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CONTROLLED g-FRAMES and THEIR g-DUALS

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Abstract In this paper, we prove some properties of controlled g-frames. A necessary and sufficient condition for a controlled g-Bessel sequence to be a controlled g-frame is obtained. In addition, we give sufficient conditions for the sum of two (C, D)-controlled g-frames to be a (C, D)-controlled g-frame. Then, we define the controlled g-dual of a controlled g-frame and prove some conditions under which two g-Bessel sequences befit controlled g-dual frames. Thereafter, we obtain necessary conditions under which a combination of two (C, D)-controlled g-duals of controlled g-frame $\{\Lambda_j\}_{j\in J}$ will again be a (C, D)-controlled g-duals of $\{\Lambda_j\}_{j\in J}$. We conclude this paper by providing the invertible operators that preserve the (C, D)-controlled g-dual of a controlled g-frame.

1 Introduction

While studying various problems in the non-harmonic Fourier series, Duffin and Schaeffer [8] discovered the concept of frames in 1952. In 1986, Daubechies, Grossman and Meyer [6] reacquainted with the notion of frames. Frames have numerous applications in pure and applied mathematics. Over a period of time, various extensions and generalizations of frames in Hilbert and Banach spaces like g-frames [15, 24, 25], Fusion frames [4], K-frames [9, 10, 21], K-fusion frames [3], Atomic subspaces for operators [2], K-g-frames [16, 17] and Fusion Banach frames [11, 12, 13, 14], F-Fusion Banach Frames [19] have been studied.

P. Balaz [1] in 2010 came up with the idea of controlled frames to improve the numerical efficiency of the iterative algorithm for inverting the frame operator on abstract Hilbert space. Since then controlled frames have been studied by various researchers. Rahimi et al. [20] coined the concept of controlled g-frames which is an extension of g-frames and controlled frames. The duals of a frame play an important role in the reconstruction of vectors in terms of frame elements. Keeping this in mind Dehghan and Fard [7] defined the concept of g-duals of frames in Hilbert spaces and it was observed that we can achieve more reconstruction formulas to obtain signals. This concept was further extended by Ramezani and Nazari [22] to g-duals of generalized frames. Ramezani [23] broadened the concept of g-dual of a frame to controlled g-dual frames for a C-controlled frame. Controlled g-frames have the dynamism of theoretically foundational and practical applications. With the ongoing research on controlled g-frames and their g-duals, we studied some interesting properties of controlled g-frames with two controller operators which are bounded and invertible. The inference that has been flagged in this paper is the construction of g-duals of controlled g-frames and it was observed that g-duals of controlled frames are generalizations of dual g-frames. Section 2 deals with the basic definitions and related results. In Section 3, we discuss several properties of controlled g-frames with two positive controller operators and give a characterization for a controlled g-Bessel sequence to be controlled g-frame. The concept of the generalized dual of controlled g-frames or simply g-dual of controlled g-frames is given in Section 4. Also, we obtain a reconstruction formula for the elements of the Hilbert space H in terms of the adjoint of controlled g-frame elements using g-dual of controlled g-frames.

Throughout this paper, H will denote a separable Hilbert space and $B(H_1, H_2)$ denotes the collection of all bounded linear operators from H_1 to H_2 and $B(H)$ is the collection of all bounded linear operators on H. T^* denotes the adjoint of the operator T. $GL(H)$ denotes the set of all bounded linear operators which have bounded inverse. $GL^+(H)$ denotes the set of all positive operators in $GL(H)$. A bounded operator $S : H \to H$ is positive if $\langle Sx, x\rangle > 0$, for all $0 \neq x \in H$. Every bounded positive operator on a complex Hilbert space is self-adjoint. Also, we have assumed in this paper that two positive operators commute. Let $\{H_j : j \in J\}$ be a sequence of closed subspaces of H, where J is a countable indexing set. Then the sequence space

$$l^2\left(\{H_j\}_{j\in J}\right) = \left\{\{x_j\}_{j\in J} : x_j \in H_j, j \in J, \sum_{j\in J} \|x_j\|^2 < \infty\right\}$$

with the inner product given by $\langle\{x_j\}_{j\in J}, \{y_j\}_{j\in J}\rangle = \sum_{j\in J}\langle x_j, y_j\rangle_{H_j}$ is a separable Hilbert space.

2 Preliminaries

In this section, we present basic definitions and results related to frames, controlled frames, g-frames and controlled g-frames and their g-duals.

Definition 2.1. [5] A sequence $\{f_j\}_{j\in J}$ in H is said to be a frame for H, if there exist constants $0 < A \le B < \infty$ such that for all $x \in H$,

$$A\|x\|^2 \le \sum_{j\in J} |\langle x, f_j\rangle|^2 \le B\|x\|^2. \tag{2.1}$$

The constants A and B are called the lower and upper frame bounds, of the frame $\{f_j\}_{j\in J}$, respectively. A frame $\{f_j\}_{j\in J}$ is said to be a tight frame if $A = B$ in (2.1) and a Parseval frame if $A = B = 1$. If only the right hand inequality in (2.1) holds, then $\{f_j\}_{j\in J}$ is known as a Bessel sequence.

