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Clifford fractional Mellin transform in $Cl_{(3,1)}$ with applications and graphical interpretation

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Abstract Two-dimensional Clifford fractional Mellin transform (2D-CFrMT) is defined in the space-time algebra $Cl_{(3,1)}$ with orthonormal vector basis. The properties of 2D-CFrMT are established. The inversion formula for 2D-CFrMT is also constructed. Parseval's type property and convolution theorem are studied. The graphical interpretations with an example are explored. Applications are demonstrated using partial differential equations involving Heaviside step function, potential of an infinite wedge and an electrical circuit.

1 Introduction

In [1], the author discussed the development and progress of geometric algebra. Basic knowledge of geometric algebra and calculus is considered in the present text from [2]. Author in [4] developed Clifford algebra by applying geometrical ideas through Mathematics and Physics. In [8], authors have illustrated as to how Clifford's geometric algebra utilizes multivectors to represent space-time, providing an elegant mathematical framework for studying relativistic phenomena. Examples with relativistic phenomena in geometric algebra have been considered. In [11], the author has demonstrated the relativistic physics in the Clifford algebra $Cl_{(1,3)}$ sense.
In [5], the author described a non-commutative generalization of the complex Fourier-Mellin transform to Clifford algebra valued functions over the domain $\mathbb{R}^{(p,q)}$ considering values in $Cl_{(p,q)},\ p+q=2$. In [6], the author has generalized the classical Fourier-Mellin transform and investigated the properties of the quaternionic Fourier-Mellin transform. In [16] authors have defined the Fourier transform and proved the Plancherel and inversion formulas in quaternionic field. Author in [17] discussed in details, the Clifford algebra with a basis of $(p+q)$ dimensional Minkowski space-time.
[7] authors have considered a version of the fractional Clifford-Fourier transform (FrCFT) and studied several properties and applications to partial differential equations in Clifford analysis. In [10], the Mellin transform method is applied to fractional differential equations with a right-sided derivative and variable potential. In [13], a new two-dimensional quaternion fractional Fourier transform was developed. The properties such as linearity, shifting, and derivatives of the quaternion-valued function were studied by the authors. The convolution theorem and inversion formula are also established. An application related to two-dimensional quaternion Fourier transform is also demonstrated. Methods of obtaining solution to a differential equation with two integral boundary conditions are discussed in [14].
In the present study, authors have developed 2D-CFrMT considering an orthonormal basis. The properties like the inversion formula, Parseval's and convolution theorem for 2D-CFrMT are established. Applications in various fields, such as electrical circuits, potential infinite wedge and Heaviside step function, have been demonstrated using 2D-CFrMT.

2 2D-CFrMT

Consider orthonormal vector basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^{(3,1)}$ with 2^4-dimensional basis as in [15].

Definition 2.1. For $h \in L^2\left(\mathbb{R}^{(3,1)}; Cl_{(3,1)}\right)$ two-dimensional Clifford fractional Mellin transform (2D-CFrMT) of order p, q in $Cl_{(3,1)}$ is defined by from [12, 13]:

$$M^{p,q}\{h(t,\vec{x}); s_p, s_q\} = M^{p,q}\{h(\mathbf{x})\} = \tilde{h}(s_p, s_q) = \int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+} t^{s_p-1} h(t,\vec{x})\, \vec{x}^{s_q-1} dt d^3\vec{x} \tag{2.1}$$

$$s_p = a - e_1 \omega_1^{1/p}, s_q = b - i_3 \vec{\omega}^{1/q} \tag{2.2}$$

where $a_1 < a < a_4$, $b_1 < b < b_4$.

Example 2.2. To find Clifford Mellin transform of $e^{-n\mathbf{x}}$.

Solution. On considering the definition from (2.2) and substituting $h(t,\vec{x}) = e^{-n\mathbf{x}}$, we get

$$M^{p,q}\{h(t,\vec{x}); s_p, s_q\} = \frac{\Gamma s_p \Gamma s_q}{(ne_1)^{s_p}(ni_3)^{s_q+2}}. \tag{2.3}$$

For different values of n, there exists different relations of s_p and s_q; such as $s_p = s_q,\ s_p < s_q$ and $s_p > s_q$ can be evaluated as per the requirement.
It may be noted that $e^{-n\mathbf{x}}$ is solved by applying fractional Clifford Mellin transform and the result obtained is shown in (2.3).

Theorem 2.3. ***Left linearity**:*
For $h_1, h_2 \in L^2\left(\mathbb{R}^{(3,1)}; Cl_{(3,1)}\right)$ and k_1, k_2 multi-vector constants in $Cl_{(3,1)}$

$$M^{p,q}\{k_1 h_1(\mathbf{x}) + k_2 h_2(\mathbf{x})\} = k_1 M^{p,q}\{h_1(\mathbf{x})\} + k_2 M^{p,q}\{h_2(\mathbf{x})\} \tag{2.4}$$

is known as 2D-CFrMT left linearity theorem.

Proof. Considering left-hand-side (2.4) and from (2.1) proof is obvious. □

Theorem 2.4. ***Right linearity**: For $h_1, h_2 \in L^2\left(\mathbb{R}^{(3,1)}; Cl_{(3,1)}\right)$ and k_1, k_2 multi-vector constants in $Cl_{(3,1)}$*

$$M^{p,q}\{h_1(\mathbf{x}) k_1 + h_2(\mathbf{x}) k_2\} = M^{p,q}\{h_1(\mathbf{x})\} k_1 + M^{p,q}\{h_2(\mathbf{x})\} k_2 \tag{2.5}$$

is known as 2D-CFrMT right linearity theorem.

Proof. Considering left-hand-side (2.4) and from (2.1), (2.5) is proved. □

Theorem 2.5. ***Scaling property:***
For $h \in L^2\left(\mathbb{R}^{(3,1)}; Cl_{(3,1)}\right)$ and $'a'$ multi-vector constant in $Cl_{(3,1)}$

$$M^{p,q}\{h[a(\mathbf{x})]\} = \frac{1}{a^{s_p+s_q+2}} M^{p,q}\{h(\mathbf{x})\}. \tag{2.6}$$

can be established as 2D-CFrMT Scaling property.

