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On Narayana-Lucas Hybrinomial Sequence

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Abstract The main objective of this study is to introduce and analyze the Narayana-Lucas hybrinomial sequence. We thoroughly analyze different elements of this sequence, including the recurrence relation, matrix representation, generating function, Binet's formula, exponential generating function, and Poisson generating function. In addition, we investigate numerous well-known identities, including the Catalan identity, the Cassini identity, the d'Ocagne identity, the Gelin-Cesaro identity, and the Melham identity, related to this newly formed sequence. Finally, we provide the source Maple 13 code.

1 Introduction

In the 14th century, Indian mathematician Narayana studied the problem of a herd of cows and calves, which resulted in the development of the Narayana numbers[1, 2]. The problem posed was as follows: At the beginning of each year, a cow produces one calf. Starting from the fourth year, each calf then produces one calf at the beginning of each subsequent year. How many calves are there in total after 20 years?[1]. This problem bears resemblance to the Fibonacci rabbit problem, and just as Fibonacci solved the rabbit problem[3], the Narayana cow problem can be similarly addressed. In Narayana sequence, denoted by N_m, the recurrence relation is defined as $N_{m+3} = N_{m+2} + N_m$ for all $m \geq 0$, with initial conditions $N_0 = 2, N_1 = 3$, and $N_2 = 4$ [4]. Recently, there has been significant interest in the Narayana sequence and Narayana-like sequences along with their generalization, as evidenced by various studies [4, 5, 6, 7, 8, 9, 10, 11]. Lucas numbers are the numbers in the integer sequence defined by the recurrence relation $L_m = L_{m-1} + L_{m-2}$ for all $m > 1$ with initial conditions $L_0 = 2$ and $L_1 = 1$. The Lucas numbers are also closely related to the Fibonacci sequence. In [12] the author defined a new recurrence relation which is called k-Lucas numbers. In [7] author described the generalization of Narayana's numbers as well as two further special cases, the Narayana-Lucas Sequence and Narayana-Perrin sequence, and also defined some of their identities. The recurrence relation for Narayana-Lucas sequence is defined as $U_{m+3} = U_{m+2} + U_m$ for all $m \geq 0$ with initial conditions $U_0 = 3, U_1 = 1, U_2 = 1$.
In the work by Özedmir [13], the pioneering exploration into hybrid numbers was presented. Hybrid numbers constitute a novel amalgamation of real, complex, hyperbolic, and dual numbers.

The collection of hybrid numbers, denoted as H, is formally characterized as follows:

$$H = \{z = a + b\iota + c\epsilon + dh; a, b, c, d \in \mathbb{R}\}$$

Here, the operators ι, ϵ, and h are introduced with specific properties: $\iota^2 = -1$, $\epsilon^2 = 0$, $h^2 = 1$, and $\iota h = -h\iota = \epsilon + \iota$. The relationship of a hybrid number z with its conjugate is expressed as:

$$\overline{z} = \overline{a + b\iota + c\epsilon + dh} = a - b\iota - c\epsilon - dh.$$

The character of hybrid numbers z is defined as the real numbers $C(z) = z\overline{z} = \overline{z}z = a^2 + b^2 - 2bc - d^2$ and the norm of hybrid numbers z is defined as $\sqrt{|C(z)|}$ and denoted by $\|z\|$ [13]. Hybrid numbers with various sequences such as Jacobsthal and Jacobsthal hybrid numbers, have earned a lot of interest recently [14]. In [15, 16, 17, 18, 19] the authors expanded the Padvon, Perrin, Fibonacci, Lucas and k- Pell hybrid sequence and studied their several identities.
In the development of science and technology, integer sequences have played a significant role and are extensively used, particularly in mathematics and various other fields of science. Hybrid numbers represent a fresh extension encompassing real, complex, hyperbolic and dual numbers. They find extensive utility across diverse mathematical domains and hold pragmatic significance in scientific investigations, design endeavors, and speculative areas of physical science. These sequences find applications in various fields such as algebra, number theory and geometry [20, 21].

In the context of classical studies, the behavior of the Narayana sequence is almost similar to the Fibonacci sequence but distinct in terms of their order of occurrence, as well as their treatment of limiting ratios. The study of the Narayana sequence and the Narayana-Lucas hybrinomial sequence can lead to various practical and theoretical applications in distinct fields. Examples of direct and indirect applications of these sequences in different fields may include graph theory, number theory, group theory, cryptography, architecture, stereographic techniques, and more. The novelty in our results is that, due to the additional complexity introduced by the Narayana-Lucas hybrinomial sequence, its study becomes a matter of interest in this direction.