Definition 2.2. [7] A frame $\{g_j\}_{j\in J}$ is called a g-dual frame of the frame $\{f_j\}_{j\in J}$ for H if there exists an invertible operator $\Xi \in B(H)$ such that for all $x \in H$,

$$x = \sum_{j\in J}\langle \Xi x, g_j\rangle f_j.$$

Definition 2.3. [1] Let $C \in GL(H)$. A sequence $\{f_j\}_{j\in J}$ in H is a C-controlled frame, if there exist constants $0 < A \le B < \infty$ such that for all $x \in H$,

$$A\|x\|^2 \le \sum_{j\in J}\langle x, f_j\rangle\langle Cf_j, x\rangle \le B\|x\|^2.$$

Definition 2.4. [23] Let $C \in GL(H)$. Suppose that $\{f_j\}_{j\in J}$ is a C-controlled frame and $\{g_j\}_{j\in J}$ is a Bessel sequence in H. Then $\{g_j\}_{j\in J}$ is said to be g-dual of $\{f_j\}_{j\in J}$, if there exist an invertible operator $\Xi \in B(H)$ such that for all $x \in H$,

$$x = \sum_{j\in J}\langle \Xi x, g_j\rangle Cf_j.$$

Definition 2.5. [24] A sequence $\{\Lambda_j \in B(H, H_j) : j \in J\}$ is said to be a g-frame for H with respect to $\{H_j\}_{j\in J}$, if there exist constants $0 < A \le B < \infty$ such that for all $x \in H$,

$$A\|x\|^2 \le \sum_{j\in J} \|\Lambda_j x\|^2 \le B\|x\|^2. \tag{2.2}$$

The constants A and B are called the lower and upper frame bounds of g-frame $\{\Lambda_j\}_{j\in J}$, respectively. The g-frame $\{\Lambda_j\}_{j\in J}$ is said to be tight if $A = B$ and a Parseval g-frame, if $A = B = 1$ in (2.2). If only the right hand inequality in (2.2) holds, then $\{\Lambda_j\}_{j\in J}$ is called a g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$.

Definition 2.6. [22] Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for H with respect to $\{H_j\}_{j\in J}$. A g-frame $\{\Gamma_j\}_{j\in J}$ is called g-dual of g-frame $\{\Lambda_j\}_{j\in J}$ for H with respect to $\{H\}_{j\in J}$, if there exist an invertible operator $\Xi \in B(H)$ such that for all $x \in H$,

$$x = \sum_{j\in J} \Lambda_j^* \Gamma_j(\Xi x).$$

Definition 2.7. [7] Let $C, D \in GL^+(H)$. A sequence $\{\Lambda_j \in B(H, H_j) : j \in J\}$ is called a (C, D)-controlled g-frame for H with respect to $\{H_j\}_{j\in J}$, if there exist constants $0 < A \leq B < \infty$ such that for all $x \in H$,

$$A\|x\|^2 \leq \sum_{j\in J} \langle \Lambda_j Cx, \Lambda_j Dx\rangle \leq B\|x\|^2. \tag{2.3}$$

The constants A and B are called as the lower and upper frame bounds for (C, D)-controlled g-frame $\{\Lambda_j\}_{j\in J}$, respectively. If only the right hand inequality in (2.3) holds, then $\{\Lambda_j\}_{j\in J}$ is called a (C, D)-controlled g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$. If $D = I_H$, then $\{\Lambda_j\}_{j\in J}$ is called a C-controlled g-frame for H with respect to $\{H_j\}_{j\in J}$. If $C = D$, then $\{\Lambda_j\}_{j\in J}$ is a C^2 (or (C, C))-controlled g-frame for H with respect to $\{H_j\}_{j\in J}$.

For a (C, D)-controlled g-Bessel sequence $\{\Lambda_j\}_{j\in J}$ of H with respect to $\{H_j\}_{j\in J}$ with bound B_Λ, the operator $T_{(C,D)} : l^2(\{H_j\}_{j\in J}) \to H$ defined as $T_{(C,D)}(\{x_j\}_{j\in J}) = \sum_{j\in J}(CD)^{\frac{1}{2}}\Lambda_j^* x_j$, for all $\{x_j\}_{j\in J} \in \{H\}_{j\in J}$ is well-defined, bounded and its adjoint $T^*_{(C,D)} : H \to l^2(\{H_j\}_{j\in J})$ given by $T^*_{(C,D)}(x) = \{\Lambda_j(DC)^{\frac{1}{2}}x\}_{j\in J}$, for all $x \in H$ is a bounded linear operator. $T_{(C,D)}$ is called the synthesis operator and $T^*_{(C,D)}$ is called the analysis operator of $\{\Lambda_j\}_{j\in J}$. The frame operator of (C, D)-controlled g-frame $\{\Lambda_j\}_{j\in J}$ is defined as $S_{(C,D)}(x) = \sum_{j\in J} D\Lambda_j^*\Lambda_j Cx$, for all $x \in H$. Moreover, $S_{(C,D)} = CS_\Lambda D$, where S_Λ is the frame operator of g-frame $\{\Lambda_j\}_{j\in J}$. The operator $S_{(C,D)}$ is positive, self-adjoint and invertible.