Proof. From (2.1) and considering $a(\mathbf{x}) = \mathbf{y}$

$$M^{p,q}\{h[a(\mathbf{x})]\} = \int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+} t^{s_p-1} h[a(\mathbf{x})]\, \vec{x}^{s_q-1} dt d^3\vec{x}$$

$$M^{p,q}\left\{h\left(\mathbf{y}\right)\right\}=\frac{1}{a^{4}}\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}\left(\frac{t}{a}\right)^{s_p-1}h\left(\mathbf{y}\right)\left(\frac{\vec{x}}{a}\right)^{s_q-1}dtd^3\vec{y}$$

$$=\frac{1}{a^{s_p-1}a^{s_q-1}a^{4}}\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p-1}h\left(\mathbf{y}\right)x^{s_q-1}dtd^3\vec{y}$$

$$=\frac{1}{a^{s_p+s_q+2}}\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p-1}h\left(\mathbf{y}\right)\vec{x}^{s_q-1}dtd^3\vec{y}$$

$$=\frac{1}{a^{s_p+s_q+2}}M^{p,q}\left\{h\left(\mathbf{x}\right)\right\}.$$

Thus (2.6) is proved. □

Theorem 2.6. ***Shifting property:***
For $M^{p,q}\left\{h\left(\mathbf{x}\right)\right\}=\tilde{h}\left(s_p,s_q\right)$; $h\in L^2\left(\mathbb{R}^{(3,1)};Cl_{(3,1)}\right)$ and l,k multi-vector constants in $Cl_{(3,1)}$, the shifting property for 2D-CFrMT can be stated as

$$M^{p,q}\left\{t^{l}\left[h\left(\mathbf{x}\right)\right]x^{k}\right\}=\tilde{h}\left(s_p+l,s_q+k\right). \tag{2.7}$$

Proof. From left-hand-side of (2.1) and (2.7)

$$M^{p,q}\left\{t^{l}\left[h\left(\mathbf{x}\right)\right]x^{k}\right\}=\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p-1}\left\{t^{l}\left[h\left(\mathbf{x}\right)\right]\vec{x}^{k}\right\}\vec{x}^{s_q-1}dtd^3\vec{x}$$

$$=\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p+l-1}h\left(\mathbf{x}\right)x^{s_q+k-1}dtd^3\vec{x}$$

$$=\tilde{h}\left(s_p+l,s_q+k\right).$$

Hence (2.7) proved. □

3 2D-CFrMT inversion formula

Theorem 3.1. *Let $M^{p,q}\left\{h\left(\mathbf{x}\right)\right\}=M^{p,q}\left\{h\left(t,\vec{x}\right);s_p,s_q\right\}$; for $a_1\le a\le a_4, b_1\le b\le b_4$ and c_1 and c_2 be real numbers, then from [12]*

$$h\left(\mathbf{x}\right)=\frac{1}{\left(2\pi\right)^{4}i_4}\lim_{c_1\to\infty}\lim_{c_2\to\infty}\int\limits_{a-e_1c_1}^{a+e_1c_1}\int\limits_{b-i_3c_2}^{b+i_3c_2}M^{p,q}\left\{h\left(\mathbf{x}\right)\right\}t^{-s_p}\vec{x}^{-s_q}ds_pd^3s_q. \tag{3.1}$$

Proof. To prove (3.1), we need to show (3.2)

$$\lim_{c_1\to\infty}\lim_{c_2\to\infty}\left\langle\frac{1}{\left(2\pi\right)^{4}i_4}\int\limits_{a-e_1c_1}^{a+e_1c_1}\int\limits_{b-i_3c_2}^{b+i_3c_2}M^{p,q}\left\{h\left(\mathbf{x}\right)\right\}t^{-s_p}\vec{x}^{-s_q}ds_pd^3s_q,\,g\left(\mathbf{x}\right)\right\rangle=\left\langle h,g\right\rangle \tag{3.2}$$

Considering

$$\left\langle\frac{1}{\left(2\pi\right)^{4}i_4}\int\limits_{a-e_1c_1}^{a+e_1c_1}\int\limits_{b-i_3c_2}^{b+i_3c_2}M^{p,q}\left\{h\left(\mathbf{x}\right)\right\}t^{-s_p}\vec{x}^{-s_q}ds_pd^3s_q,\,g\right\rangle$$

$$=\frac{1}{\left(2\pi\right)^{4}i_4}\int\limits_{\mathbb{R}^{(3,1)}}\int\limits_{\mathbb{R}^{(3,1)}}g\left(\int\limits_{-c_1}^{c_1}\int\limits_{-c_2}^{c_2}M^{p,q}\left\{h\left(\mathbf{x}\right)\right\}t^{-s_p}\vec{x}^{-s_q}d\omega_1d^3\vec{\omega}\right)dtd^3\vec{x}.$$

Change of order of integration

$$\left\langle \frac{1}{(2\pi)^4 i_4} \int_{a-e_1c_1}^{a+e_1c_1} \int_{b-i_3c_2}^{b+i_3c_2} M^{p,q}\{h(\mathbf{x})\} t^{-s_p} \vec{x}^{-s_q} ds_p d^3 s_q,\, g \right\rangle$$

$$= \frac{1}{(2\pi)^4} \int_{-c_1}^{c_1} \int_{-c_2}^{c_2} \left\langle f, y_1^{s_p-1} \vec{y}^{s_q-1} \right\rangle \left(\int_{\mathbb{R}^{(3,1)}} \int_{\mathbb{R}^{(3,1)}} g\, t^{-s_p} \vec{x}^{-s_q} dt d^3\vec{x} \right) d\omega_1 d^3\vec{\omega}$$

$$= \left\langle h, \frac{1}{(2\pi)^4} \int_{-c_1}^{c_1} \int_{-c_2}^{c_2} y_1^{s_p-1} \vec{y}^{s_q-1} \left(\int_{\mathbb{R}^{(3,1)}} \int_{\mathbb{R}^{(3,1)}} g\, t^{-s_p} \vec{x}^{-s_q} dt d^3\vec{x} \right) d\omega_1 d^3\vec{\omega} \right\rangle$$

$$= \left\langle h, \frac{1}{(2\pi)^4} \int_{-c_1}^{c_1} \int_{-c_2}^{c_2} g \int_{\mathbb{R}^{(3,1)}} \int_{\mathbb{R}^{(3,1)}} y_1^{s_p-1} \vec{y}^{s_q-1}\, t^{-s_p} \vec{x}^{-s_q} dt d^3\vec{x}\; d\omega_1 d^3\vec{\omega} \right\rangle$$

$$= \left\langle h, \frac{1}{(2\pi)^4} \int_{-c_1}^{c_1} \int_{-c_2}^{c_2} g \int_{\mathbb{R}^{(3,1)}} \int_{\mathbb{R}^{(3,1)}} y_1^{s_p-1} \vec{y}^{s_q-1}\, x_1^{-s_p} \vec{x}^{s_q} dt d^3\vec{x}\; d\omega_1 d^3\vec{\omega} \right\rangle$$

considering $y_1 = e^{-m_1}$, $\vec{y} = e^{-\vec{m}}$, $t = e^{-n_1}$, $\vec{x} = e^{-\vec{n}}$

$$= \left\langle h, \frac{1}{(2\pi)^4} \int_{-c_1}^{c_1} \int_{-c_2}^{c_2} g \int_{\mathbb{R}^{(3,1)}} \int_{\mathbb{R}^{(3,1)}} e^{-m_1 {}_1^{(s_p-1)}} e^{-\vec{m}(s_q-1)}\; e^{n_1}\, e^{\vec{n}} dn_1 d^3\vec{n}\, d\omega_1 d^3\vec{\omega} \right\rangle$$

