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THE PARABOLIC TRIGONOMETRIC IDENTITIES USING DUAL NUMBERS AND THEIR APPLICATIONS

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Abstract Analogous to complex numbers, we have derived the parabolic trigonometric identities using dual numbers. Using the exponential function, we have derived the equation of the parabolic Euler's identity. We have developed the matrix representation of the commonly known elliptic and hyperbolic Euler's identities, and used the same technique to develop matrix representation for the parabolic one. These matrices correspond to the rotation matrices in each case. To develop the parabolic Pythagorean identity, we have created another form of the rotation matrices using the Cayley transformation. Using the Pythagorean identity, we have defined the parabolic "unit circle", similar to the elliptic and hyperbolic ones. Lastly, we have discussed some practical applications of the Euler's identity involving both its equation and matrix form.

1 Introduction

The elliptic and hyperbolic trigonometry is much known to us and their identites are widely studied as a part of trigonometry. But, the parabolic form has existed as a denegerate case. So, we have made an attempt to establish similar identities for the parabolic one. The motivation behind our work is to establish the parabolic Euler's and Pythagorean identities analogous to their much known elliptic and hyperbolic counterparts. The novelty in our work is the expression of these much known identities in parabolic form with the matrix algebra playing a significant role.

Brewer, [7] had stated that up to isomorphism, there exists only three 2-dimensional commutative algebras with a unit, over the real numbers which are the spaces of complex numbers, dual numbers and double numbers, each of them termed as elliptic, parabolic and hyperbolic cases (abbreviated as EPH) respectively by Kisil in his works, [13, 14, 15, 17]. The Euler's identity is well known in the elliptic case defined through trigonometric functions and can be analogously defined in the hyperbolic case through hyperbolic functions. But, the same analogy cannot be applied in the parabolic case as the parabolic trigonometric functions are not yet known to us. So, it can be simply defined using the definition of an exponential function. Kisil, in his work, [16] has also shown a matrix representation of the identities in each of the three EPH cases and, using the Cayley transformation [2, 12, 21] developed the rotation matrices for each case. We have used this method to develop two parabolic rotation matrices which are conjugate to each other. Similarly, the Pythagorean identity is also known for the elliptic and hyperbolic cases. To find a similar identity in the parabolic case, we have derived inspiration from another work, [15] of Kisil in which he has used the Erlangen Program of Felix Klein, [3, 4, 5, 6, 18, 22]. Erlangen program is a technique developed in the year 1872 mainly used for the development of non-

Euclidean geometry, [23, 24] under which geometry is defined as the study of those properties that do not change under a group action, [10]. In his work, Kisil has taken Möbius action as the group action performed by the group $SL(2;\mathbb{R})$, [19] on the three EPH cases. For obtaining the parabolic Pythagorean identity, we have done a similar action on the dual unit ε performed by the parabolic rotation matrices. For some related study see [11, 20].

1.1 Preliminaries

The following terms are required in our paper:

Definition 1.1. (Group action) A group action of a group $\mathbb{G}$ on a set $\mathbb{X}$ is a map from $\mathbb{G} \times \mathbb{X}$ to $\mathbb{X}$ (written as $g.x$ for all $g \in \mathbb{G},\ x \in \mathbb{X}$) such that, [10]
(i) $g_1.(g_2.x) = (g_1 g_2).x$ for all $g_1, g_2 \in \mathbb{G},\ x \in \mathbb{X}$ and
(ii) $e.x = x$ for all $x \in \mathbb{X}$ (e is the identity element of $\mathbb{G}$).

Definition 1.2. (Dual numbers) The numbers of the form $a + b\varepsilon$, where $\varepsilon^2 = 0,\ a, b \in \mathbb{R}$ are called *dual numbers*, where ε is the dual unit, [9].

Definition 1.3. (Double numbers) The numbers of the form $a + b\epsilon$, where $\epsilon^2 = 1,\ a, b \in \mathbb{R}$ are called *double numbers*, where ϵ is the double unit, [9].

We shall use the common notation $\mathbb{R}^\sigma = \{a + \iota b\ ;\ a, b \in \mathbb{R}\}$ to denote the complex, dual and double numbers where $\sigma = \iota^2 = -1, 0, 1$ and are referred to as elliptic ($\mathbb{R}^e$), parabolic ($\mathbb{R}^p$) and hyperbolic ($\mathbb{R}^h$) cases respectively, [15].

Definition 1.4. (Exponential of a matrix) The exponential of an $n \times n$ matrix X, denoted by e^X or exp X is defined by a power series given by, [12]

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}, \tag{1.1}$$

where X^0 is the identity matrix I and, X^m is the repeated matrix product of X with itself.

