifications: 16N60, 16U80, 16W25.

Keywords: Prime ideal, Commutativity, Generalized derivation.

The authors are grateful to the referee for his/her valuable suggestions and remarks that definitely improved the paper.

The authors would like to thank the Integral University, India for the continuous support and encouragement to carry out this research work under the manuscript communication number IU/R & D/ 2022- MCN 0001363.

Abstract. Let R be a ring and P be a prime ideal of R. An additive mapping $d : R \to R$ is called a derivation if for any $p, q \in R, d(pq) = d(p)q + pd(q)$. In this paper, we investigate the commutativity of the factor ring R/P satisfying certain differential identities. More precisely, there is no primeness or semi-primeness assumption on the considered ring R.

1 Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$. Cite that an ideal P of R is said to be prime if $P \neq R$ and for $a, b \in R, aRb \subseteq P$ implies that $a \in P$ or $b \in P$. The ring R is called a prime ring if $a, b \in R, aRb = (0)$ implies $a = 0$ or $b = 0$. The Lie product of two elements x and y of R is $[x, y] = xy - yx$, while the symbol $x \circ y$ denotes a Jordan product which is defined as $xy + yx$. An additive mapping $d : R \to R$ is a derivation on R if it satisfies $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive map $F : R \to R$ is said to be a generalized derivation associated with a derivation d on R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Generally, we do not mention the derivation d associated with a generalized derivation F, rather we call F, a generalized derivation. It is noteworthy that the concept of generalized derivation includes the concept of derivation and generalized inner derivation and that of the left multipliers when $d = 0$.

Several authors subsequently proved commutativity theorems for prime rings admitting derivations which are centralizing on R. This work was initiated by Posner [4] who proved that a prime ring R admitting a non zero centralizing derivation is commutative. Since then a number of authors have extended the Posner's result in several directions. In [5], Vukman proved that if R admits a non zero derivation d such that the mapping $x \to [d(x), x]$ is centralizing on R, then R is commutative provided the characteristic of R is different from 2 and 3.

In this paper, we continue this line of investigation by considering more general situations. More precisely, we are interested in the study of rings given as a quotient R/P where R is an arbitrary ring and P is a prime ideal of R. In this work we are using a derivation and a generalized derivation on R (and not on R/P) which satisfies certain differential identities on R, without assuming R to be a prime ring.

Many authors have recently examined the rings given as the quotient R/P. In [7], M.S. Khan et. al. studied about the action of generalized derivations on prime ideals of an arbitrary ring with involution. A generalization of Posner's theorem for the quotient ring R/P is given by Almahdi et. al. in [1].

2 Commutativity of rings admitting generalized derivation

In the year 1992, Daif and Bell [3] obtained commutativity of semi-prime ring R satisfying differential identity $d([x,y]) = \pm[x,y]$ for all $x,y \in R$. Later on, many authors explored commutativity of prime and semi-prime rings satisfying various conditions on rings (for reference see [2], [6], [7] and [8] where further references can be found).

In the present paper, we study these differential identities in the setting of generalized derivation on an arbitrary ring R. Infact we obtained the following results

Theorem 2.1. *Let R be a ring and P be a prime ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F([p,q]) - [p,q] \in P$, then either $d(R) \subseteq P$ or R/P is commutative.*

Proof. For any p,q in R, we have $F([p,q]) - [p,q] \in P$, which gives

$$F(p)q + pd(q) - F(q)p - qd(p) - [p,q] \in P \quad \text{for all } p,q \in R. \tag{2.1}$$

Replacing q by qr gives

$$F(p)qr + pd(q)r + pqd(r) - F(q)rp - qd(r)p - qrd(p) - q[p,r] - [p,q]r \in P \quad \text{for all } p,q,r \in R. \tag{2.2}$$

Using (2.1), we have

$$F(q)pr + qd(p)r + pqd(r) - F(q)rp - qd(r)p - qrd(p) - q[p,r] \in P \quad \text{for all } p,q,r \in R. \tag{2.3}$$

Above equation results in

$$F(q)[p,r] + q[d(p),r] + q[p,d(r)] + [p,q]d(r) - q[p,r] \in P \quad \text{for all } p,q,r \in R. \tag{2.4}$$