Proposition 2.8. *[5] Let H and K be Hilbert spaces and suppose that $U : K \to H$ is a bounded operator with a closed range. Then there exists a bounded operator $U^+ : H \to K$ satisfying $ker(U^+) = range(U)^\perp, range(U^+) = ker(U)^\perp$ and $UU^+x = x$, for all $x \in range(U)$. The operator U^+ is called the pseudo-inverse of U.*

Proposition 2.9. *[5] Let $U : H \to H$ be a linear operator. Then the following are equivalent:*

(i) There exists two constants $0 \leq m \leq M < \infty$, such that $mI_H \leq U \leq MI_H$.

(ii) U is positive and there exists constants $0 \leq m \leq M < \infty$ such that $m\|x\|^2 \leq \|U^{\frac{1}{2}}x\|^2 \leq M\|x\|^2$.

(iii) $U \in GL^+(H)$.

3 Properties of controlled frames

In this section, we discuss various properties of controlled g-frames with two controller operators. A necessary and sufficient condition for a controlled g-Bessel sequence to be a controlled g-frame is obtained. Further, we construct new controlled g-frames from a given controlled g-frame.

Theorem 3.1. *Let $C, D \in GL^+(H)$. A (C, D)-controlled g-Bessel sequence $\{\Lambda_j\}_{j\in J}$ with Bessel bound B_Λ is a (C, D)-controlled g-frame for H with respect to $\{H_j\}_{j\in J}$ if and only if the operator $T_{(C,D)} : l^2(\{H_j\}_{j\in J}) \to H$ given by $T_{(C,D)}(\{x_j\}_{j\in J}) = \sum_{j\in J}(CD)^{\frac{1}{2}}\Lambda_j^* x_j$ is a well-defined bounded and surjective linear operator.*

Proof. As the frame operator $S_{(C,D)} = T_{(C,D)}T^*_{(C,D)}$ of a (C,D)-controlled g-frame $\{\Lambda_j\}_{j\in J}$ is invertible, therefore it follows that $T_{(C,D)}$ is surjective. Conversely, suppose that $\{\Lambda_j\}_{j\in J}$ is a (C,D)-controlled g-Bessel sequence with bessel bound B_Λ and $T_{(C,D)}$ is a well defined bounded and surjective linear operator. So, by Proposition 2.8, its pseudoinverse $T^+_{(C,D)}$ exists and $T_{(C,D)}T^+_{(C,D)}x = x$, for all $x \in H$.
Thus,

$$\|x\|^4 \le |\langle T^+_{(C,D)}x, T^*_{(C,D)}x\rangle|^2 \le \|T^+_{(C,D)}\|^2\|x\|^2 \sum_{j\in J}\langle \Lambda_j Cx, \Lambda_j Dx\rangle.$$

Hence, for all $x \in H$,

$$\|T^+_{(C,D)}\|^{-2}\|x\|^2 \le \sum_{j\in J}\langle \Lambda_j Cx, \Lambda_j Dx\rangle \le B_\Lambda\|x\|^2.$$

□

Remark 3.2. However, if the operator $T_{(C,D)}$ is not surjective, then the (C,D)-controlled g-Bessel sequence $\{\Lambda_j\}_{j\in J}$ for H with respect to $\{H_j\}_{j\in J}$ need not be a (C,D)-controlled g-frame for H with respect to $\{H_j\}_{j\in J}$ as is evident from the following example.

Example 3.3. Suppose that $H = \mathbf{R}^4$ and $\{e_j\}_{j\in J}$ be an orthonormal basis of H. Let $H_j = span\{e_j\}$ and $0 \neq \alpha \in \mathbf{R}$. For $1 \le j \le 4$, define $\Lambda_j : H \to H_j$ as $\Lambda_1 x = (\langle x, e_2\rangle + \langle x, e_3\rangle)e_1$; $\Lambda_2 x = \langle x, e_2\rangle e_2$; $\Lambda_3 x = \langle x, e_3\rangle e_3$; $\Lambda_4 x = \langle x, e_4\rangle e_4$ and $C, D : H \to H$ as $Cx = \alpha x$ and $Dx = \alpha^{-1}x$. Then, for any $x \in H$, $\sum_{j=1}^4\langle \Lambda_j Cx, \Lambda_j Dx\rangle \le 3\|x\|^2$. Thus, $\{\Lambda_j\}_{j\in J}$ is a (C,D)-controlled g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$, but is not a (C,D)-controlled g-frame for H with respect to $\{H_j\}_{j\in J}$ as $\|e_1\| = 1$ and $\sum_{j=1}^4\langle \Lambda_j Ce_1, \Lambda_j De_1\rangle = 0$. Also, the operator $T_{(C,D)} : l^2(\{H_j\}_{j\in j}) \to H$ given by $T_{(C,D)}(\{x_j\}_{j\in J}) = (0, x_2, x_3, x_4)$, for all $\{x_j\}_{j\in J} \in l^2(\{H_j\}_{j\in j})$ is not surjective as for $e_1 \in H$, there does not exist any $\{x_j\}_{j\in J} \in l^2(\{H_j\}_{j\in j})$ such that $T_{(C,D)}(\{x_j\}_{j\in J}) = e_1$.