$$= \left\langle h, \frac{1}{(2\pi)^4} \int_{-c_1}^{c_1} \int_{-c_2}^{c_2} g \int_{\mathbb{R}^{(3,1)}} \int_{\mathbb{R}^{(3,1)}} e^{-{}_1^{(s_p-1)(m_1-n_1)}} e^{-(s_q-1)(\vec{m}-\vec{n})} dn_1 d^3\vec{n}\, d\omega_1 d^3\vec{\omega} \right\rangle$$

substituting $s_p = a - e_1\omega_1^{1/p}$; $s_q = b - i_3\vec{\omega}^{1/q}$

$$= \left\langle h, \frac{1}{(2\pi)^4} \int_{-c_1}^{c_1} \int_{-c_2}^{c_2} g \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+} e^{(a-e_1\omega_1)(m_1-n_1)} e^{(b-i_3\vec{\omega})(\vec{m}-\vec{n})}\; e^{-(n_1-m_1)} e^{-(\vec{n}-\vec{m})} dn_1 d^3\vec{n} d\omega_1 d^3\vec{\omega} \right\rangle$$

$$= \left\langle h, \frac{1}{(2\pi)^4} \int_{-c_1}^{c_1} \int_{-c_2}^{c_2} g \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+} \left(e^{am_1} e^{-an_1} e^{-e_1\omega_1 m_1} e^{-e_1\omega_1 n_1}\right) \left(e^{b\vec{m}} e^{-b\vec{n}} e^{-i_3\vec{\omega}\vec{m}} e^{i_3\vec{\omega}\vec{n}}\right) e^{-(n_1-m_1)} e^{-(\vec{n}-\vec{m})} dn_1 d^3\vec{n} d\omega$$

$$= \left\langle h, \frac{1}{(2\pi)^4} \int_{\mathbb{R}_+^3} \int_{\mathbb{R}_+} g \int_{-c_1}^{c_1} \int_{-c_2}^{c_2} e^{(a+1)m_1} e^{(b+1)\vec{m}} \left[e^{(-a+e_1\omega_1-1)n_1} e^{-e_1\omega_1 m_1}\; e^{(-b+i_3\vec{\omega}-1)\vec{n}} e^{-i_3\vec{\omega}\vec{m}} \right] dn_1 d^3\vec{n} d\omega_1 d^3\vec{\omega} \right\rangle$$

as $c_1, c_2 \to \infty$, second term of the inner product tends to $g(\mathbf{x})$.
Thus (3.1) is proved. □

4 2D-CFrMT Parseval's theorem

Theorem 4.1. *For $M^{p,q}\{h(\mathbf{x})\} = \tilde{h}(s_p, s_q)$ and $M^{p,q}\{g(\mathbf{x})\} = \tilde{g}(s_p, s_q)$, from [9] 2D-CFrMT Parseval's type theorem can be stated as*

$$M^{p,q}\{h(\mathbf{x})\, g(\mathbf{x})\} = \frac{1}{(2\pi)^4 i_4} \lim_{c_1\to\infty} \lim_{c_2\to\infty} \int_{a-e_1c_1}^{a+e_1c_1} \int_{b-i_3c_2}^{b+i_3c_2} \tilde{h}\left(s_{p'}, s_{q'}\right) \tilde{g}\left(s_p - s_{p'}, s_q - s_{q'}\right) ds_{p'} d^3 s_{q'}. \tag{4.1}$$

Proof. Using definition (2.1)

$$M^{p,q}\left\{h\left(\mathbf{x}\right)g\left(\mathbf{x}\right)\right\}=\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p-1}h\left(\mathbf{x}\right)g\left(\mathbf{x}\right)\vec{x}^{s_q-1}dtd^3\vec{x}$$

$$=\frac{1}{(2\pi)^4 i_4}\lim_{c_1\to\infty}\lim_{c_2\to\infty}\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p-1}g\left(\mathbf{x}\right)\vec{x}^{s_q-1}dtd^3\vec{x}\left[\int\limits_{a-e_1c_1}^{a+e_1c_1}\int\limits_{b-i_3c_1}^{b+i_3c_1}\tilde{h}\left(s_{p'},s_{q'}\right)t^{-s_{p'}}x^{-s_{q'}}ds_{p'}d^3s_{q'}\right]$$

$$=\frac{1}{(2\pi)^4 i_4}\lim_{c_1\to\infty}\lim_{c_2\to\infty}\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p-1}g\left(\mathbf{x}\right)x^{s_q-1}dtd^3\vec{x}\left[\int\limits_{a-e_1c_1}^{a+e_1c_1}\int\limits_{b-i_3c_1}^{b+i_3c_1}\tilde{h}\left(s_{p'},s_{q'}\right)t^{-s_{p'}}x^{-s_{q'}}ds_{p'}d^3s_{q'}\right]$$

$$=\frac{1}{(2\pi)^4 i_4}\lim_{c_1\to\infty}\lim_{c_2\to\infty}\int\limits_{a-e_1c_1}^{a+e_1c_1}\int\limits_{b-i_3c_2}^{b+i_3c_2}\tilde{h}\left(s_{p'},s_{q'}\right)ds_{p'}d^3s_{q'}\left[\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p-s_{p'}-1}g\left(\mathbf{x}\right)x^{s_q-s_{q'}-1}dtd^3\vec{x}\right]$$

Hence, (4.1) is proved. □

5 2D-CFrMT Scalar product

Theorem 5.1. *The scalar product of two Clifford valued functions* $h_1,h_2\in L^2\left(\mathbb{R}^{(3,1)};Cl_{(3,1)}\right)$, $M^{p,q}\left\{h_1\left(\mathbf{x}\right)\right\}=\tilde{h}_1\left(s_p,s_q\right)$ *and* $M^{p,q}\left\{h_2\left(\mathbf{x}\right)\right\}=\tilde{h}_2\left(s_p,s_q\right)$ *is given by*

$$\langle h_1,h_2\rangle=\frac{1}{(2\pi)^4 i_4}\left\langle\tilde{h}_1\left(s_p,s_q\right),\tilde{h}_2\left(s_p,s_q\right)\right\rangle. \tag{5.1}$$