Now, replacing r by rp, we get

$$F(q)[p,r]p + qr[d(p),p] + q[d(p),r]p + [p,q]d(r)p + [p,q]rd(p) + q[p,d(r)]p + qr[p,d(p)] + q[p,r]d(p) - q[p,r]p \in P \quad \text{for all } p,q,r \in R. \tag{2.5}$$

Using (2.4), we find that

$$qr[d(p),p] + [p,q]rd(p) + qr[p,d(p)] + q[p,r]d(p) \in P \quad \text{for all } p,q,r \in R.$$

That is

$$[p,q]rd(p) + q[p,r]d(p) \in P \quad \text{for all } p,q,r \in R. \tag{2.6}$$

Now, replacing q by q_1q, we obtain

$$q_1[p,q]rd(p) + [p,q_1]qrd(p) + q_1q[p,r]d(p) \in P \quad \text{for all } p,q,q_1,r \in R. \tag{2.7}$$

By (2.6), we have

$$[p,q_1]qrd(p) \in P \quad \text{for all } p,q_1 \in R. \tag{2.8}$$

That is

$$[p,q_1]Rd(p)Rd(p) \subseteq P \quad \text{for all } p,q_1 \in R. \tag{2.9}$$

By using the fact that P is prime, we get

$$[p,q_1] \in P \text{ or } d(p) \in P \quad \text{for all } p,q_1 \in R. \tag{2.10}$$

Consequently, R is a union of two additive subgroups G_1 and G_2, where

$$G_1 = \{p \in R | d(p) \in P\} \text{ and } G_2 = \{p \in R | [R,p] \subset P\}.$$

Since a group cannot be a union of two of its proper subgroups, As a result, we must determine that either $R = G_1$ or $R = G_2$. Hence, either $d(R) \subseteq P$ or R/P is commutative. □

Theorem 2.2. *Let R be a ring and P be a prime ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F([p,q]) + [p,q] \in P$, then either $d(R) \subseteq P$ or R/P is commutative.*

Proof. For any p, q in R, we have $F([p,q]) + [p,q] \in P$, which gives

$$F(p)q + pd(q) - F(q)p - qd(p) + [p,q] \in P \quad \text{for all } p, q \in R. \tag{2.11}$$

Replacing q by qr we obtain

$$\begin{aligned} &F(p)qr + pd(q)r + pqd(r) - F(q)rp - qd(r)p - qrd(p) \\ &+ q[p,r] + [p,q]r \in P \quad \text{for all } p, q, r \in R. \end{aligned} \tag{2.12}$$

Using (2.11), we have

$$\begin{aligned} &F(q)[p,r] + q[d(p),r] + q[p,d(r)] + [p,q]d(r) \\ &+ q[p,r] \in P \quad \text{for all } p, q, r \in R. \end{aligned} \tag{2.13}$$

Replacing r by rp, the equation (2.13) gives

$$\begin{aligned} &F(q)[p,r]p + qr[d(p),p] + q[d(p),r]p + q[p,d(r)p + rd(p)] \\ &+ [p,q](d(r)p + rd(p)) + q[p,r]p \in P \quad \text{for all } p, q, r \in R. \end{aligned} \tag{2.14}$$

Using (2.13), we find that

$$q[p,r]d(p) + [p,q]rd(p) \in P \quad \text{for all } p, q, r \in R. \tag{2.15}$$

Replacing q by q_1q, we get

$$q_1q[p,r]d(p) + q_1[p,q]rd(p) + [p,q_1]qrd(p) \in P \quad \text{for all } p, q_1, q, r \in R. \tag{2.16}$$

Using (2.15), we have

$$[p,q_1]qrd(p) \in P \quad \text{for all } p, q_1, q, r \in R. \tag{2.17}$$

That is

$$[p,q_1]Rd(p)Rd(p) \subseteq P \quad \text{for all } p, q_1 \in R. \tag{2.18}$$

By using the fact that P is prime, we get

$$[p,q_1] \in P \text{ or } d(p) \in P \quad \text{for all } p, q_1 \in R.$$

Following on the same lines as above after (2.10), we find that either $d(R) \subseteq P$ or R/P is commutative. □