In the present paper, our focus revolves around the Narayana-Lucas hybrinomial sequence. We delve an insight into its properties, including the determination of generating functions, Binet's formula, and the exploration of renowned identities such as Catalan's identity, Cassini's identity, d'Ocagne identity, Gelin-Cesaro identity, and Melham's identity. By investigating these aspects, our aim is to enhance the understanding and applicability of the Narayana-Lucas hybrinomial sequence in different fields of science.

2 Naryana-Lucas Hybrinomial Sequence

The present section aims to define the Narayana-Lucas hybrinomial sequence and then some simple findings related to Narayana-Lucas hybrinomial sequence.

Definition 2.1. Narayana polynomial sequence in [4] is defined by

$$N_m(r) = \begin{cases} 2, & m=0, \\ 3, & m=1, \\ 4, & m=2, \\ rN_{m-1}(r) + N_{m-3}(r), & m \geq 3. \end{cases}$$

The first few terms of $N_m(r)$ are :
$2, 3, 4, 4r+2, 4r^2+2r+3, 4r^3+2r^2+3r+4, \cdots$

Definition 2.2. Narayana-Lucas polynomial sequence in [7] is defined by

$$U_m(r) = \begin{cases} 3, & \text{m=0}, \\ 1, & \text{m=1}, \\ 1, & \text{m=2}, \\ rU_{m-1}(r) + U_{m-3}(r), & m \geq 3. \end{cases}$$

The first few terms of $U_m(r)$ are
$3, 1, 1, r+3, r^2+3r+1, r^3+3r^2+r+1, \cdots$

Definition 2.3. Narayana-Lucas hybrinomial sequence denoted by $\{HU_m(r)\}$ is defined by

$$HU_m(r) = U_m(r) + U_{m+1}(r)\iota + U_{m+2}(r)\epsilon + U_{m+3}(r)h, \tag{2.1}$$

where $U_m(r)$ is the Narayana-Lucas polynomial sequence. Consequently, certain initial values for the Narayana-Lucas hybrinomial sequence are:

$$\begin{aligned} HU_0(r) =& 3 + \iota + \epsilon + (r+3)h, \\ HU_1(r) =& 1 + \iota + (r+3)\epsilon + (r^2+3r+1)h, \\ HU_2(r) =& 1 + (r+3)\iota + (r^2+3r+1)\epsilon + (r^3+3r^2+r+1)h, \\ HU_3(r) =& (r+3) + (r^2+3r+1)\iota + (r^3+3r^2+r+1)\epsilon \\ &+ (r^4+3r^3+r^2+2r+3)h. \end{aligned}$$

Considering the definition of the Narayana-Lucas hybrinomial sequence for the case $r=1$, it becomes apparent that

$$\begin{aligned} HU_0(1) =& 3 + \iota + \epsilon + 4h, \\ HU_1(1) =& 1 + \iota + 4\epsilon + 5h, \\ HU_2(1) =& 1 + 4\iota + 5\epsilon + 6h, \\ HU_3(1) =& 4 + 5\iota + 6\epsilon + 10h. \end{aligned}$$

Therefore, we can say that the Narayana-Lucas hybrinomial sequence $\{HU_m(r)\}$ is a generalization of the Narayana-Lucas hybrid numbers.
The norm of the Narayana-Lucas hybrinomial sequence can be easily deduced from the definition of a hybrinomial sequence (by following the similar steps mentioned in [4]), which is given by:

$$\begin{aligned} \|HU_m(r)\| =& \sqrt{C(HU_m(r))} \\ =& \sqrt{|U_m^2(r) + U_{m+1}^2(r) - 2U_{m+1}(r)U_{m+2}(r) - U_{m+3}^2(r)|} \\ \|HU_m(r)\|^2 =& |U_m^2(r) + U_{m+1}^2(r) - 2U_{m+1}(r)U_{m+2}(r) - U_{m+3}^2(r)| \\ =& |(U_{m+3}(r) - rU_{m+2}(r))^2 + U_{m+1}^2(r) - 2U_{m+1}(r)U_{m+2}(r) \\ & -U_{m+3}^2(r)| \\ =& |U_{m+3}^2(r) + r^2U_{m+2}^2(r) - 2rU_{m+3}(r)U_{m+2}(r) + U_{m+1}^2(r) \\ & -2U_{m+1}(r)U_{m+2}(r) - U_{m+3}^2(r)| \\ =& |U_{m+1}^2(r) + U_{m+2}(r)(r^2U_{m+2}(r) - 2rU_{m+3}(r) - 2U_{m+1}(r))| \\ =& |U_{m+1}^2(r) - U_{m+2}(r)(r^2U_{m+2}(r) + 2rU_{m+1}(r) + 2U_m(r))| \\ \|HU_m(r)\|^2 =& |U_{m+1}^2(r) - U_{m+2}(r)(r^2U_{m+2}(r) + 2rU_{m+1}(r) + 2U_m(r))|. \end{aligned}$$