Definition 1.5. (Möbius transformation) A transformation $\phi : \mathbb{R}^\sigma \to \mathbb{R}^\sigma$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z \longmapsto \frac{az+b}{cz+d} = \phi(z);\ ad - bc \neq 0,\ a, b, c, d \in \mathbb{R},\ z \in \mathbb{R}^\sigma$$

is called a Möbius Transformation.

2 Describing rotations in each of the three EPH cases

2.1 Elliptic case

The elliptic case is isomorphic to the set of all complex numbers which can be parametrically written as,

$$z = \cos t + i \sin t;\ \ x = \cos t,\ y = \sin t,\ \text{where } t \in \mathbb{R}.$$

The above equation can also be expressed in the form of an identity called the Euler's identity is given by

$$e^{it} = \cos t + i \sin t. \tag{2.1}$$

The above identity has a geometrical interpretation as well given by the following lemma.

Lemma 2.1. *Multiplication by e^{it} is an isometric rotation of the plane $\mathbb{R}^2$, where the elliptic distance is given by $(x + iy)(x - iy) = x^2 + y^2$, [16].*

Proof. Let $z' = x' + iy'$ be the new coordinates of a point $z = x + iy$ after multiplication by e^{it}. Then,

$$\begin{aligned} z' &= ze^{it} \\ \Rightarrow x' + iy' &= (x+iy)(\cos t + i\sin t) \\ \Rightarrow x' + iy' &= (x\cos t - y\sin t) + i(x\sin t + y\cos t) \\ \therefore x' &= x\cos t - y\sin t \text{ and } y' = x\sin t + y\cos t. \end{aligned}$$

$$\begin{aligned} \text{Now, } (x'+iy')(x'-iy') &= x'^2 + y'^2 \\ &= (x\cos t - y\sin t)^2 + (x\sin t + y\cos t)^2 \\ &= (x^2+y^2)(\cos^2 t + \sin^2 t) \\ &= x^2 + y^2 = (x+iy)(x-iy) \\ \text{i.e., } (x'+iy')(x'-iy') &= (x+iy)(x-iy). \end{aligned}$$ □

Hence, the elliptic distance is preserved under multiplication by e^{it}.

2.2 Hyperbolic case

The hyperbolic case is isomorphic to the set of double numbers. Replacing i by ϵ in Eq.(2.1) we deduce the Euler's identity in the hyperbolic case as follows:

$$\begin{aligned} e^{\epsilon t} &= 1 + (\epsilon t) + \frac{(\epsilon t)^2}{2!} + \frac{(\epsilon t)^3}{3!} + \frac{(\epsilon t)^4}{4!} + \frac{(\epsilon t)^5}{5!} + \frac{(\epsilon t)^6}{6!} + \frac{(\epsilon t)^7}{7!} + \frac{(\epsilon t)^8}{8!} + \dots \\ &= \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{t^8}{8!} + \dots\right) + \epsilon\left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^5}{5!}\dots\right) \quad (\because\ \epsilon^{2n} = 1\ \forall\, n \geq 1) \\ &= \cosh t + \epsilon \sinh t. \end{aligned}$$

Hence, the Euler's identity in the hyperbolic case is given by

$$e^{\epsilon t} = \cosh t + \epsilon \sinh t. \tag{2.2}$$

The above identity also has a geometrical implication similar to the elliptic case which can be described by the following lemma.

Lemma 2.2. *Multiplication by $e^{\epsilon t}$ is a map of double numbers into itself, where the hyperbolic distance given by $(x+\epsilon y)(x-\epsilon y) = x^2 - y^2$ is preserved. Geometrically, this can also be viewed as a hyperbolic rotation, [16].*

Proof. Let $z' = x' + \epsilon y'$ be the new coordinates of a point $z = x + \epsilon y$ after multiplication by $e^{\epsilon t}$. Then,

$$\begin{aligned} z' &= ze^{\epsilon t} \\ \Rightarrow x' + \epsilon y' &= (x+\epsilon y)(\cosh t + \epsilon \sinh t) \\ \Rightarrow x' + \epsilon y' &= (x\cosh t + y\sinh t) + \epsilon(x\sinh t + y\cosh t) \\ \therefore x' &= x\cosh t + y\sinh t \text{ and } y' = x\sinh t + y\cosh t. \end{aligned}$$

$$\begin{aligned} \text{Now, } (x'+\epsilon y')(x'-\epsilon y) &= x'^2 - y'^2 \\ &= (x\cosh t + y\sinh t)^2 - (x\sinh t + y\cosh t)^2 \\ &= (x^2-y^2)(\cosh^2 t - \sinh^2 t) \\ &= x^2 - y^2 = (x+\epsilon y)(x-\epsilon y) \\ \text{i.e., } (x'+\epsilon y')(x-\epsilon y') &= (x+\epsilon y)(x-\epsilon y). \end{aligned}$$ □

Hence, the hyperbolic distance is preserved under multiplication by $e^{\epsilon t}$.