Proof. For $h_1,h_2\in L^2\left(\mathbb{R}^{(3,1)};Cl_{(3,1)}\right)$, we have

$$\langle h_1,h_2\rangle=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}}\langle h_1\left(\mathbf{x}\right)\bar{h}_2\left(\mathbf{x}\right)\rangle dtd^3\vec{x}$$

$$=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}}\left\langle\frac{1}{(2\pi)^4 i_4}\lim_{c_1\to\infty}\lim_{c_2\to\infty}\int\limits_{a-e_1c_1}^{a+e_1c_1}\int\limits_{b-i_3c_2}^{b+i_3c_2}M^{p,q}\left\{h_1\left(\mathbf{x}\right)\right\}t^{-s_p}\vec{x}^{-s_q}ds_pd^3s_q\,\bar{h}_2\left(\mathbf{x}\right)\right\rangle dtd^3\vec{x}$$

$$=\frac{1}{(2\pi)^4 i_4}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}}\left\langle\lim_{c_1\to\infty}\lim_{c_2\to\infty}\int\limits_{a-e_1c_1}^{a+e_1c_1}\int\limits_{b-i_3c_2}^{b+i_3c_2}M^{p,q}\left\{h_1\left(\mathbf{x}\right)\right\}t^{-s_p}\vec{x}^{-s_q}ds_pd^3s_q\,\bar{h}_2\left(\mathbf{x}\right)\right\rangle dtd^3\vec{x}$$

$$=\frac{1}{(2\pi)^4 i_4}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}}\left\langle M^{p,q}\left\{h_1\left(\mathbf{x}\right)\right\}\lim_{c_1\to\infty}\lim_{c_2\to\infty}\int\limits_{a-e_1c_1}^{a+e_1c_1}\int\limits_{b-i_3c_2}^{b+i_3c_2}\bar{h}_2\left(\mathbf{x}\right)t^{-s_p}\vec{x}^{-s_q}ds_pd^3s_q\right\rangle dtd^3\vec{x}$$

$$=\frac{1}{(2\pi)^4 i_4}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}}\langle M^{p,q}\left\{h_1\left(\mathbf{x}\right)\right\}M^{p,q}\left\{\bar{h}_2\left(\mathbf{x}\right)\right\}\rangle dtd^3\vec{x}$$

$$=\frac{1}{(2\pi)^4 i_4}\langle M^{p,q}\left\{h_1\left(\mathbf{x}\right)\right\},M^{p,q}\left\{h_2\left(\mathbf{x}\right)\right\}\rangle.$$

Hence, the theorem holds. □

6 2D-CFrMT Convolution theorem

Theorem 6.1. *If* $M^{p,q}\{h_1(\mathbf{x})\} = \tilde{h}_1(s_p, s_q)$ *and* $M^{p,q}\{h_2(\mathbf{x})\} = \tilde{h}_2(s_p, s_q)$ *, then 2D-CFrMT convolution theorem can be stated as*

$$M^{p,q}\{h_1(\mathbf{x}) * h_2(\mathbf{x})\} = \tilde{h}_1(s_p, s_q)\, \tilde{h}_2(s_p, s_q)\,. \tag{6.1}$$

Proof. For $\mathbf{x} = (t, \vec{x})$; $\mathbf{y} = (y_1, \vec{y})$; $\mathbf{u} = (u_1, \vec{u})$

$$M^{p,q}\{h_1(\mathbf{x}) * h_2(\mathbf{x})\} = M^{p,q}\left[\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+} h_1(\mathbf{u})\, h_2\left(\frac{\mathbf{x}}{\mathbf{u}}\right) \frac{du_1 d^3\vec{u}}{u_1 \vec{u}^3}\right]$$

$$= \int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+} t^{s_p - 1}\left[\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+} h_1(\mathbf{u})\, h_2\left(\frac{\mathbf{x}}{\mathbf{u}}\right) \frac{du_1 d^3\vec{u}}{u_1} \vec{u}^3\right] \vec{x}^{s_q - 1} dt d^3\vec{x}.$$

Considering $\frac{\mathbf{x}}{\mathbf{u}} = \mathbf{y}$

$$\begin{aligned} M^{p,q}\{h_1(\mathbf{x}) * h_2(\mathbf{x})\} &= \int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+} (u_1 y_1)^{s_p - 1} h_1(\mathbf{u})\, du_1 d^3\vec{u} \left[\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+} h_2(\mathbf{y}) (\vec{u}\vec{y})^{s_q - 1} dy_1 d^3\vec{y}\right] \\ &= \int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+} (u_1)^{s_p - 1} h_1(\mathbf{u})\, (\vec{u})^{s_q - 1} du_1 d^3\vec{u} \left[\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+} (y_1)^{s_p - 1} h_2(\mathbf{y}) (\vec{y})^{s_q - 1} dy_1 d^3\vec{y}\right] \\ &= \tilde{h}_1(s_p, s_q)\, \tilde{h}_2(s_p, s_q)\,. \end{aligned}$$

Thus, the theorem holds. □

7 Differentiation of 2D-CFrMT

Theorem 7.1. *For* $h \in L^2\left(\mathbb{R}^{(3,1)}; Cl_{(3,1)}\right)$ *first order derivative of* $h(t, \vec{x})$ *w.r.t* $'t'$ *and* $\vec{x}$ *respectively is obtained using [9] and [13]*

i) $M^{p,q}\left\{\frac{\partial}{\partial t} h(t, \vec{x})\right\} = -(s_p - 1)\, \tilde{h}(s_p - 1, s_q)\,.$

ii) $M^{p,q}\left\{\frac{\partial^2}{\partial t^2} h(t, \vec{x})\right\} = (s_p - 1)(s_p - 2)\, \tilde{h}(s_p - 2, s_q)\,.$

iii) $M^{p,q}\left\{\frac{\partial^n}{\partial t^n} h(t, \vec{x})\right\} = (-1)^n (s_p - 1)(s_p - 2)\cdots(s_p - n)\, \tilde{h}(s_p - n, s_q)\,.$

iv) $M^{p,q}\left\{\frac{\partial}{\partial \vec{x}} h(t, \vec{x})\right\} = -(s_q - 1)\, \tilde{h}(s_p, s_q - 1)\,.$

v) $M^{p,q}\left\{\frac{\partial^2}{\partial^2 \vec{x}} h(t, \vec{x})\right\} = (s_q - 1)(s_q - 2)\, \tilde{h}(s_p, s_q - 2)\,.$

vi) $M^{p,q}\left\{\frac{\partial^m}{\partial^m \vec{x}} h(t, \vec{x})\right\} = (-1)^n (s_q - 1)(s_q - 2)\cdots(s_q - m) \tilde{h}(s_p, s_q - m)\,.$

vii) $M^{p,q}\left\{t \frac{\partial}{\partial t} h(t, \vec{x})\right\} = -(s_p)\, \tilde{h}(s_p, s_q)\,.$

viii) $M^{p,q}\left\{\vec{x} \frac{\partial}{\partial \vec{x}} h(t, \vec{x})\right\} = -(s_q)\, \tilde{h}(s_p, s_q)\,.$

Proof. Consider left-hand-side of theorem 7.1.i) and from (2.1), we get

i) $M^{p,q}\left\{\frac{\partial}{\partial t}h\left(t,\vec{x}\right)\right\}=\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p-1}\frac{\partial}{\partial t}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1}dtd^3\vec{x}.$