Lemma 2.4. *Let $\{HU_m(r)\}$ represents the Narayana-Lucas hybrinomial sequence, where m is a non-negative integer. Then the recurrence relation is given by:*

$$HU_m(r) = rHU_{m-1}(r) + HU_{m-3}(r).$$

Proof. From (2.1), we have

$$\begin{aligned} HU_m(r) - rHU_{m-1}(r) - HU_{m-3}(r) &= (U_m(r) - rU_{m-1}(r) + U_{m-3}(r)) + \\ (U_{m+1}(r) - rU_m(r) + U_{m-2}(r))\iota &+ (U_{m+2}(r) - rU_{m+1}(r) - U_{m-1}(r))\epsilon \\ &+ (U_{m+3}(r) - rU_{m+2}(r) - U_m(r))h, \end{aligned}$$

as $U_m(r)$ forms a Narayana polynomial sequence, the result is that the right-hand side of the aforementioned equation equals zero. This leads us to the conclusion that

$$HU_m(r) - rHU_{m-1}(r) - HU_{m-3}(r) = 0.$$

Thus the proof is completed. □

2.1 Matrix representation of Narayana-Lucas Hybrinomial Sequence

Özedmir [13] provide the following relation to describe the matrix representation of the hybrinomial sequence :

$$M(a + b\iota + c\epsilon + dh) = \begin{bmatrix} a+c & b-c+d \\ c-b+d & a-c \end{bmatrix},$$

Now, we represent the Narayana-Lucas hybrinomial sequence of order m in the 2×2 matrix as:

$$M_{HU_m}(r) = \begin{bmatrix} U_m(r) + U_{m+2}(r) & U_{m+1}(r) + (r-1)U_{m+2}(r) + U_m(r) \\ (r-1)U_{m+1}(r) + U_{m-1}(r) + U_{m+3}(r) & U_m(r) - U_{m+2}(r) \end{bmatrix}.$$

The hybrid matrix corresponding to the Narayana-Lucas hybrinomial sequence $\{HU_m(r)\}$ shall be referred to as $\{M_{HU_m(r)}\}$.

Proposition 2.5. *Consider $M_{HU_m}(r)$ as the matrix representation of the Narayana-Lucas hybrinomial sequence $\{HU_m(r)\}$. In this context, the following relationship is valid:*

$$\|HU_m(r)\|^2 = \det(M_{HU_m}(r)).$$

3 Generating Function, Binet's formula and exponential generating function for Narayana-Lucas Hybrinomial Sequence

The aim of this section is to represent the generating function and Binet's formula for the Narayana-Lucas hybrinomial sequence, which is further followed by an important corollary representing the exponential generating function corresponding to the newly formed Narayana Lucas hybrinomial sequence.

Theorem 3.1. *Let $\{HU_m(r)\}$ be the Narayana-Lucas hybrinomial sequence. Then the generating function for $\{HU_m(r)\}$ can be written as:*

$$\begin{aligned} g(t) &= \sum_{m=0}^{\infty} HU_m(r)t^m \\ &= \frac{HU_0(r) + (HU_1(r) - HU_0(r))t + (HU_2(r) - HU_1(r))t^2}{1 - rt - t^3}. \end{aligned}$$

Proof. Let us consider the following formal power series to be the generating function for the Narayana-Lucas hybrinomial sequence as:

$$g(t) = \sum_{m=0}^{\infty} HU_m(r)t^m = HU_0(r) + HU_1(r)t + HU_2(r)t^2 + \dots$$

Then we have,

$$\begin{aligned} rtg(t) &= rHU_0(r)t + rHU_1(r)t^2 + rHU_2(r)t^3 + \dots \\ t^3g(t) &= HU_0(r)t^3 + HU_1(r)t^4 + HU_2(r)t^5 + \dots \end{aligned}$$