2.3 Parabolic case

The parabolic case is isomorphic to the set of dual numbers. Replacing i by ε in Eq.(2.1), we deduce the Euler's identity in the parabolic case as follows :

$$\begin{aligned} e^{\varepsilon t} &= 1 + (\varepsilon t) + \frac{(\varepsilon t)^2}{2!} + \frac{(\varepsilon t)^3}{3!} + \frac{(\varepsilon t)^4}{4!} + \frac{(\varepsilon t)^5}{5!} + \frac{(\varepsilon t)^6}{6!} + \frac{(\varepsilon t)^7}{7!} + \frac{(\varepsilon t)^8}{8!} + \ldots \\ &= 1 + \varepsilon t. \quad (\because\ \varepsilon^n = 0\ \forall\, n \geq 2) \end{aligned}$$

Hence, the Euler's identity in the parabolic case is given by

$$e^{\varepsilon t} = 1 + \varepsilon t. \tag{2.3}$$

Then, parabolic rotations defined by multiplication of the above identity with dual numbers is given by the following action, [16]

$$e^{\varepsilon t}.(x + \varepsilon y) \longmapsto x + \varepsilon(xt + y) \tag{2.4}$$

for

$$\begin{aligned} e^{\varepsilon t}.(x + \varepsilon y) &\longmapsto (1 + \varepsilon t)(x + \varepsilon y) \\ &= x + \varepsilon y + \varepsilon xt + \varepsilon^2 ty \\ &= x + \varepsilon(xt + y). \ \ (\text{as } \varepsilon^2 = 0) \end{aligned}$$

Remark 2.3. We observe the following:

(i) Comparing the parabolic Euler's identity given by Eq.(2.3) with that of the previous identities derived in elliptic(Eq.(2.1)) and hyperbolic(Eq.(2.2)) cases, we get the parabolic trigonometric functions:

$$\text{cosp}\, t = 1 \ \text{ and } \ \text{sinp}\, t = t.$$

(ii) Polar decomposition of complex numbers is given by $z = re^{i\theta}$, where r is the modulus and θ is the argument of the complex number. Then, rewriting a dual number $x + \varepsilon y$ as

$$x + \varepsilon y = x\left(1 + \varepsilon \frac{y}{x}\right) = xe^{\varepsilon(y/x)}, \ \ (\text{as } e^{\varepsilon t} = 1 + \varepsilon t)$$

we get the modulus and argument of a dual number as

$$|x + \varepsilon y| = x \ \text{ and } \ \arg\,(x + \varepsilon y) = \frac{y}{x}.$$

(iii) Parabolic distance (analogous to the elliptic and hyperbolic cases) is given by $(x + \varepsilon y)(x - \varepsilon y) = x^2$, which is independent of y.

But all the above observations are quite trivial and not much significant. So, we shall make an attempt to explore more of these rotations, particularly the parabolic rotations in a non-trivial manner.

3 Matrix form of the EPH Euler identities

The hypercomplex units i, ε and ϵ in each of the three EPH cases can be well represented through traceless square matrices belonging to the Lie algebra $\mathfrak{sl}(2;\mathbb{R})$ of the matrix Lie group $SL(2;\mathbb{R})$ as, [16]

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \ \varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \ \epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

such that the matrix multiplication implies

$$i^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I,$$

$$\varepsilon^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O,$$

$$\epsilon^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1, \end{pmatrix} = I, \quad \text{where I is the } 2 \times 2 \text{ identity matrix.}$$

We shall call the above three matrices as the hypercomplex unit matrices.

Lemma 3.1. *The Euler's identity given by the elliptic, hyperbolic and parabolic unit matrices i, ϵ and ε respectively are the standard elliptic, hyperbolic and parabolic rotation matrices belonging to $SL(2;\mathbb{R})$.*

Proof. (i) **Elliptic case:**

$$\begin{aligned} e^{it} &= \exp\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t \right\} \\ &= I + \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}^2 + \frac{1}{3!} \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}^3 + \frac{1}{4!} \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}^4 + \ldots \text{(From Eq.(1.1))} \\ &= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots & t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots \\ -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \ldots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots \end{pmatrix} \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \end{aligned}$$

Therefore, the Euler's identity in elliptic case can be represented in matrix form as $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$, which is the standard elliptic rotation matrix.