Theorem 2.3. *Let R be a ring, P be a prime ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F(pq) - F(qp) \in P$, then either $d(R) \subseteq P$ or R/P is commutative.*

Proof. For any p, q in R, we have $F(pq) - F(qp) \in P$, which gives

$$F(p)q + pd(q) - F(q)p - qd(p) \in P \quad \text{for all } p, q \in R. \tag{2.19}$$

Replacing q by qp, we have

$$\begin{aligned} &F(p)qp - F(q)pp - qd(p)p + pd(q)p \\ &+ pqd(p) - qpd(p) \in P \quad \text{for all } p, q \in R. \end{aligned} \tag{2.20}$$

Using (2.19), we have

$$pqd(p) - qpd(p) \in P \quad \text{for all } p, q \in R. \tag{2.21}$$

Which implies

$$[p,q]d(p) \in P \quad \text{for all } p, q \in R. \tag{2.22}$$

Replacing q by qr, we get

$$q[p,r]d(p)+[p,q]rd(p)\in P \quad \text{for all } p,q,r\in R. \tag{2.23}$$

Using (2.22), we get

$$[p,q]rd(p)\in P \quad \text{for all } p,q,r\in R. \tag{2.24}$$

That is

$$[p,q]Rd(p)\subseteq P \quad \text{for all } p,q\in R. \tag{2.25}$$

Consequently, using similar arguments after (2.10), either $d(R)\subseteq P$ or R/P is commutative. □

Theorem 2.4. *Let R be a ring and P be a prime ideal of R. If R admits a generalized derivation F associated with a derivation d such that $[F(p),q]-[p,F(q)]\in P$, then either $d(R)\subseteq P$ or R/P is commutative.*

Proof. For any p,q in R, we have

$$[F(p),q]-[p,F(q)]\in P \quad \text{for all } p,q\in R. \tag{2.26}$$

Replacing q by qr, we obtain

$$\begin{aligned} &q([F(p),r]-[p,F(r)])+([F(p),q]-[p,F(q)])r+q[p,F(r)]\\ &-F(q)[p,r]-q[p,d(r)]-[p,q]d(r)\in P \quad \text{for all } p,q,r\in R. \end{aligned} \tag{2.27}$$

Using (2.26), we have

$$F(q)[p,r]+q[p,d(r)]+[p,q]d(r)-q[p,F(r)]\in P \quad \text{for all } p,q,r\in R. \tag{2.28}$$

Replacing r by rp, we get

$$\begin{aligned} &F(q)[p,r]p+q[p,d(r)]p+[p,q]d(r)p\\ &+[p,q]rd(p)-q[p,F(r)]p\in P \quad \text{for all } p,q,r\in R. \end{aligned} \tag{2.29}$$

Using (2.28), we get

$$[p,q]rd(p)\in P \quad \text{for all } p,q,r\in R. \tag{2.30}$$

That is

$$[p,q]Rd(p)\subseteq P \quad \text{for all } p,q\in R. \tag{2.31}$$

Hence, by similar arguments after (2.10), either $d(R)\subseteq P$ or R/P is commutative. □

Theorem 2.5. *Let R be a ring and P be a prime ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F(p\circ q)-p\circ q\in P$, then either $d(R)\subseteq P$ or R/P is commutative.*

Proof. For any p,q in R, we have $F(p\circ q)-p\circ q\in P$, which is

$$F(p)q+pd(q)+F(q)p+qd(p)-p\circ q\in P \quad \text{for all } p,q\in R. \tag{2.32}$$

Replacing q by qp, we get

$$\begin{aligned} &F(p)qp+pd(q)p+pqd(p)+F(q)p^2+qd(p)p\\ &+qpd(p)-(p\circ q)p\in P \quad \text{for all } p,q\in R. \end{aligned} \tag{2.33}$$

Using (2.32), we obtain

$$(p\circ q)d(p)\in P \quad \text{for all } p,q\in R. \tag{2.34}$$

Replacing q by rq, we get

$$r(p\circ q)d(p)+[p,r]qd(p)\in P \quad \text{for all } p,q,r\in R. \tag{2.35}$$

Using (2.34), we find that

$$[p,r]qd(p)\in P \quad \text{for all } p,q,r\in R.$$

Which is

$$[p,r]Rd(p)\subset P \quad \text{for all } p,r\in R. \tag{2.36}$$

Using similar arguments as used in the proof of previous theorem we get the required result. □