Therefore we get

$$\begin{aligned} g(t) - rtg(t) - t^3g(t) =& (HU_0(r) + HU_1(r)t + HU_2(r)t^2 + \dots) - (rHU_0(r)t + rHU_1(r)t^2 + rU_2(r)t^3 + \dots) \\ & - (HU_0(r)t^3 + HU_1(r)t^4 + HU_2(r)t^5 + \dots) \\ g(t)(1 - rt - t^3) =& HU_0(r) + (HU_1(r) - rHU_0(r))t + (HU_2(r) - rHU_1(r))t^2 \end{aligned}$$

$$g(t) = \frac{HU_0(r) + (HU_1(r) - rHU_0(r))t + (HU_2(r) - rHU_1(r))t^2}{(1 - rt - t^3)}$$

$$g(t) = \sum_{m=0}^{\infty} HU_n(r)t^m$$
$$= \frac{HU_0(r) + (HU_1(r) - rHU_0(r))t + (HU_2(r) - rHU_1(r))t^2}{(1 - rt - t^3)}.$$

Thus the proof is completed. □

Now, Binet's formula is provided in the following theorem for this sequence:

Theorem 3.2. *Let $\{HU_m(r)\}$ be the Narayana-Lucas hybrinomial sequence. Then the Binet's formula for $\{HU_m(r)\}$ can be written as:*

$$HU_m = (1 + \mu\iota + \mu^2\epsilon + \mu^3 h)\mu^m + (1 + \nu\iota + \nu^2\epsilon + \nu^3 h)\nu^m + (1 + \lambda\iota + \lambda^2\epsilon + \lambda^3 h)\lambda^m,$$

where,

$$\mu = \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}},$$

$$\nu = \frac{1}{3} + w\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + w^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}},$$

$$\lambda = \frac{1}{3} + w^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + w\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}},$$

where

$$w = \frac{-1 + \iota\sqrt{3}}{2} = \exp\left(\frac{2\pi\iota}{3}\right),$$

and μ, ν, λ are the characteristics roots of equation $1 - rt - t^3$.

Proof. By the definition of the Narayana-Lucas hybrinomial sequence from (2.1) and Narayana-Lucas sequences. From[**?**], we have

$$U_m = \mu^m + \nu^m + \lambda^m,$$

where

$$\mu = \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}},$$

$$\nu = \frac{1}{3} + w\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + w^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}},$$

$$\lambda = \frac{1}{3} + w^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}} + w\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{\frac{1}{3}}.$$

Thus, we get

$$\begin{aligned}HU_m(r) =& (\mu^m + \nu^m + \lambda^m) + (\mu^{m+1} + \nu^{m+1} + \lambda^{m+1})\iota + (\mu^{m+2} + \nu^{m+2} + \lambda^{m+2})\epsilon \\ &+ (\mu^{m+3} + \nu^{m+3} + \lambda^{m+3})h \\ =& \mu^m + \iota\mu^{m+1} + \epsilon\mu^{m+2} + h\mu^{m+3} + \nu^m + \iota\nu^{m+1} + \epsilon\nu^{m+2} + h\nu^{m+3} \\ &+ \lambda^m + \iota\lambda^{m+1} + \epsilon\lambda^{m+2} + h\lambda^{m+3} \\ HU_m(r) =& (1 + \iota\mu + \epsilon\mu^2 + h\mu^3)\mu^m + (1 + \iota\nu + \epsilon\nu^2 + h\nu^3)\nu^m \\ &+ (1 + \iota\lambda + \epsilon\lambda^2 + h\lambda^3)\lambda^m.\end{aligned}$$

Thus the proof is completed. □

Corollary 3.3. *Consider the Narayana-Lucas hybrinomial sequence denoted as $\{HU_m(r)\}$. Then the exponential generating function corresponding to $\{HU_m(r)\}$ can be expressed as follows:*

$$\sum_{m=0}^{\infty} HU_m(r)\frac{t^m}{m!} = (1 + \mu\iota + \mu^2\epsilon + \mu^3 h)e^{\mu t} + (1 + \nu\iota + \nu^2\epsilon + \nu^3 h)e^{\nu t}$$
$$+ (1 + \lambda\iota + \lambda^2\epsilon + \lambda^3 h)e^{\lambda t}.$$