(ii) **Hyperbolic case:**

$$\begin{aligned} e^{\epsilon t} &= \exp\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t \right\} \\ &= I + \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^2 + \frac{1}{3!} \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^3 + \frac{1}{4!} \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}^4 + \ldots \text{(From Eq.(1.1))} \\ &= \begin{pmatrix} 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots & t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots \\ t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots & 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots \end{pmatrix} \\ &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}. \end{aligned}$$

Therefore, the Euler's identity in hyperbolic case can be represented in matrix form as $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, which is the standard hyperbolic rotation matrix. One can easily check that both the obtained matrices belong to $SL(2;\mathbb{R})$.

(iii) **Parabolic case:** Using the same technique as in the previous two cases, we obtain the parabolic Euler's identity belonging to $SL(2;\mathbb{R})$ as follows:

$$\begin{aligned} e^{\varepsilon t} &= \exp\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t \right\} \\ &= I + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}^2 + \frac{1}{3!}\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}^3 + \frac{1}{4!}\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}^4 + \ldots \text{(From Eq.(1.1))} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \ldots \\ &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore, the Euler's identity in parabolic case can be represented in matrix form as $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. We will denote the set of matrices of this form by N, and use them to represent the parabolic rotation matrix. □

Remark 3.2. The matrix representation of the dual unit ε is not unique, for it may be equally well represented by the matrix $\varepsilon' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ since, $\varepsilon'^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which gives us

$$\begin{aligned} e^{\varepsilon' t} &= \exp\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} t \right\} \\ &= I + \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}^2 + \frac{1}{3!}\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}^3 + \frac{1}{4!}\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}^4 + \ldots \text{(From Eq.(1.1))} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \ldots \\ &= \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}. \end{aligned}$$

Therefore, we have another form of matrix representation of the Euler's identity in parabolic case given by the matrix $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. We will denote the set of matrices of this form by N'.

Hence, we have both upper and lower triangular matrix forms given by the matrices N and N' respectively for the Euler's identity in parabolic case. This non-unique representation in the parabolic case distinguishes it from the other two cases and gives us a scope for further research.

Lemma 3.3. *The subgroup N' is the stabilizer of ε under the $SL(2;\mathbb{R})$-action.*

Proof. Performing the Möbius action of $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \in N'$ on ε, we get

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} .\varepsilon \longmapsto = \frac{\varepsilon}{1+\varepsilon t}\left(\frac{1-\varepsilon t}{1-\varepsilon t}\right) = \frac{\varepsilon - \varepsilon^2 t}{1-\varepsilon^2 t^2} = \varepsilon. \quad (\because\ \varepsilon^2 = 0)$$

Hence, N' is the stabilizer of ε. But, one can check that it does not fix its reciprocal i.e., $\dfrac{1}{\varepsilon}$. □

In the next section, we shall be discussing another matrix form of the identities given by Eqs.(2.1),(2.2) and (2.3) using the Cayley transform. The reason behind creating a new form is the very trivial action of the matrices N and N' on $(-\varepsilon)$ and $\left(-\dfrac{1}{\varepsilon}\right)$ respectively which yield the orbits $u = -vt$ and $u = vt$. Hence, we need a stronger form of the parabolic rotation matrices in order to develop the Pythagorean identity which we obtain in the next section.

4 Constructing a new form of the EPH rotation matrices

Another matrix form of the elliptic rotation matrices was given by Kisil, [16] using the Cayley transform,

$$\frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix},$$

where the matrix $C_i = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$ is the Cayley transform from the upper-half plane to the unit disk. Elliptic rotations can be well understood through the above matrix, for it rotates an element $z \in \mathbb{R}^e$ by an angle $2t$ as given by the following equation,

$$\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.z \longmapsto \frac{e^{it}z}{e^{-it}} = e^{2it}z,$$

and also preserves the elliptic distance. Hence, we consider this matrix as the rotation matrix in the elliptic case.

Its hyperbolic variant, $C_\epsilon = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix}$ produces a matrix form of the hyperbolic rotation matrix given by,

$$\frac{1}{2}\begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix}\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix} = \begin{pmatrix} e^{\epsilon t} & 0 \\ 0 & e^{-\epsilon t} \end{pmatrix}.$$

Similar to the previous case, hyperbolic rotations can be well understood through the above matrix, for it rotates an element $z \in \mathbb{R}^h$ by an angle $2t$ as given by the following equation,

$$\begin{pmatrix} e^{\epsilon t} & 0 \\ 0 & e^{-\epsilon t} \end{pmatrix}.z \longmapsto \frac{e^{\epsilon t}z}{e^{-\epsilon t}} = e^{2\epsilon t}z,$$

and also preserves the hyperbolic distance. Hence, we consider this matrix as the rotation matrix in the hyperbolic case.