On integration by parts follows

$$\begin{aligned}&M^{p,q}\left\{\frac{\partial}{\partial t}h\left(t,\vec{x}\right)\right\}\\&=\int\limits_{\mathbb{R}^3_+}\left\{t^{s_p-1}h\left(t,\vec{x}\right)-\int\limits_{\mathbb{R}_+}\left(s_p-1\right)t^{s_p-1-1}h\left(t,\vec{x}\right)\,dt\right\}\vec{x}^{s_q-1}d^3\vec{x}\\&=-\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}\left(s_p-1\right)t^{s_p-1-1}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1}dtd^3\vec{x}\\&=-\left(s_p-1\right)\tilde{h}\left(s_p-1,s_q\right)\end{aligned}$$

provided $t^{s_p-1}h\left(t,\vec{x}\right)\vec{x}^{s_q-1}$ vanishes as $t\to 0$ and $t\to\infty$.

ii) From left-hand-side of theorem 7.1.ii) and (2.1)

$$M^{p,q}\left\{\frac{\partial^2}{\partial t^2}h\left(t,\vec{x}\right)\right\}=\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p-1}\frac{\partial^2}{\partial t^2}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1}dtd^3\vec{x}.$$

Integrating using by parts rule, we get

$$\begin{aligned}&M^{p,q}\left\{\frac{\partial^2}{\partial t^2}h\left(t,\vec{x}\right)\right\}\\&=\int\limits_{\mathbb{R}^3_+}\left\{t^{s_p-1}\frac{\partial}{\partial t}h\left(t,\vec{x}\right)-\int\limits_{\mathbb{R}_+}\left(s_p-1\right)t^{s_p-1-1}\frac{\partial}{\partial t}h\left(t,\vec{x}\right)\,dt\right\}\vec{x}^{s_q-1}d^3\vec{x}\\&=-\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}^3_+}\left(s_p-1\right)t^{s_p-1-1}\frac{\partial}{\partial t}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1}d^3\vec{x}\,dt\\&=-\left(s_p-1\right)\left(s_p-2\right)\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p-1-2}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1}\,dtd^3\vec{x}\\&=\left(s_p-1\right)\left(s_p-2\right)\tilde{h}\left(s_p-2,s_q\right)\end{aligned}$$

provided $\left[t^{s_p-2}h\left(t,\vec{x}\right)\vec{x}^{s_q-1}\right]$ vanishes as $t\to 0$ and $t\to\infty$.

iii) Taking left-hand-side of theorem 7.1.iii) and from (2.1)

$$M^{p,q}\left\{\frac{\partial^{k-1}}{\partial t^{k-1}}h\left(t,\vec{x}\right)\right\}=\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p-1}\frac{\partial^{k-1}}{\partial t^{k-1}}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1}dtd^3\vec{x}.$$

Using integration by parts, we get

$$\begin{aligned}&M^{p,q}\left\{\frac{\partial^{k-1}}{\partial t^{k-1}}h\left(t,\vec{x}\right)\right\}\\&=\int\limits_{\mathbb{R}^3_+}\left\{t^{s_p-1}\frac{\partial^{k-2}}{\partial t^{k-2}}h\left(t,\vec{x}\right)-\int\limits_{\mathbb{R}_+}\left(s_p-1\right)t^{s_p-1-1}\frac{\partial^{k-2}}{\partial t^{k-2}}h\left(t,\vec{x}\right)\,dt\right\}\vec{x}^{s_q-1}d^3\vec{x}\\&=-\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}^3_+}\left(s_p-1\right)t^{s_p-1-1}\frac{\partial^{k-2}}{\partial t^{k-2}}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1}d^3\vec{x}\,dt\end{aligned}$$

repeating integration by parts, we get

$$= (-1)^{k-1} (s_p - 1) \cdots (s_p - k + 1) \int\limits_{\mathbb{R}_+^3} \int\limits_{\mathbb{R}_+} t^{s_p - 1 - k + 1} h(t, \vec{x})\ \vec{x}^{s_q - 1}\, dt d^3\vec{x}$$

$$= (-1)^{k-1} (s_p - 1) \cdots (s_p - k + 1) \tilde{h}(s_p - k + 1, s_q)$$

provided $t^{s_p-k+1} h(t, \vec{x})\, \vec{x}^{s_q-1}$ vanishes as $t \to 0$ and $t \to \infty$.
Thus, by the method of mathematical induction, the result holds for $n = k$ for all n.

iv) Considering left-hand-side of theorem7.1.iv) and (2.1)

$$M^{p,q}\left\{\frac{\partial}{\partial \vec{x}} h(t, \vec{x})\right\} = \int\limits_{\mathbb{R}_+^3} \int\limits_{\mathbb{R}_+} t^{s_p-1} \frac{\partial}{\partial \vec{x}} h(t, \vec{x})\ \vec{x}^{s_q-1} dt d^3\vec{x}.$$

On integration by parts, we obtain

$$\begin{aligned} &M^{p,q}\left\{\frac{\partial}{\partial \vec{x}} h(t, \vec{x})\right\} \\ &= \int\limits_{\mathbb{R}_+} t^{s_p-1} \left\{ \int\limits_{\mathbb{R}_+^3} h(t, \vec{x})\ \vec{x}^{s_q-1} d^3\vec{x} - \int\limits_{\mathbb{R}_+^3} (s_q - 1) h(t, \vec{x})\ \vec{x}^{s_q-1-1} d^3\vec{x} \right\} dt \\ &= -\int\limits_{\mathbb{R}_+^3} \int\limits_{\mathbb{R}_+} (s_q - 1) t^{s_p-1} h(t, \vec{x})\ \vec{x}^{s_q-1-1} dt d^3\vec{x} \\ &= -(s_q - 1)\tilde{h}(s_p, s_q - 1) \end{aligned}$$

provided $t^{s_p-1} h(t, \vec{x})\, \vec{x}^{s_q-1}$ vanishes as $t \to 0$ and $t \to \infty$.

v) From (2.1) and left-hand-side of theorem 7.1.v), follows

$$M^{p,q}\left\{\frac{\partial^2}{\partial^2 \vec{x}} h(t, \vec{x})\right\} = \int\limits_{\mathbb{R}_+^3} \int\limits_{\mathbb{R}_+} t^{s_p-1} \frac{\partial^2}{\partial^2 \vec{x}} h(t, \vec{x})\ \vec{x}^{s_q-1} dt d^3\vec{x}.$$