Proof. Through the utilization of Binet's formula for the Narayana-Lucas hybrinomial sequence, we obtain:

$$\begin{aligned}\sum_{m=0}^{\infty} HU_m(r)\frac{t^m}{m!} = &\sum_{m=0}^{\infty}[(1+\mu\iota+\mu^2\epsilon+\mu^3 h)\mu^m + (1+\nu\iota+\nu^2\epsilon+\nu^3 h)\nu^m \\ &+(1+\lambda\iota+\lambda^2\epsilon+\lambda^3 h)\lambda^m]\frac{t^m}{m!}, \\ = &\Big[(1+\mu\iota+\mu^2\epsilon+\mu^3 h)\Big]\sum_{m=0}^{\infty}\frac{(\mu t)^m}{m!} + \Big[(1+\nu\iota+\nu^2\epsilon+\nu^3 h)\Big]\sum_{m=0}^{\infty}\frac{(\nu t)^m}{m!} \\ &+\Big[(1+\lambda\iota+\lambda^2\epsilon+\lambda^3 h)\Big]\sum_{m=0}^{\infty}\frac{(\lambda t)^m}{m!}, \\ = &(1+\mu\iota+\mu^2\epsilon+\mu^3 h)e^{\mu t} + (1+\nu\iota+\nu^2\epsilon+\nu^3 h)e^{\nu t} + (1+\lambda\iota+\lambda^2\epsilon+\lambda^3 h)e^{\lambda t}.\end{aligned}$$

Thus we get the result. □

Corollary 3.4. *Consider the Narayana-Lucas hybrinomial sequence denoted as $\{HU_m(r)\}$. Then the Poisson generating function corresponding to $\{HU_m(r)\}$ can be expressed as follows:*

$$\sum_{m=0}^{\infty} HU_m(r)\frac{t^m}{m!}e^{-t} = \frac{1}{e^t}\left((1+\mu\iota+\mu^2\epsilon+\mu^3 h)e^{\mu t} + (1+\nu\iota+\nu^2\epsilon+\nu^3 h)e^{\nu t} + (1+\lambda\iota+\lambda^2\epsilon+\lambda^3 h)e^{\lambda t}\right).$$

Proof. Through the utilization of Binet's formula for the Narayana-Lucas hybrinomial sequence, we obtain:

$$\begin{aligned}\sum_{m=0}^{\infty} HU_m(r)\frac{t^m}{m!}e^{-t} = &\sum_{m=0}^{\infty}[(1+\mu\iota+\mu^2\epsilon+\mu^3 h)\mu^m + (1+\nu\iota+\nu^2\epsilon+\nu^3 h)\nu^m \\ &+(1+\lambda\iota+\lambda^2\epsilon+\lambda^3 h)\lambda^m]\frac{t^m}{m!}e^{-t}, \\ = &\frac{1}{e^t}\Big[(1+\mu\iota+\mu^2\epsilon+\mu^3 h)\Big]\sum_{m=0}^{\infty}\frac{(\mu t)^m}{m!} + \frac{1}{e^t}\Big[(1+\nu\iota+\nu^2\epsilon+\nu^3 h)\Big]\sum_{m=0}^{\infty}\frac{(\nu t)^m}{m!} \\ &+\frac{1}{e^t}\Big[(1+\lambda\iota+\lambda^2\epsilon+\lambda^3 h)\Big]\sum_{m=0}^{\infty}\frac{(\lambda t)^m}{m!}, \\ = &\frac{1}{e^t}\left((1+\mu\iota+\mu^2\epsilon+\mu^3 h)e^{\mu t} + (1+\nu\iota+\nu^2\epsilon+\nu^3 h)e^{\nu t} + (1+\lambda\iota+\lambda^2\epsilon+\lambda^3 h)e^{\lambda t}\right).\end{aligned}$$

Thus we get the result. □

4 Some identities involving Narayana-Lucas Hybrinomial Sequence

Theorem 4.1. *(Catalan's Identity) Let $\{HU_m(r)\}$ is the Narayana-Lucas hybrinomial sequence. For any integers m and s where $m \geq s$, the subsequent identity is valid:*

$$\begin{aligned}HU_{m-s}(r)HU_{m+s}(r) - (HU_m(r))^2 = &\delta_1\delta_2\mu^m\nu^m(\mu^s\nu^{-s}-2)\mu^{m+1}\nu^{m-1} + \delta_2\delta_1\nu^{m+1}\mu^{m-1} + \delta_1\delta_3\mu^m\lambda^m \\ &(\mu^s\lambda^{-s}-2) + \delta_3\delta_1\lambda^{m+1}\mu^{m-1} + \delta_2\delta_3\nu^m\lambda^m(\nu^s\lambda^{-s}-2) + \delta_3\delta_2\lambda^{m+1} \\ &\nu^{m-1},\end{aligned}$$

where

$$\begin{aligned}\delta_1 &= (1+\mu\iota+\mu^2\epsilon+\mu^3 h), \\ \delta_2 &= (1+\nu\iota+\nu^2\epsilon+\nu^3 h), \\ \delta_3 &= (1+\lambda\iota+\lambda^2\epsilon+\lambda^3 h).\end{aligned}$$