In the same way, Kisil also developed a parabolic rotation matrix using the parabolic variant $C_\varepsilon = \begin{pmatrix} 1 & -\varepsilon \\ -\varepsilon & 1 \end{pmatrix}$ which produced the matrix form [16],

$$\begin{pmatrix} 1 & -\varepsilon \\ -\varepsilon & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} = \begin{pmatrix} e^{\varepsilon t} & t \\ 0 & e^{-\varepsilon t} \end{pmatrix}.$$

.

Another form of the above matrix can be generated by the other form of the dual unit matrix, given by $\varepsilon' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ which produces the matrix [16],

$$\begin{pmatrix} 1 & -\varepsilon \\ -\varepsilon & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} = \begin{pmatrix} e^{-\varepsilon t} & 0 \\ t & e^{\varepsilon t} \end{pmatrix}.$$

We denote the upper triangular form of matrices $\begin{pmatrix} e^{\varepsilon t} & t \\ 0 & e^{-\varepsilon t} \end{pmatrix}$ by P and the lower triangular form of matrices $\begin{pmatrix} e^{-\varepsilon t} & 0 \\ t & e^{\varepsilon t} \end{pmatrix}$ by P'. Now, analogous to the elliptic and hyperbolic cases discussed above, we can identify parabolic rotations through both the matrix representations given by P and P'. Hence, we consider these matrices as the rotation matrices in the parabolic case.

Remark 4.1. All the EPH rotation matrices belong to the matrix Lie group $SL(2;\mathbb{R}^\sigma)$.

Lemma 4.2. *The subgroup P is conjugate to the subgroup P'.*

Proof. We need to show that there exists a non-singular matrix M such that $P' = MPM^{-1}$. Now, we have the matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ such that

$$MPM^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{\varepsilon t} & t \\ 0 & e^{-\varepsilon t} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} e^{-\varepsilon t} & 0 \\ t & e^{\varepsilon t} \end{pmatrix} = P'. \qquad \square$$

Hence, P and P' are conjugate to each other.

Lemma 4.3. *Under the Möbius action, the matrix P' fixes the zero divisors in dual numbers.*

Proof. The zero divisors of dual numbers are given by the set $\mathbb{D} = \{\varepsilon v : v \in \mathbb{R}\}$. Then, performing the Möbius action of P' on $\varepsilon v \in \mathbb{D}$ we get,

$$\begin{pmatrix} e^{-\varepsilon t} & 0 \\ t & e^{\varepsilon t} \end{pmatrix} .\varepsilon v \longmapsto \frac{\varepsilon v e^{-\varepsilon t}}{\varepsilon v t + e^{\varepsilon t}} = \frac{\varepsilon v(1-\varepsilon t)}{\varepsilon v t + (1+\varepsilon t)} = \varepsilon v.$$

Hence, points of the form εv i.e., the zero divisors of dual numbers are fixed. $\square$

Theorem 4.4. *Coordinate-wise addition of elements of $\mathbb{R}^\sigma$ is invariant under the elliptic and hyperbolic rotations but not under the parabolic one, [16].*

Proof. We know, the rotations in the three EPH cases can be well defined by their respective rotation matrices. We shall now investigate each case separately.

(i) **Elliptic case:** Let $a+ib,\ c+id \in \mathbb{R}^e$. Then

$$\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} .(a+ib) \longmapsto e^{2it}(a+ib),$$

$$\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} .(c+id) \longmapsto e^{2it}(c+id),$$

$$\text{and} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} .\{(a+ib)+(c+id)\} \longmapsto e^{2it}(a+ib) + e^{2it}(c+id).$$

$$\therefore \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} .\{(a+ib)+(c+id)\} = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} .(a+ib) \right\} + \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} .(c+id) \right\}.$$

Hence, coordinate-wise addition is invariant under elliptic rotation.

(ii) **Hyperbolic case:** Let $a+\epsilon b,\ c+\epsilon d \in \mathbb{R}^h$. Then,

$$\begin{pmatrix} e^{\epsilon t} & 0 \\ 0 & e^{-\epsilon t} \end{pmatrix} .(a+\epsilon b) \longmapsto e^{2\epsilon t}(a+\epsilon b),$$

$$\begin{pmatrix} e^{\epsilon t} & 0 \\ 0 & e^{-\epsilon t} \end{pmatrix} .(c+\epsilon d) \longmapsto e^{2\epsilon t}(c+\epsilon d),$$