On performing integration by parts, we get

$$\begin{aligned} &M^{p,q}\left\{\frac{\partial^2}{\partial^2 \vec{x}} h(t, \vec{x})\right\} \\ &= \int\limits_{\mathbb{R}_+} t^{s_p-1} \left\{ \int\limits_{\mathbb{R}_+^3} \frac{\partial}{\partial \vec{x}} h(t, \vec{x})\ \vec{x}^{s_q-1} d^3\vec{x} - \int\limits_{\mathbb{R}_+^3} (s_q - 1) \frac{\partial}{\partial \vec{x}} h(t, \vec{x})\ \vec{x}^{s_q-1-1} d^3\vec{x} \right\} dt \\ &= -\int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}_+^3} (s_q - 1) t^{s_p-1} \frac{\partial}{\partial \vec{x}} h(t, \vec{x})\ \vec{x}^{s_q-1-1} d^3\vec{x}\, dt \\ &= -(s_q - 1) \int\limits_{\mathbb{R}_+} t^{s_p-1} \left\{ \int\limits_{\mathbb{R}_+^3} h(t, \vec{x})\ \vec{x}^{s_q-1-1} d^3\vec{x} - \int\limits_{\mathbb{R}_+^3} (s_q - 1 - 2) h(t, \vec{x})\ \vec{x}^{s_q-1-2} d^3\vec{x} \right\} dt \\ &= (s_q - 1)(s_q - 2)\tilde{h}(s_p, s_q - 2) \end{aligned}$$

provided $t^{s_p-1} \frac{\partial}{\partial \vec{x}} h(t, \vec{x})\, \vec{x}^{s_q-1}$ and $t^{s_p-1} h(t, \vec{x})\, \vec{x}^{s_q-1}$ vanishes as $t \to 0$ and $t \to \infty$.

vi) From (2.1) and left-hand-side of theorem 7.1.iv), we obtain

$$M^{p,q}\left\{\frac{\partial^{k-1}}{\partial \vec{x}^{k-1}} h(t, \vec{x})\right\} = \int\limits_{\mathbb{R}_+^3} \int\limits_{\mathbb{R}_+} t^{s_p-1} \frac{\partial^{k-1}}{\partial \vec{x}^{k-1}} h(t, \vec{x})\ \vec{x}^{s_q-1} dt d^3\vec{x}.$$

As solved in theorem 7.1.v), one can obtain

$$M^{p,q}\left\{\frac{\partial^{k-1}}{\partial\vec{x}^{k-1}}h\left(t,\vec{x}\right)\right\}$$

$$=\int\limits_{\mathbb{R}_+}t^{s_p-1}\left\{\int\limits_{\mathbb{R}^3_+}\frac{\partial^{k-2}}{\partial\vec{x}^{k-2}}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1}d^3\vec{x}-\int\limits_{\mathbb{R}^3_+}\left(s_q-1\right)\frac{\partial^{k-2}}{\partial\vec{x}^{k-2}}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1-1}d^3\vec{x}\right\}dt$$

$$=-\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}^3_+}\left(s_q-1\right)t^{s_p-1}\frac{\partial^{k-2}}{\partial\vec{x}^{k-2}}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1-1}d^3\vec{x}\,dt$$

provided $t^{s_p-1}\frac{\partial}{\partial\vec{x}}h\left(t,\vec{x}\right)\vec{x}^{s_q-1}$ and $t^{s_p-1}h\left(t,\vec{x}\right)\vec{x}^{s_q-1}$ vanishes as $t\to 0$ and $t\to\infty$.
On repeating the integration by parts rule, we get

$$M^{p,q}\left\{\frac{\partial^{k-1}}{\partial\vec{x}^{k-1}}h\left(t,\vec{x}\right)\right\}$$

$$=\left(-1\right)^{k-1}\left(s_q-1\right)\cdots\left(s_q-k+1\right)\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p-1}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1-k+1}\,dtd^3\vec{x}$$

$$=\left(-1\right)^{k-1}\left(s_q-1\right)\cdots\left(s_q-k+1\right)\tilde{h}\left(s_p-k+1,s_q\right)$$

provided $t^{s_p-k+1}h\left(t,\vec{x}\right)\vec{x}^{s_q-1}$ vanishes as $t\to 0$ and $t\to\infty$.
Thus, by the method of mathematical induction, the result holds for $n=k$ for all n.

vii) From left-hand-side of theorem 7.1.vii) and (2.1)

$$M^{p,q}\left\{t\frac{\partial}{\partial t}h\left(t,\vec{x}\right)\right\}=\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t\left(t^{s_p-1}\frac{\partial}{\partial t}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1}\right)dtd^3\vec{x}$$

$$=\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p}\frac{\partial}{\partial t}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1}dtd^3\vec{x}.$$

By parts rule gives

$$M^{p,q}\left\{t\frac{\partial}{\partial t}h\left(t,\vec{x}\right)\right\}$$

$$=\int\limits_{\mathbb{R}^3_+}\left\{t^{s_p}h\left(t,\vec{x}\right)\,-\int\limits_{\mathbb{R}_+}\left(s_p\right)t^{s_p-1}h\left(t,\vec{x}\right)\,dt\right\}\vec{x}^{s_q-1}d^3\vec{x}$$

$$=-\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}\left(s_p\right)t^{s_p-1}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1}dtd^3\vec{x}$$

$$=-\left(s_p\right)\tilde{h}\left(s_p,s_q\right)$$

provided $t^{s_p}h\left(t,\vec{x}\right)\vec{x}^{s_q-1}$ vanishes as $t\to 0$ and $t\to\infty$.

viii) From (2.1) and left-hand-side of theorem 7.1.viii)

$$M^{p,q}\left\{\vec{x}\frac{\partial}{\partial\vec{x}}h\left(t,\vec{x}\right)\right\}=\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p-1}\,\vec{x}\frac{\partial}{\partial\vec{x}}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1}\,dtd^3\vec{x}$$

$$=\int\limits_{\mathbb{R}^3_+}\int\limits_{\mathbb{R}_+}t^{s_p-1}\frac{\partial}{\partial\vec{x}}h\left(t,\vec{x}\right)\,\vec{x}^{s_q}dtd^3\vec{x}.$$

On integration by parts, we get

$$
\begin{aligned}
&M^{p,q}\left\{\vec{x}\frac{\partial}{\partial\vec{x}}h\left(t,\vec{x}\right)\right\}\\
&=\int\limits_{\mathbb{R}_+} t^{s_p-1}\left\{\int\limits_{\mathbb{R}_+^3} h\left(t,\vec{x}\right)\,\vec{x}^{s_q}d^3\vec{x}-\int\limits_{\mathbb{R}_+^3}\left(s_q\right)h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1}d^3\vec{x}\right\}dt\\
&=-\int\limits_{\mathbb{R}_+^3}\int\limits_{\mathbb{R}_+}\left(s_q\right)t^{s_p-1}h\left(t,\vec{x}\right)\,\vec{x}^{s_q-1}dtd^3\vec{x}\\
&=-\left(s_q\right)\tilde{h}\left(s_p,s_q\right)
\end{aligned}
$$

provided $t^{s_p-1}h\left(t,\vec{x}\right)\vec{x}^{s_q-1}$ vanishes as $t\to 0$ and $t\to\infty$.