In the next two theorems, we construct new (C,D)-controlled g-frames from an existing (C,D)-controlled g-frame.

Theorem 3.4. *Let $C, D \in GL^+(H)$ and $\{\Lambda_j\}_{j\in J}$ be a (C,D)-controlled g-frame. Then $\{\Lambda_j S^{-1}_{(C,D)}\}_{j\in J}$ is a (C,D)-controlled g-frame and $\{\Lambda_j S^{\frac{-1}{2}}_{(C,D)}\}_{j\in J}$ is a (C,D)-controlled Parseval g-frame for H with respect to $\{H_j\}_{j\in J}$.*

Proof. As $\{\Lambda_j\}_{j\in J}$ is a (C,D)- controlled g-frame, therefore there exists constants $0 < A \le B < \infty$ such that for all $x \in H$,

$$A\|x\|^2 \le \sum_{j\in J}\langle \Lambda_j Cx, \Lambda_j Dx\rangle \le B\|x\|^2.$$

One may note that for any $x \in H$, we have

$$\begin{aligned}
\sum_{j\in J}\langle \Lambda_j S^{-1}_{(C,D)}Cx, \Lambda_j S^{-1}_{(C,D)}Dx\rangle &= \sum_{j\in J}\langle S_{(C,D)}S^{-1}_{(C,D)}x, S^{-1}_{(C,D)}x\rangle \\
&\le B\|S^{-1}_{(C,D)}x\|^2 \\
&\le BA^{-2}\|x\|^2.
\end{aligned}$$

Additionally, for all $x \in H$,

$$\sum_{j\in J}\langle \Lambda_j S^{-1}_{(C,D)}Cx, \Lambda_j S^{-1}_{(C,D)}Dx\rangle \ge A\|S^{-1}_{(C,D)}x\|^2 \ge AB^{-2}\|x\|^2.$$

Thus, for all $x \in H$, we have

$$AB^{-2}\|x\|^2 \le \sum_{j\in J}\langle \Lambda_j S^{-1}_{(C,D)}Cx, \Lambda_j S^{-1}_{(C,D)}Dx\rangle \le BA^{-2}\|x\|^2.$$

So, $\{\Lambda_j S_{(C,D)}\}_{j\in J}$ is a (C,D)-controlled g-frame for H with respect to $\{H_j\}_{j\in J}$. Further, for any $x \in H$,

$$\sum_{j\in J}\langle \Lambda_j S_{(C,D)}^{\frac{-1}{2}} Cx, \Lambda_j S_{(C,D)}^{\frac{-1}{2}} Dx\rangle = \langle S_{(C,D)}^{\frac{-1}{2}} x, S_{(C,D)}^{\frac{-1}{2}} x\rangle = \|x\|^2.$$

Hence, $\{\Lambda_j S_{(C,D)}^{\frac{-1}{2}}\}_{j\in J}$ is a (C,D)-controlled Parseval g-frame for H with respect to $\{H_j\}_{j\in J}$. □

Theorem 3.5. *Let $C, D \in GL^+(H)$, $\{\Lambda_j\}_{j\in J}$ be a (C,D)-controlled g-frame for H with respect to $\{H_j\}_{j\in J}$. If $U \in B(H)$ is invertible that commutes with C and D, then $\{\Lambda_j U\}_{j\in J}$ is also a (C,D)-controlled g-frame.*

Proof. By definition of (C,D)-controlled g-frame, there exists constants $0 < A \leq B < \infty$ such that for all $x \in H$,

$$A\|x\|^2 \leq \sum_{j\in J}\langle \Lambda_j Cx, \Lambda_j Dx\rangle \leq B\|x\|^2.$$

Thus, for any $x \in H$,

$$A\|x\|^2 \leq \|U^{-1}\|^2 \sum_{j\in J}\langle \Lambda_j CUx, \Lambda_j DUx\rangle \leq B\|U^{-1}\|^2\|U\|^2\|x\|^2.$$

Hence, for all $x \in H$.

$$A\|U^{-1}\|^{-2}\|x\|^2 \leq \sum_{j\in J}\langle \Lambda_j UCx, \Lambda_j UDx\rangle \leq B\|U\|^2\|x\|^2.$$

□

In the next example, we see that the Theorem 3.5 may not hold if U does not commute with C and D.