$$\text{and} \begin{pmatrix} e^{\epsilon t} & 0 \\ 0 & e^{-\epsilon t} \end{pmatrix} .\{(a+\epsilon b)+(c+\epsilon d)\} \longmapsto e^{2\epsilon t}(a+\epsilon b) + e^{2\epsilon t}(c+\epsilon d).$$

$$\therefore \begin{pmatrix} e^{\epsilon t} & 0 \\ 0 & e^{-\epsilon t} \end{pmatrix}.\{(a+\epsilon b)+(c+\epsilon d)\} = \left\{ \begin{pmatrix} e^{\epsilon t} & 0 \\ 0 & e^{-\epsilon t} \end{pmatrix}.(a+\epsilon b)\right\} + \left\{ \begin{pmatrix} e^{\epsilon t} & 0 \\ 0 & e^{-\epsilon t} \end{pmatrix}.(c+\epsilon d)\right\}.$$

Hence, coordinate-wise addition is invariant under hyperbolic rotation.

(iii) **Parabolic case:** Let $a+\varepsilon b,\ c+\varepsilon d \in \mathbb{R}^p$. Then using the matrix P we have,

$$\begin{pmatrix} e^{\varepsilon t} & t \\ 0 & e^{-\varepsilon t} \end{pmatrix}.(a+\varepsilon b) \longmapsto e^{2\varepsilon t}(a+\varepsilon b) + te^{\varepsilon t}, \tag{4.1}$$

$$\begin{pmatrix} e^{\varepsilon t} & t \\ 0 & e^{-\varepsilon t} \end{pmatrix}.(c+\varepsilon d) \longmapsto e^{2\varepsilon t}(c+\varepsilon d) + te^{\varepsilon t}, \tag{4.2}$$

and $$\begin{pmatrix} e^{\varepsilon t} & t \\ 0 & e^{-\varepsilon t} \end{pmatrix}.\{(a+\varepsilon b)+(c+\varepsilon d)\} \longmapsto e^{2\varepsilon t}\{(a+\varepsilon b)+(c+\varepsilon d)\} + te^{\varepsilon t}. \tag{4.3}$$

Now, adding Eqs.(4.1) and (4.2) we get

$$\begin{pmatrix} e^{\varepsilon t} & t \\ 0 & e^{-\varepsilon t} \end{pmatrix}.\{(a+\varepsilon b)+(c+\varepsilon d)\} \longmapsto e^{2\varepsilon t}\{(a+\varepsilon b)+(c+\varepsilon d)\} + 2te^{\varepsilon t},$$

which is not the same as Eq.(4.3). Hence, coordinate-wise addition is not invariant under parabolic rotation. The same can be proved by using the matrix P'.

□

5 Pythagorean identity in the parabolic case

One way of obtaining the elliptic and hyperbolic Pythagorean identities is through their respective commonly known rotation matrices $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $\begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$; $\theta \in \mathbb{R}$, by equating the determinant of these matrices to 1. But, such an approach with the parabolic rotation matrices P and P' will give us no result. So, we try an alternative approach to derive the parabolic Pythagorean identity through the Möbius action of P and P'.

Theorem 5.1. *Orbits of P-action on $(-\varepsilon)$ and $P'-$action on $\left(-\frac{1}{\varepsilon}\right)$ are of the same form.*

Proof. (i) Action of P: Performing the Möbius action of the matrix P on $(-\varepsilon)$, we get

$$\begin{pmatrix} e^{\varepsilon t} & t \\ 0 & e^{-\varepsilon t} \end{pmatrix}.(-\varepsilon) \longmapsto \frac{-\varepsilon e^{\varepsilon t} + t}{e^{-\varepsilon t}} = t + \varepsilon(t^2 - 1) = u + \varepsilon v.$$

$$\therefore\ u = t,\ v = t^2 - 1.$$

The orbit of this action is given by $u^2 - v = 1$ and can be seen in Figure 1.

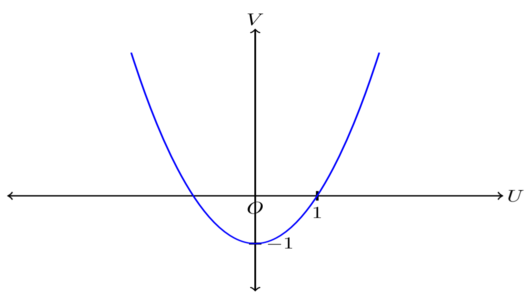

Figure 1. Orbit of the P-action on $-\varepsilon$.