Proof. Using theorem (3.2), we have

$$\begin{aligned}HU_{m-s}(r)HU_{m+s}(r) - (HU_m(r))^2 = &(\delta_1\mu^{m-s}+\delta_2\nu^{m-s}+\delta_3\lambda^{m-s})(\delta_1\mu^{m+s}+\delta_2\nu^{m+s}+\delta_3\lambda^{m+s}) \\ &-(\delta_1\mu^m+\delta_2\nu^m+\delta_3\lambda^m)^2,\end{aligned}$$

where

$$\begin{aligned}\delta_1 &= (1+\mu\iota+\mu^2\epsilon+\mu^3 h), \\ \delta_2 &= (1+\nu\iota+\nu^2\epsilon+\nu^3 h), \\ \delta_3 &= (1+\lambda\iota+\lambda^2\epsilon+\lambda^3 h).\end{aligned}$$

Now we get

$$\begin{aligned}HU_{m-s}(r)HU_{m+s}(r)-(HU_m(r))^2 =&(\delta_1^2\mu^{2m}+\delta_1\delta_2\mu^{m+s}\nu^{m-s}+\delta_1\delta_3\mu^{m+s}\lambda^{m-s}+\delta_2\delta_1\nu^{m+s}\mu^{m-s}\\&+\delta_2^2\nu^{2m}+\delta_2\delta_3\nu^{m+s}\lambda^{m-s}+\delta_3\delta_1\lambda^{m+s}\mu^{m-s}+\delta_3\delta_2\lambda^{m+s}\nu^{m-s}\\&+\delta_3^2\lambda^{2m})-(\delta_1^2\mu^{2m}+\delta_2^2\nu^{2m}+\delta_3^2\lambda^{2^m}+2\delta_1\delta_2\mu^m\nu^m+2\delta_1\delta_3\\&\mu^m\lambda^m+2\delta_2\delta_3\nu^m\lambda^m)\\=&(\delta_1\delta_2\mu^{m+s}\nu^{m-s}+\delta_2\delta_1\nu^{m+s}\mu^{m-s}-2\delta_1\delta_2\mu^m\nu^m)+(\delta_1\delta_3\mu^{m+s}\\&\lambda^{m-s}+\delta_3\delta_1\lambda^{m+s}\mu^{m-s}-2\delta_1\delta_3\mu^m\lambda^m)+(\delta_2\delta_3\nu^{m+s}\lambda^{m-s}\\&+\delta_3\delta_2\lambda^{m+s}\nu^{m-s}-2\delta_2\delta_3\nu^m\lambda^m)\\=&\delta_1\delta_2\mu^m\nu^m(\mu^s\nu^{-s}-2)\mu^{m+1}\nu^{m-1}+\delta_2\delta_1\nu^{m+1}\mu^{m-1}+\delta_1\delta_3\mu^m\lambda^m\\&(\mu^s\lambda^{-s}-2)+\delta_3\delta_1\lambda^{m+1}\mu^{m-1}+\delta_2\delta_3\nu^m\lambda^m(\nu^s\lambda^{-s}-2)+\delta_3\delta_2\lambda^{m+1}\\&\nu^{m-1}.\end{aligned}$$

Thus the proof is completed. □

Theorem 4.2 (Cassini's Identity). *Let $\{HU_m(r)\}$ is the m^{th} Narayana-Lucas hybrinomial sequence. For m to be any positive integer, the following identity holds:*

$$\begin{aligned}HU_{m-1}(r)HU_{m+1}(r)-(HU_m(r))^2 =&(\delta_1\delta_2\mu^m\nu^m(\mu\nu^{-1}-2)\mu^{m+1}\nu^{m-1}+\delta_2\delta_1\nu^{m+1}\mu^{m-1})+(\delta_1\delta_3\mu^m\lambda^m\\&(\mu\lambda^{-1}-2)+\delta_3\delta_1\lambda^{m+1}\mu^{m-1})+(\delta_2\delta_3\nu^m\lambda^m(\nu\lambda^{-1}-2)+\delta_3\delta_2\lambda^{m+1}\\&\nu^{m-1}-2\delta_2\delta_3\nu^m\lambda^m),\end{aligned}$$

where

$$\begin{aligned}\delta_1 =&(1+\mu\iota+\mu^2\epsilon+\mu^3h),\\ \delta_2 =&(1+\nu\iota+\nu^2\epsilon+\nu^3h),\\ \delta_3 =&(1+\lambda\iota+\lambda^2\epsilon+\lambda^3h).\end{aligned}$$