Example 3.6. Let $H = \mathbf{R}^3$ and $\{e_j\}_{j\in J}$ be an orthonormal basis of H. Let $H_1 = span\{e_2, e_3\}$, $H_2 = span\{e_1, e_3\}$ and $H_3 = span\{e_1, e_2\}$. For $1 \leq j \leq 3$, define $\Lambda_j : H \to H_j$ as $\Lambda_1 x = \langle x, e_2\rangle e_3$; $\Lambda_2 x = \langle x, e_3\rangle e_1$; $\Lambda_3 x = 2\langle x, e_1\rangle e_2$; and $C, D : H \to H$ as $Cx = \frac{1}{2}\langle x, e_1\rangle e_1 + \frac{1}{3}\langle x, e_2\rangle e_2 + \frac{1}{4}\langle x, e_3\rangle e_3$, $Dx = 2\langle x, e_1\rangle e_1 + 3\langle x, e_2\rangle e_2 + 3\langle x, e_3\rangle e_3$ and $U : H \to H$ as $Ux = \langle x, e_1\rangle e_1 + (2\langle x, e_1\rangle + \langle x, e_2\rangle)e_2 + \langle x, e_3\rangle e_3$. Then, the operators $C, D \in GL^+(H)$ and $U \in GL(H)$ does not commute with C and D. For any $x \in H$, $\frac{1}{4}\|x\|^2 \leq \sum_{j=1}^{3}\langle \Lambda_j Cx, \Lambda_j Dx\rangle \leq 4\|x\|^2$. Therefore, it follows that $\{\Lambda_j\}_{j\in J}$ is a (C,D)-controlled g-frame for H with respect to $\{H\}_{j\in J}$, but $\{\Lambda_j U\}_{j\in J}$ is not a (C,D)-controlled g-frame for H with respect to $\{H\}_{j\in J}$. Since for $x = e_3$, $\|x\|^2 = 1$ and $\sum_{j=1}^{3}\langle \Lambda_j UCx, \Lambda_j UDx\rangle = \frac{3}{4}$.

Definition 3.7. Let $C, D \in GL^+(H), \Lambda = \{\Lambda_j\}_{j\in J}$ and $\Gamma = \{\Gamma_j\}_{j\in J}$ be two g-Bessel sequences for H with respect to $\{H_j\}_{j\in J}$. We define the operator $S_{D\Lambda\Gamma C}$ on H as $S_{D\Lambda\Gamma C}x = \sum_{j\in J} D\Lambda_j^*\Gamma_j Cx$, for all $x \in H$.

It can easily be verified that $S_{D\Lambda\Gamma C}$ is a well defined, bounded, linear operator and $\|S_{D\Lambda\Gamma C}\| = \sqrt{B_\Lambda B_\Gamma}$, where B_Λ and B_Γ are the g-Bessel bounds of $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$, respectively. For $C = D = I_H$, we have $S_{\Lambda\Gamma}x = \sum_{j\in J}\Lambda_j^*\Gamma_j x$. Moving forward, we look for the conditions under which the sum of two (C,D)-controlled g-frames is again a (C,D)-controlled g-frame.

Theorem 3.8. *Let $C, D \in GL^+(H)$, $\Lambda = \{\Lambda_j\}_{j\in J}$ and $\Gamma = \{\Gamma_j\}_{j\in J}$ be (C,D)-controlled g-frames for H with respect to $\{H\}_{j\in J}$. If $S_{\Lambda\Gamma}$ and $S_{\Gamma\Lambda}$ commute with C, D and $S_{\Lambda\Gamma} + S_{\Gamma\Lambda} \in GL^+(H)$. Then $\{\Lambda_j + \Gamma_j\}_{j\in J}$ is a (C,D)-controlled g-frame.*

Proof. As Λ and Γ are (C,D)-controlled g-frames, there exists $0 < A_1 \leq B_1 < \infty$ and $0 < A_2 \leq B_2 < \infty$ such that for all $x \in H$, we have

$$A_1\|x\|^2 \leq \sum_{j\in J}\langle \Lambda_j Cx, \Lambda_j Dx\rangle \leq B_1\|x\|^2$$

and

$$A_2\|x\|^2 \leq \sum_{j\in J}\langle \Gamma_j Cx, \Gamma_j Dx\rangle \leq B_2\|x\|^2.$$

Note that $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are g-Bessel sequences and it can easily be verified that $S_{D\Lambda\Gamma C}+S_{C\Gamma\Lambda D}$ is a positive operator. Therefore, by Proposition 2.9, there exists $m > 0$ such that for all $x \in H$,

$$\langle S_{D\Lambda\Gamma C} + S_{C\Gamma\Lambda D}x, x\rangle \geq m\|x\|^2.$$