(ii) Action of P': We next consider the Möbius action of the matrix P'. As we have proved previously in Lemma 4.3 that P' fixes the zero divisors in dual numbers so, we cannot perform its action on ε and $-\varepsilon$ since, both of them are zero divisors and will be fixed by the matrix P'. Instead, we perform its action on $\left(-\frac{1}{\varepsilon}\right)$ to get,

$$\begin{pmatrix} e^{-\varepsilon t} & 0 \\ t & e^{\varepsilon t} \end{pmatrix} \cdot \left(-\frac{1}{\varepsilon}\right) \longmapsto \frac{-\frac{1}{\varepsilon}e^{-\varepsilon t}}{-\frac{t}{\varepsilon}+e^{\epsilon t}} = \frac{1}{t} + \varepsilon\left(\frac{1}{t^2}-1\right) = u + \varepsilon v.$$

$$\therefore\ u = \frac{1}{t},\ v = \frac{1}{t^2} - 1.$$

which gives us the same orbit $u^2 - v = 1$ as in the previous case and consequently has the same figure (see Figure 1).

Hence, the orbits obtained by the respective P and $P'-$actions are of the same form. □

Remark 5.2. We do not perform actions of P and P' on ε and $\frac{1}{\varepsilon}$ respectively since their orbits are not of the same form and differ in nature.

Using the common orbit $u^2 - v = 1$, we shall now define the Pythagorean identity for the parabolic case. We take the parabolic cosine and sine as $u = \text{cosp}\,\theta$ and $v = \text{sinp}\,\theta$ respectively, which gives us the parabolic Pythagorean identity,

$$\text{cosp}^2\,\theta - \text{sinp}\,\theta = 1.$$

Remark 5.3. We know that both the elliptic and hyperbolic Pythagorean identities represent the elliptic "unit circle" and hyperbolic "unit circle" and are given by the equations $u^2 + v^2 = 1$ and $u^2 - v^2 = 1$ respectively. Following the same analogy, we can represent the parabolic "unit circle" using the parabolic Pythagorean identity. So, irrespective of the matrices P and P', we can express the equation of the parabolic "unit circle" by the common orbit, [16]

$$\mathbf{u^2 - v = 1},$$

which is an upward facing parabola (see Figure 1) with vertex at $(0, -1)$, focus at $\left(0, \frac{-3}{4}\right)$ and equation of directrix given by $v + \frac{5}{4} = 0$.

6 Applications of the parabolic Euler's identity

6.1 Galilean Relativity Principle

If we denote an element in space-time by $(t, x) \in \mathbb{R}^2$ where t is the time component and x denotes the position then, according to the Galilean relativity principle, the laws of classical mechanics will be invariant under the following transformation of the transfer matrix N', [17]

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} t \\ x + vt \end{pmatrix} = G_v \begin{pmatrix} t \\ x \end{pmatrix},\ \text{where } G_v = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \in N'.$$

In other words, we can use the parabolic Euler's identity given by the matrix N' (see Lemma 3.1) to transfer events from one frame to another moving with a constant speed v w.r.t the first one.

6.2 Hamiltonian mechanics

Dual numbers itself along with the parabolic Euler's identity play a great role in solving for the momentum and displacement in the coupled equations of a Hamiltonian. The Hamiltonian H, [8] for a particle with mass m under a uniform gravitational field g is given by,

$$H = \frac{p^2}{2m} + mgx,$$

where p denotes the momentum. Then the coupled equations corresponding to the above Hamiltonian are given by

$$m\frac{dx}{dt} = p, \quad \frac{dp}{dt} = -mg$$

Suppose, α be a constant having dimensions of length. Then,

$$\frac{dp}{dt} + \varepsilon\alpha\frac{dx}{dt} = -mg + \frac{\varepsilon\alpha p}{m} = -mg + \frac{\varepsilon\alpha}{m}(p + \varepsilon\alpha x) \qquad (\because \varepsilon^2 = 0)$$

Thus, the coupled equations can be combined and rewritten using the property of $\varepsilon^2 = 0$ of the dual unit ε as,

$$\frac{d}{dt}(p + \varepsilon\alpha x) = \frac{\varepsilon\alpha}{m}(p + \varepsilon\alpha x) - mg,$$

which is a linear first order differential equation of the form

$$\frac{dy}{dt} + P(t)y = Q(t)$$

Then, using the method of integrating factors we get the solution $\left(y = e^{-\int P(t)} \int e^{\int P(t)} Q(t)dt\right)$ of this differential equation as,

$$p + \varepsilon\alpha x = e^{\frac{\varepsilon\alpha t}{m}}\left(-\int mge^{-\frac{\varepsilon\alpha t}{m}}dt\right)$$