□

Example 7.2. To find derivative of 2D-CFrMT of $e^{-n\mathbf{x}}$.

Solution. On considering the definition from (2.2) and substituting $h(t,\vec{x})=e^{-n\mathbf{x}}$, we get

$$M^{p,q}\left\{h\left(t,\vec{x}\right);s_p,s_q\right\}=\frac{\Gamma s_p\Gamma s_q}{\left(ne_1\right)^{s_p}\left(ni_3\right)^{s_q+2}} \tag{7.1}$$

Using Theorem 7.1.i) taking partial derivative w.r.t t gives

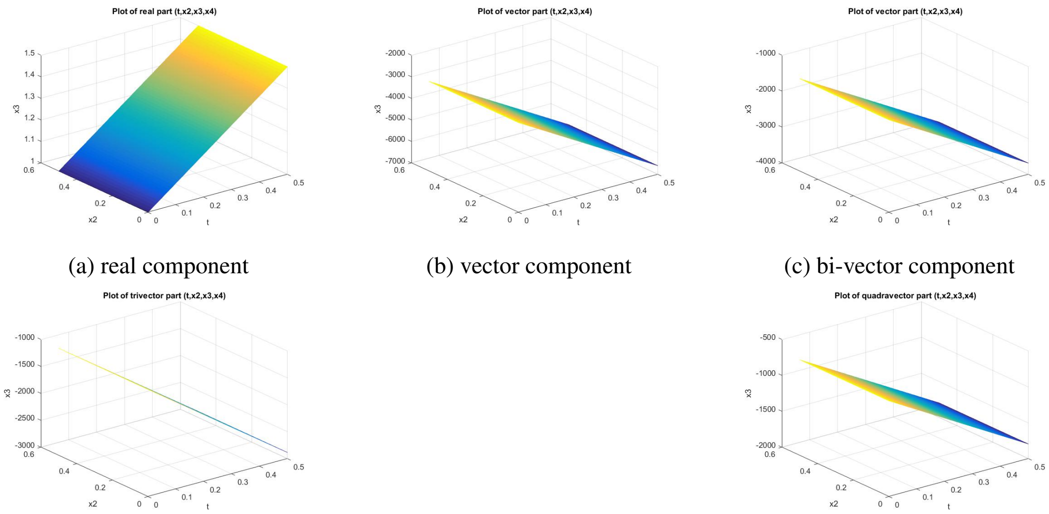
$$M^{p,q}\left\{\frac{\partial}{\partial t}h\left(t,\vec{x}\right)\right\}=-\left(s_p-1\right)\frac{\Gamma(s_p-1)\Gamma s_q}{\left(ne_1\right)^{s_p-1}\left(ni_3\right)^{s_q+2}}. \tag{7.2}$$

and partial derivative w.r.t $\vec{x}$ using Theorem 7.1.iv) gives

$$M^{p,q}\left\{\frac{\partial}{\partial\vec{x}}h\left(t,\vec{x}\right)\right\}=-\left(s_q-1\right)\frac{\Gamma s_p\Gamma(s_q-1)}{\left(ne_1\right)^{s_p}\left(ni_3\right)^{s_q+1}}. \tag{7.3}$$

8 Analysis of 2D-CFrMT using graph

In figure graph of 2D-CFrMT is shown by considering $h(t, \vec{x}) = 1$ in (2.1). The components of $h(t, \vec{x})$ in real, vector, bi-vector, tri-vector and quatra-vector are shown below.

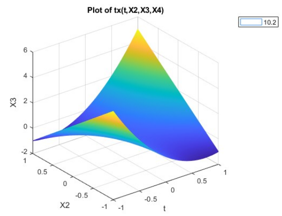
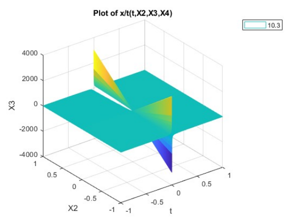
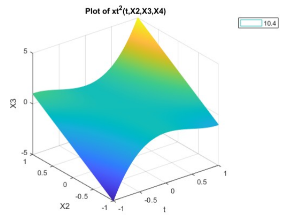
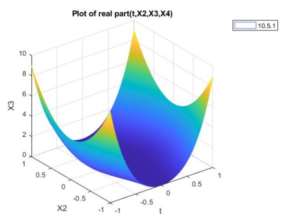
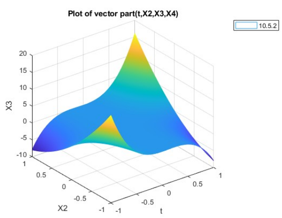
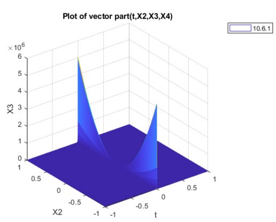
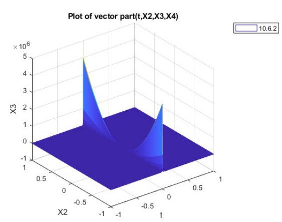


(a) real component (b) vector component (c) bi-vector component

(d) tri-vector component (e) quadra-vector component

Figure 1: Graph of 2D-CFrMT

Table 1

Sr.	Functions	Real component	Bi-vector component
1	$t^{-s_p+2}\vec{x}^{-s_q+2}$		
2	$t^{-s_p}\vec{x}^{s_q+2}$		
3	$t^{-s_p+3}\vec{x}^{s_q+2}$		
4	$t^{-s_p+3}\vec{x}^{s_q+3}$		
5	$t^{-s_p-1}\vec{x}^{s_q+3}$		

By considering $h(t,\vec{x}) = t^{-s_p+2}\vec{x}^{-s_q+2}, t^{-s_p}\vec{x}^{s_q+2}$ and $t^{-s_p+3}\vec{x}^{s_q+2}$, we get the bi-vector component where as when $h(t,\vec{x}) = t^{-s_p+3}\vec{x}^{s_q+3}$ and $t^{-s_p-1}\vec{x}^{s_q+3}$ the real and the bi-vector components are obtained as shown in Table 1.

9 Applications

This section demonstrates the practical application of the 2D-CFr Mellin transform in various real-world circumstances. In section (9.1), a partial differential equation using the 2D-CFr Mellin transform is solved, explicitly involving the Heaviside step function. In section (9.2), 2D-CFr Mellin transform to the potential infinite wedge is illustrated. The partial differential equation using the 2D-CFr Mellin transform is theoretical and demonstrates its practical application in circuit analysis in section (9.3). The responding current $j\left(t,\vec{x}\right)$ in a circuit with a series connection of a time-varying inductance is determined.