Thus,

$$\begin{aligned}\sum_{j\in J}\langle(\Lambda_j+\Gamma_j)Cx,(\Lambda_j+\Gamma_j)Dx\rangle &= \sum_{j\in J}\langle \Lambda_j Cx, \Lambda_j Dx\rangle + \sum_{j\in J}\langle \Gamma_j Cx, \Gamma_j Dx\rangle \\ &+ \langle S_{\Lambda C\Gamma D} + S_{\Gamma D\Lambda C}x, x\rangle \\ &\geq (A_1 + A_2 + m)\|x\|^2.\end{aligned}$$

Moreover,

$$\sum_{j\in J}\langle(\Lambda_j+\Gamma_j)Cx,(\Lambda_j+\Gamma_j)Dx\rangle \leq (B_1 + B_2 + 2\sqrt{B_\Lambda B_\Gamma})\|x\|^2,$$

where B_Λ and B_Γ are the g-Bessel bounds of $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ respectively. Hence, the result follows. □

4 Controlled g-dual frames of generalized controlled frames

S.M.Ramezani and A. Nazari [22] introduced the concept of g-dual of g-frames in Hilbert spaces. In this section, we define the g-dual of controlled g-frames and establish relationships between controlled g-frames and their g-duals.

Definition 4.1. [18] Let $C, D \in GL^+(H)$. A (D,D)-controlled g-Bessel sequence $\{\Gamma_j\}_{j\in J}$ is called a (C,D)-controlled dual g-frame of a (C,C)-controlled g-frame $\{\Lambda_j\}_{j\in J}$ for H with respect to $\{H_j\}_{j\in J}$ if for all $x \in H$,

$$x = \sum_{j\in J} C\Lambda_j^*\Gamma_j Dx.$$

Note that every (C,C)-controlled g-frame $\{\Lambda_j\}_{j\in J}$ has atleast one (C,D)-controlled dual g-frame. Indeed, $\{\Gamma_j\}_{j\in J} = \{\Lambda_j CS^{-1}_{(C,C)}D^{-1}\}_{j\in J}$, is a (D,D)-controlled g-Bessel sequence and $x = \sum_{j\in J} C\Lambda_j^*\Gamma_j Dx$, for all $x \in H$. Thus, $\{\Gamma_j\}_{j\in J}$ is a (C,D)-controlled dual g-frame of $\{\Lambda_j\}_{j\in J}$.

Now, we define (C,D)-controlled g-dual of a (C,C)-controlled g-frame $\{\Lambda_j\}_{j\in J}$.

Definition 4.2. Let $C, D \in GL^+(H)$ and $\{\Lambda_j\}_{j\in J}$ a (C,C)-controlled g-frame. A (D,D)-controlled g-Bessel sequence $\{\Gamma_j\}_{j\in J}$ is is said to be a (C,D)-controlled g-dual of $\{\Lambda_j\}_{j\in J}$, if there exists an operator $\Xi \in GL(H)$ such that for all $x \in H$,

$$x = \sum_{j\in J} C\Lambda_j^*\Gamma_j D(\Xi x).$$

When $\Xi = I_H$, then $\{\Gamma_j\}_{j\in J}$ is a (C,D)-controlled dual g-frame of $\{\Lambda_j\}_{j\in J}$. Further, if $\Xi = C = D = I_H$, then $\{\Gamma_j\}_{j\in J}$ is an g-dual of g-frame of $\{\Lambda_j\}_{j\in J}$. Hence, the controlled g-dual of a controlled g-frame is the generalization of a dual g-frame.

Now, we prove some equivalent conditions under which the g-Bessel sequences $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ becomes the controlled g-dual frames.

Proposition 4.3. *Let $C, D \in GL^+(H)$. Suppose $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are g-Bessel sequences in H with respect to $\{H_j\}_{j\in J}$. Then the following are equivalent:*

(i) there exist $\Xi \in GL(H)$ such that for all $x \in H$,

$$x = \sum_{j \in J} C\Lambda_j^* \Gamma_j D(\Xi x);$$

(ii) there exist $\Xi \in GL(H)$ such that for all $x \in H$,

$$x = \sum_{j \in J} D\Gamma_j^* \Lambda_j C(\Xi^* x);$$

(iii) there exist $\Xi \in GL(H)$ such that for all $x \in H$,

$$\langle x, y \rangle = \sum_{j \in J} \langle \Gamma_j D(\Xi x), \Lambda_j C y \rangle;$$

(iv) there exist $\Xi \in GL(H)$ such that for all $x \in H$,

$$\langle x, y \rangle = \sum_{j \in J} \langle \Lambda_j C(\Xi^* x), \Gamma_j D y \rangle.$$