Here, we use the Euler's identity $e^{\varepsilon t} = 1 + \varepsilon t$ (see Section 2.3) to get,

$$\begin{aligned} p + \varepsilon\alpha x &= \left(1 + \frac{\varepsilon\alpha t}{m}\right)\left(-\int mg\left(1 - \frac{\varepsilon\alpha t}{m}\right)dt\right) \\ &= \left(1 + \frac{\varepsilon\alpha t}{m}\right)\left(-mgt + \frac{\varepsilon\alpha g t^2}{2} + c_1 + \varepsilon c_2\right) \\ &= (-mgt + c_1) + \varepsilon\alpha\left(\frac{c_2}{\alpha} - \frac{gt^2}{2} + \frac{c_1 t}{m}\right), \end{aligned}$$

where c_1 and c_2 are real constants. Equating the components we get,

$$p = -mgt + c_1 \text{ and } x = \frac{c_2}{\alpha} - \frac{gt^2}{2} + \frac{c_1 t}{m}.$$

The constants c_1 and $\frac{c_2}{\alpha}$ represent the initial values of p and x respectively. Hence, using the Euler's identity $e^{\varepsilon t}$, we were able to obtain the displacement x and momentum p from the Hamiltonian equation of a point mass.

6.3 Heat equation in one-dimenison

We know, the time-dependent PDE for the heat distribution function $f(x,t)$ in a homogeneous one-dimensional long metal rod in the x-direction is given by, [1, 17]

$$(\delta_t - k\delta_x^2)f(x,t) = 0, \text{ where } x,\ t(>0) \in \mathbb{R},$$

and k is a real constant which measures the heat conductivity of the medium. For the initial-value $f(x,0) = g(x)$, the solution is given by the convolution

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} exp\left(-\frac{(x-y)^2}{4kt}\right)g(y)\,dy,$$

with the function $exp\left(-\frac{x^2}{4kt}\right)$, which is called the heat kernel, [17].

Lemma 6.1. *The heat kernel is invariant under Möbius action of the parabolic Euler's identity N' given by*

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} .(x+\varepsilon t) \longmapsto \frac{x+\varepsilon t}{c(x+\varepsilon t)+1}.$$

Proof. Suppose, $(x'+\varepsilon t')$ is the new coordinate of a point $(x+\varepsilon t)$ after transformation by N' (see Lemma 3.1). Then,

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} .(x+\varepsilon t) \longmapsto \frac{x+\varepsilon t}{c(x+\varepsilon t)+1} = \frac{x}{1+cx} + \varepsilon \frac{t}{(1+cx)^2} = x' + \varepsilon t'.$$

$$\therefore\ x' = \frac{x}{1+cx} \text{ and } t' = \frac{t}{(1+cx)^2}.$$

From the above equation, we deduce the equation of the N'-orbits as

$$x'^2 = \frac{x^2}{t} t'$$
$$\Rightarrow \frac{x'^2}{t'} = \frac{x^2}{t} = \text{constant}$$

Suppose, we express the constant solution in the form

$$\frac{x'^2}{t'} = \frac{x^2}{t} = 4k \ln\left(\frac{1}{C}\right), \text{ for some } C > 0 \in \mathbb{R},$$

where k is the thermal conductivity constant. Then,

$$\frac{x^2}{t} = 4k \ln\left(\frac{1}{C}\right)$$
$$\Rightarrow -\frac{x^2}{4kt} = \ln C$$
$$\Rightarrow \exp\left(-\frac{x^2}{4kt}\right) = C.$$

Hence, we express the N'-orbits in the form of heat kernel given by $\exp\left(-\frac{x^2}{4kt}\right) = C$. The heat kernel has a constant value i.e., it remains unchanged or invariant under the N'-action. In other words, the contour of the heat kernel can be equally expressed using the orbit of the parabolic Euler's identity given by N'. □

7 Conclusion and future work

We tried to show some trigonometrical aspects of the degenerate parabolic case analogous to the more famous elliptic and hyperbolic cases. We derived the Euler's identity and its matrix form as well for the parabolic case, along with their already known elliptic and hyperbolic counterparts. Using the Cayley transform, we constructed a new form of the EPH rotation matrices. The reason behind this was to devise a method to derive the two parabolic rotation matrices and ultimately, the Pythagorean identity. Also, zero divisors of dual numbers are fixed by the lower-triangular rotation matrices. Although, coordinate-wise addition is preserved in elliptic and hyperbolic rotations, it is not the same for parabolic rotations. We obtained the parabolic Pythagorean identity through the Möbius action of the parabolic rotation matrices. Using this identity, we tried to define the parabolic "unit circle". The importance of our results is that they might contribute significantly to the development of the non-Euclidean parabolic geometry particularly through the rotation matrices that we discussed. We also tried to show some other physical applications of the Euler's identity. In future, we shall try to develop more geometrical and physical applications related to the parabolic trigonometric identities.

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