Theorem 2.6. *Let R be a ring and P be a prime ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F([p,q]) - p \circ q \in P$, then either $d(R) \subseteq P$ or R/P is commutative.*

Proof. For any p,q in R, we have $F([p,q]) - p \circ q \in P$, which is

$$F(p)q + pd(q) - F(q)p - qd(p) - p \circ q \in P \quad \text{for all } p,q \in R. \tag{2.37}$$

Replacing q by qr, we get

$$\begin{aligned} &F(p)qr + pd(q)r + pqd(r) - F(q)rp - qd(r)p - qrd(p) \\ &- (p \circ q)r + q[p,r] \in P \quad \text{for all } p,q,r \in R. \end{aligned} \tag{2.38}$$

Using (2.37), we obtain

$$F(q)[p,r] + q[d(p),r] + q[p,d(r)] + [p,q]d(r) + q[p,r] \in P \quad \text{for all } p,q,r \in R. \tag{2.39}$$

Now, replacing r by rp, we find that

$$\begin{aligned} &F(q)[p,r]p + qr[d(p),p] + q[d(p),r]p + q[p,d(r)]p + q[p,rd(p)] \\ &+ [p,q]d(r)p + [p,q]rd(p) + q[p,r]p \in P \quad \text{for all } p,q,r \in R. \end{aligned} \tag{2.40}$$

Using (2.39), we get

$$[p,q]rd(p) + q[p,r]d(p) \in P \quad \text{for all } p,q,r \in R. \tag{2.41}$$

Replacing q by q_1q, we get

$$[p,q_1]qrd(p) \in P \quad \text{for all } p,q,q_1,r \in R. \tag{2.42}$$

That is

$$[p,q_1]Rd(p)Rd(p) \subseteq P \quad \text{for all } p,q_1 \in R. \tag{2.43}$$

By using the fact that P is prime, we get

$$[p,q_1] \in P \text{ or } d(p) \in P \quad \text{for all } p,q_1 \in R.$$

Following on the same lines as above, we obtain that either $d(R) \subseteq P$ orR/P is commutative. □

Theorem 2.7. *Let R be a ring and P be a prime ideal of R. If R admits a generalized derivation F associated with a derivation d such that $F(p \circ q) - [p,q] \in P$, then either $d(R) \subseteq P$ or R/P is commutative.*

Proof. For any p,q in R, we have $F(p \circ q) - [p,q] \in P$, which is

$$F(p)q + pd(q) + F(q)p + qd(p) - [p,q] \in P \quad \text{for all } p,q \in R. \tag{2.44}$$

Replacing q by qr, we get

$$\begin{aligned} &F(p)qr + pd(q)r + pqd(r) + F(q)rp + qd(r)p + qrd(p) \\ &- q[p,r] - [p,q]r \in p \quad \text{for all } p,q,r \in R. \end{aligned} \tag{2.45}$$

Using (2.44), we get

$$\begin{aligned} &F(q)[r,p] + q[r,d(p)] + (p \circ q)d(r) \\ &- q[p,d(r)] - q[p,r] \in P \quad \text{for all } p,q,r \in R. \end{aligned} \tag{2.46}$$

Now, replacing r by p

$$(p \circ q)d(p) \in P \quad \text{for all } p,q,r \in R. \tag{2.47}$$

Replacing q by rq, we obtain

$$r(p \circ q)d(p) + [p,r]qd(p) \in P \quad \text{for all } p,q \in R. \tag{2.48}$$

Using (2.47), we get

$$[p, r]qd(p) \in P \quad \text{for all } p, q, r \in R. \tag{2.49}$$

That is

$$[p, r]Rd(p) \subseteq P \quad \text{for all } p, r \in R. \tag{2.50}$$

By using the fact that P is prime, we get

$$[p, r] \in P \text{ or } d(p) \in P \quad \text{for all } p, r \in R.$$

Following on the same lines as above, we conclude that either $d(R) \subseteq P$ or R/P is commutative. □

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Received: 2023-08-29
Accepted: 2023-11-04