Proof. Let (i) hold. Then for any $x \in H$, we have $x = \sum_{j \in J} C\Lambda_j^* \Gamma_j D(\Xi x)$, where Ξ is an invertible operator. Also, there exists some $y \in H$ such that $x = \Xi y$. Therefore, $y = \sum_{j \in J} C\Lambda_j^* \Gamma_j D(\Xi y) = \sum_{j \in J} C\Lambda_j^* \Gamma_j D x$. Thus, $x = \Xi y = \sum_{j \in J} \Xi C\Lambda_j^* \Gamma_j D x$. Since $\{\Gamma_j D\}_{j \in J}$ and $\{\Lambda_j C \Xi^*\}_{j \in J}$ are g-Bessel sequences, it follows that, $x = \sum_{j \in J} (\Lambda_j C \Xi^*)^* \Gamma_j D x = \sum_{j \in J} (\Gamma_j D)^* \Lambda_j C(\Xi^* x)$. Thus, $x = \sum_{j \in J} D\Gamma_j^* \Lambda_j C(\Xi^* x)$, therefore, (ii) holds. Using a similar argument, we can prove that (ii) implies (i).
Next, we prove the equivalence of conditions (i) and (iii). As $x = \sum_{j \in J} C\Lambda_j^* \Gamma_j D(\Xi x)$. Therefore, for any $y \in H$,

$$\langle x, y \rangle = \sum_{j \in J} \langle C\Lambda_j^* \Gamma_j D(\Xi x), y \rangle = \sum_{j \in J} \langle \Gamma_j D(\Xi x), \Lambda_j C y \rangle.$$

Fix $x \in H$, then for every $y \in H$,

$$\begin{aligned} \langle x - \sum_{j \in J} C\Lambda_j^* \Gamma_j D(\Xi x), y \rangle &= \langle x, y \rangle - \sum_{j \in J} \langle \Gamma_j D(\Xi x), \Lambda_j C y \rangle \\ &= \langle x, y \rangle - \langle x, y \rangle = 0. \end{aligned}$$

This is true for every $y \in H$ and hence, $x = \sum_{j \in J} C\Lambda_j^* \Gamma_j D(\Xi x)$.
Following arguments similar to the ones used above, we can prove (ii) and (iv) are equivalent, which concludes the proof. □

Corollary 4.4. *Let $C, D \in GL^+(H)$. Suppose that $\{\Lambda_j\}_{j \in J}$ and $\{\Gamma_j\}_{j \in J}$ are two g-Bessel sequences of H with respect to $\{H_j\}_{j \in J}$ such that for all $x \in H, x = \sum_{j \in J} C\Lambda_j^* \Gamma_j D(\Xi x)$, for some $\Xi \in GL(H)$. Then, $\{\Lambda_j\}_{j \in J}$ and $\{\Gamma_j\}_{j \in J}$ are (C, C)-controlled and (D, D)-controlled g-frames for H with respect to $\{H_j\}_{j \in J}$ and $x = \sum_{j \in J} D\Gamma_j^* \Lambda_j C(\Xi^* x)$, for all $x \in H$.*

Proof. Let B_Λ and B_Γ be the Bessel bound of the g-Bessel sequences $\{\Lambda_j\}_{j \in J}$ and $\{\Gamma_j\}_{j \in J}$, respectively. Then for any $x \in H$,

$$\begin{aligned} \|x\|^4 &= |\langle \sum_{j \in J} C\Lambda_j^* \Gamma_j D(\Xi x), x \rangle|^2 \\ &\leq \sum_{j \in J} \|\Gamma_j D(\Xi x)\|^2 \sum_{j \in J} \|\Lambda_j C x\|^2 \\ &\leq B_\Gamma \|\Xi\|^2 \|D\|^2 \|x\|^2 \sum_{j \in J} \langle \Lambda_j C x, \Lambda_j C x \rangle. \end{aligned}$$

Thus, for all $x \in H$,

$$\frac{1}{B_\Gamma \|C\|^2 \|D\|^2}\|x\|^2 \leq \sum_{j\in J} \langle \Lambda_j Cx, \Lambda_j Cx\rangle \leq B_\Lambda \|C\|^2 \|x\|^2.$$

Therefore, $\{\Lambda_j\}_{j\in J}$ is a (C, C)-controlled g-frame. Similarly, we can prove that $\{\Gamma_j\}_{j\in J}$ is also a (D, D)-controlled g-frame.
As $x = \sum_{j\in J} C\Lambda_j^*\Gamma_j D(\Xi x)$, therefore by Proposition 4.3, it follows that $x = \sum_{j\in J} D\Gamma_j^*\Lambda_j C(\Xi^* x)$. □

Remark 4.5. In view of Corollary 4.4, it is clear that $\{\Gamma_j\}_{j\in J}$ is a (C, D)-controlled g-dual of $\{\Lambda_j\}_{j\in J}$ and $\{\Lambda_j\}_{j\in J}$ is a (D, C)-controlled g-dual of $\{\Gamma_j\}_{j\in J}$.

Next, we generate new (C, D)-controlled g-dual of (C, C)-controlled g-frame $\{\Lambda_j\}_{j\in J}$ from a given (C, D)-controlled g-dual frame.