Proof. It can be demonstrated by substituting $s=1$ for the Narayana-Lucas hybrinomial sequence in the theorem (4.1). □

Theorem 4.3 (d'Ocagne Identity). *Let $\{HU_m(r)\}$ is the m^{th} Narayana-Lucas hybrinomial sequence. For all integers p and m with $p\geq m+1$, the following identity holds:*

$$\begin{aligned}HU_p(r)HU_{m+1}(r)-HU_{p+1}(r)HU_m(r) =&\delta_1\delta_2\mu^p\nu^m(\nu-\mu)+\delta_2\delta_1\nu^p\mu^m(\mu-\nu)+\delta_2\delta_3\nu^p\lambda^m(\lambda-\nu)\\&+\delta_3\delta_2\lambda^p\nu^m(\nu-\lambda)+\delta_1\delta_3\mu^p\lambda^m(\lambda-\mu)+\delta_3\delta_1\lambda^p\mu^m(\mu-\nu),\end{aligned}$$

where

$$\begin{aligned}\delta_1 =& (1+\mu\iota+\mu^2\epsilon+\mu^3h),\\ \delta_2 =& (1+\nu\iota+\nu^2\epsilon+\nu^3h),\\ \delta_3 =& (1+\lambda\iota+\lambda^2\epsilon+\lambda^3h).\end{aligned}$$

Proof. Using theorem (3.2), we have

$$\begin{aligned}HU_p(r)HU_{m+1}(r)-HU_{p+1}(r)HU_m(r) =&(\delta_1\mu^p+\delta_2\nu^p+\delta_3\lambda^p)+(\delta_1\mu^{m+1}+\delta_2\nu^{m+1}+\delta_3\lambda^{m+1})\\&-(\delta_1\mu^{p+1}+\delta_2\nu^{p+1}+\delta_3\lambda^m),\end{aligned}$$

where

$$\begin{aligned}\delta_1 =& (1+\mu\iota+\mu^2\epsilon+\mu^3h),\\ \delta_2 =& (1+\nu\iota+\nu^2\epsilon+\nu^3h),\\ \delta_3 =& (1+\lambda\iota+\lambda^2\epsilon+\lambda^3h).\end{aligned}$$

Now we get

$$\begin{aligned}HU_p(r)HU_{m+1}(r)-HU_{p+1}(r)HU_m(r) =&(\delta_1^2\mu^{p+m+1}+\delta_1\delta_2\mu^p\nu^{m+1}+\delta_1\delta_3\mu^p\lambda^{m+1}+\delta_2\delta_1\nu^p\mu^{m+1}\\&+\delta_2^2\nu^{p+m+1}+\delta_2\delta_3\nu^p\lambda^{m+1}+\delta_3\delta_1\lambda^p\mu^{m+1}+\delta_3\delta_2\lambda^p\nu^{m+1}\\&+\delta_3^2\lambda^{p+m+1})-(\delta_1^2\mu^{p+m+1}+\delta_1\delta_2\mu^{p+1}\nu^m\\&+\delta_1\delta_3\mu^{p+1}\lambda^m+\delta_2\delta_1\nu^{p+1}\mu^m+\delta_2^2\nu^{p+m+1}+\delta_2\delta_3\nu^{p+1}\lambda^m\\&+\delta_3\delta_1\lambda^{p+1}\mu^m+\delta_3^2\lambda^{p+m+1})\\=&\delta_1\delta_2\mu^p\nu^m(\nu-\mu)+\delta_2\delta_1\nu^p\mu^m(\mu-\nu)+\delta_2\delta_3\nu^p\lambda^m(\lambda-\nu)\\&+\delta_3\delta_2\lambda^p\nu^m(\nu-\lambda)+\delta_1\delta_3\mu^p\lambda^m(\lambda-\mu)+\delta_3\delta_1\lambda^p\mu^m(\mu-\nu).\end{aligned}$$