9.1 Heaviside step function:

Partial differential equation is solved by applying 2D-CFr Mellin trasform.
Analogous to [12], applied in Clifford algebra context considering

$$t\frac{\partial h}{\partial t}+\vec{x}\frac{\partial h}{\partial \vec{x}}=H\left(t-1\right). \tag{9.1}$$

Applying the definition (2.1) in (9.1), we get

$$-s_p\tilde{h}\left(s_p,s_q\right)-s_q\tilde{h}\left(s_p,s_q\right)=-\frac{1}{s_p}$$

$$\tilde{h}\left(s_p,s_q\right)=\frac{1}{s_p\left(s_p+s_q\right)}. \tag{9.2}$$

Using (3.1) in (9.2), the result is obtained as

$$h\left(t,\vec{x}\right)=\frac{1}{\left(2\pi\right)^4 i_4}\lim_{c_1\to\infty}\lim_{c_2\to\infty}\int\limits_{a-e_1c_1}^{a+e_1c_1}\int\limits_{b-i_3c_2}^{b+i_3c_2}\frac{1}{s_p\left(s_p+s_q\right)}t^{-s_p}\vec{x}^{-s_q}ds_pd^3s_q$$

which is the required solution of (9.1).

9.2 Potential in infinite wedge:

To find the potential which satisfies Laplace equation analogous to [3] is represented as

$$r^2\psi_{rr}+r\psi_r+\psi_{\theta\theta}=0 \tag{9.3}$$

in an infinite wedge $0<r<\infty, -\alpha<\theta<\alpha$ where $0<\alpha<2\pi$.
The boundary conditions are:
i) As $\theta\to 0+, \psi(r,\theta)$ converges to Clifford-Mellin transformable function $\psi(r)$.
ii) As $\theta\to\alpha-, \psi(r,\theta)$ converges to zero uniformly.
Applying Clifford fractional Mellin transform of the potential $\psi(r,\theta)$

$$s_p(s_p+1)\tilde{\psi}(s_p,s_q)-s_p\tilde{\psi}(s_p,s_q)+\frac{d^2\psi}{d\theta^2}=0. \tag{9.4}$$

The general solution of the transformed equation is given by

$$\tilde{\psi}(s_p,s_q)=C_1e^{is_p\theta}+C_2e^{-is_p\theta}. \tag{9.5}$$

Now applying boundary conditions to (9.5), we get

$$C_1+C_2=\tilde{h}(s_p). \tag{9.6}$$

$$C_1e^{is_p\alpha}+C_2e^{-is_p\alpha}=0. \tag{9.7}$$

On solving (9.6) and (9.7), we get

$$C_1=\frac{\tilde{h}(s_p)}{1-e^{2is_p\alpha}} \tag{9.8}$$

and

$$C_2=\frac{\tilde{h}(s_p)}{1-e^{-2is_p\alpha}}. \tag{9.9}$$

On substituting C_1, C_2 in (9.5) follows

$$\tilde{\psi}(s_p,s_q)=\frac{\sin(\alpha s_p-\theta s_q)}{\sin(\alpha s_p)}. \tag{9.10}$$

Using inversion formula (3.1), we obtain

$$h(t,\vec{x}) = \frac{1}{(2\pi)^4 i_4} \lim_{c_1\to\infty} \lim_{c_2\to\infty} \int_{a-e_1c_1}^{a+e_1c_1} \int_{b-i_3c_2}^{b+i_3c_2} \frac{\sin(\alpha s_p - \theta s_q)}{\sin(\alpha s_p)} t^{-s_p} \vec{x}^{-s_q} ds_p d^3 s_q. \tag{9.11}$$

Hence, (9.3) is solved.

9.3 Electric Circuit:

Section (9.3) is an example of a physical situation that leads to an Euler differential equation from [3].
Consider the electrical circuit consisting of a series connection of a time-varying inductance $L(t,\vec{x}) = t$ henry and a fixed resistance $R = 1$ ohm. Here, t represents time.

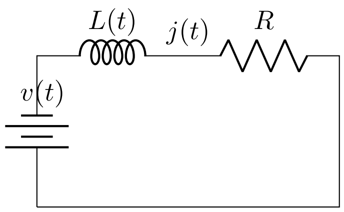


Figure 2: Electric circuit diagram

Assuming that the circuit has no initial excitation in it and that the driving voltage $v(t,\vec{x})$ is zero until some positive instant of time $t = T > 0$.
To determine the responding current $j(t,\vec{x})$, on applying Kirchhoff's law, the differential equation can be represented as:

$$\begin{aligned} v(t,\vec{x}) &= \frac{\partial L j(t,\vec{x})}{\partial t} + R j(t,\vec{x}) \\ &= \frac{\partial t j(t,\vec{x})}{\partial t} + j(t,\vec{x}) \\ &= t\frac{\partial j(t,\vec{x})}{\partial t} + 2j(t,\vec{x}). \end{aligned} \tag{9.12}$$

Considering $v((t,\vec{x})$ is a 2D-CFr Mellin-transformable function concentrated on $T \le t < \infty$.
In (9.12) 2D-CFrMT is applied using theorem 7.1(i) to get

$$\tilde{V}(s_p,s_q) = -(s_p-1)\tilde{J}(s_p,s_q) + 2\tilde{J}(s_p,s_q) \tag{9.13}$$

where

$$\tilde{V}(s_p,s_q) = M^{p,q} v(t,\vec{x}); \quad \tilde{J}(s_p,s_q) = M^{p,q} j(t,\vec{x}). \tag{9.14}$$

$\tilde{V}(s_p,s_q) = M^{p,q} v(t,\vec{x})$ is region of definition and must be a half-plane extending infinitely to the left because $v(t,\vec{x})$ is concentrated on $T \le t < \infty$.
Half-plane can be given from [3]

$$\Omega_{v(t,\vec{x})} = \left\{ s_p : \text{Re}\, s_p < \sigma_{v(t,\vec{x})} \right\}. \tag{9.15}$$

Physical considerations indicates that $j(t,\vec{x})$ will also be concentrated on $T \le t < \infty$.
Hence, the solution $j(t,\vec{x})$ is that unique 2D-CFr Mellin transformable function

$$\tilde{J}(s_p,s_q) = \frac{\tilde{V}(s_p,s_q)}{(3-s_p)}. \tag{9.16}$$

Hence, the solution is given by (9.16) which has region of definition $\{s_p : \text{Re}\, s_p < \min(\sigma_v, 3)\}$

10 Conclusion

Authors have developed a two-dimensional Clifford fractional Mellin transform (2D-CFrMT) in the present study. Properties based on 2D-CFrMT demonstrated. The inversion formula and Parseval's type properties are established. The examples related to the hence developed results are discussed. The graphical representation and application in Heaviside step function, potential of an infinite wedge and electrical circuits applying 2D-CFrMT are presented in the work.

Future work

The study can further be developed using different functions by applying the Clifford Mellin transform. Graphical interpretation and tabular data with such developed results can be interesting for researchers to analyze and relate to the recent contributions in Mathematical Physics.

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