Thus the proof is completed. □

Theorem 4.4 (Gelin-Cesaro's Identity). *Let $\{HU_m(r)\}$ is the m^{th} Narayana-Lucas hybrinomial sequence. For m to be any positive integer, the following identity holds:*

$$HU_{m+2}(r)HU_{m+1}(r)HU_{m-1}(r)HU_{m-2}(r) - (HU_m)^4(r) = (\delta_1\mu^{m+2}(r) + \delta_2\nu^{m+2}(r) + \delta_3\lambda^{m+2}(r))$$
$$(\delta_1\mu^{m+1}(r) + \delta_2\nu^{m+1}(x) + \delta_3\lambda^{m+1}(r))(\delta_1\mu^{m-1}(r) + \delta_2\nu^{m-1}(r)$$
$$+ \delta_3\lambda^{m-1}(r))(\delta_1\mu^{m-2}(r) + \delta_2\nu^{m-2}(r) + \delta_3\lambda^{m-2}(r)) - (\delta_1\mu^{m}(r) + \delta_2\nu^{m}(r) + \delta_3\lambda^{m}(r)),$$

where

$$\delta_1 = (1 + \mu\iota + \mu^2\epsilon + \mu^3 h),$$
$$\delta_2 = (1 + \nu\iota + \nu^2\epsilon + \nu^3 h),$$
$$\delta_3 = (1 + \lambda\iota + \lambda^2\epsilon + \lambda^3 h).$$

Proof. Considering Binet's formula from theorem (3.2) and making some necessary calculations, the above expression is obtained. □

Theorem 4.5 (Melham's Identity). *Let $\{HU_m(r)\}$ is the m^{th} Narayana-Lucas hybrinomial sequence. For m to be any positive integer, the following identity holds:*

$$HU_{m+1}(r)HU_{m+2}(r)HU_{m+6}(r) - (HU_{m+3})^2(r) = (\delta_1\mu^{m+1}(r) + \delta_2\nu^{m+1}(r) + \delta_3\lambda^{m+1}(r))(\delta_1\mu^{m+2}(r)$$
$$+ \delta_2\nu^{m+2}(r) + \delta_3\lambda^{m+2}(r))(\delta_1\mu^{m+6}(r) + \delta_2\nu^{m+6}(r) + \delta_3\lambda^{m+6}(r)) - (\delta_1\mu^{m+3}(r) + \delta_2\nu^{m+3}(r) + \delta_3\lambda^{m+3}(r))^2,$$

where

$$\delta_1 = (1 + \mu\iota + \mu^2\epsilon + \mu^3 h),$$
$$\delta_2 = (1 + \nu\iota + \nu^2\epsilon + \nu^3 h),$$
$$\delta_3 = (1 + \lambda\iota + \lambda^2\epsilon + \lambda^3 h).$$

Proof. Considering Binet's formula from theorem (3.2) and making some necessary calculations, the above expression is obtained. □

Algorithm 1 Maple 13 source code for the Narayana-Lucas hybrinomial sequence

```
ApplyFunction:=proc(m :: nonnegint)
option remember;
if m ≥ 3 then
    return ApplyFunction(m − 1) + ApplyFunction(m − 3);
else
    return[3, 1, 1, 4][m + 1]; # Use index m + 1 to match 0 − based indexing
end if
end proc;
epsilon_squared := 0;
h_squared: = 1;
iota_squared: = −1;
iota_ast h_equals: = −h ∗ ι = (epsilon + ι);
m: = 0; # Intial value of m
while true do

 HU :=ApplyFunction(U) + ι · ApplyFunction(U + 1) + epsilon · ApplyFunction(U + 2)
        + h · ApplyFunction(U + 3);

# Update the values of m
m: = m + 1;
 end do
```

5 Conclusion remarks

In this paper, we have introduced the Narayana-Lucas hybrinomial sequence and its recurrence relation. We also discussed the matrix representation, generating functions, Binet's formula, exponential generating functions, and Poisson generating functions for the Narayana-Lucas hybrinomial sequence. We also introduced several well-known identities, including Catalan's identity, Cassini's identity, d'Ocagne's identity, Gelin-Cesaro's identity, and Melham's identity, for this newly formed Narayana-Lucas hybrinomial sequence. In the last, we have also provided the Maple 13 source code (in Algorithm 1) for the Narayana-Lucas hybrinomial sequence.

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