Theorem 4.6. *Let $C, D \in GL^+(H)$ and $\{\Gamma_j\}_{j\in J}$ be a (C, D)-controlled g-dual of (C, C)-controlled g-frame $\{\Lambda_j\}_{j\in J}$ for H with respect to $\{H_j\}_{j\in J}$ with $\Xi \in GL(H)$ and λ be any complex number. Then the sequence $\{\psi_j\}_{j\in J}$ given by $\psi_j = \lambda\Gamma_j + (1-\lambda)\Lambda_j DS_{(C,D)}^{-1}\Xi^{-1}D^{-1}$ is a (C, D)-controlled g-dual of $\{\Lambda_j\}_{j\in J}$ with the invertible operator Ξ.*

Proof. For any $x \in H$,

$$\begin{aligned}
\sum_{j\in J} C\Lambda_j^*\psi_j D(\Xi x) &= \lambda \sum_{j\in J} C\Lambda_j^*\Gamma_j D(\Xi x) \\
&+ (1-\lambda)\sum_{j\in J} C\Lambda_j^*\Lambda_j DS_{(C,D)}^{-1}\Xi^{-1}D^{-1}D(\Xi x) \\
&= \lambda x + (1-\lambda)S_{(C,D)}S_{(C,D)}^{-1}x \\
&= \lambda x + (1-\lambda)x \\
&= x.
\end{aligned}$$

Thus, $\{\psi_j\}_{j\in J}$ is a (C, D) controlled g-dual of $\{\Lambda_j\}_{j\in J}$ with $\Xi \in GL(H)$. □

In the next result, we obtain conditions under which a combination of two (C, D)-controlled g-duals of a (C, C)-controlled g-frame $\{\Lambda_j\}_{j\in J}$ will again be a (C, D)-controlled g-dual of $\{\Lambda_j\}_{j\in J}$.

Theorem 4.7. *Let $C, D \in GL^+(H)$. Suppose that $\{\Gamma_j\}_{j\in J}$ and $\{\Theta_j\}_{j\in J}$ be (C, D)-controlled g-dual of (C, C)-controlled g-frame $\{\Lambda_j\}_{j\in J}$ for H with respect to $\{H_j\}_{j\in J}$ with $\Xi_1, \Xi_2 \in GL(H)$, respectively. Then for any complex number λ, the sequence $\{\psi_j\}_{j\in J}$ given by*

$$\psi_j = \lambda\Gamma_j\Xi_1 + (1-\lambda)\Theta_j\Xi_2$$

is a (C, D) controlled g-dual frame of $\{\Lambda_j\}_{j\in J}$ if $\Xi_1 D = D\Xi_1$ and $D\Xi_2 = \Xi_2 D$.

Proof. For any $x \in H$,

$$\begin{aligned}
\sum_{j\in J} C\Lambda_j^*\psi_j Dx &= \lambda\sum_{j\in J} C\Lambda_j^*\Gamma_j\Xi_1 Dx + (1-\lambda)\sum_{j\in J} C\Lambda_j^*\Theta_j\Xi_2 Dx \\
&= \lambda x + (1-\lambda)x = x.
\end{aligned}$$

Therefore $\{\psi_j\}_{j\in J}$ is a (C, D) controlled g-dual frame of $\{\Lambda_j\}_{j\in J}$. □

Now, we present a necessary condition under which the sum of two (C, D)-controlled g-dual frames of a (C, C)-controlled frame $\{\Lambda_j\}_{j\in J}$ turns out to be a (C, D)-controlled g-dual frame.

Theorem 4.8. *Let $C, D \in GL^+(H)$. Suppose that $\{\Gamma_j\}_{j\in J}$ and $\{\Theta_j\}_{j\in J}$ are two (C, D)-controlled g-dual of (C, C)-controlled g-frame $\{\Lambda_j\}_{j\in J}$ with respect to operator $\Xi_1, \Xi_2 \in GL(H)$, respectively. If $\Xi = \Xi_1^{-1} + \Xi_2^{-1}$ is invertible. Then, $\{\Gamma_j + \Theta_j\}_{j\in J}$ is a (C, D)-controlled g-dual frame of $\{\Lambda_j\}_{j\in J}$.*

Proof. For any $x \in H$,

$$\begin{aligned}\sum_{j\in J} C\Lambda_j^*(\Gamma_j+\Theta_j)D\Xi^{-1}x &= \sum_{j\in J} C\Lambda_j^*\Gamma_j D\Xi_1\Xi_1^{-1}\Xi^{-1}x \\ &+ \sum_{j\in J} C\Lambda_j^*\Theta_j D\Xi_2\Xi_2^{-1}\Xi^{-1}x \\ &= (\Xi_1^{-1}+\Xi_2^{-1})\Xi^{-1}x \\ &= x.\end{aligned}$$

□

Theorem 4.9. *Let $C, D, T \in GL^+(H)$. Then $\{\Gamma_j\}_{j\in J}$ is a (C,D)-controlled g-dual of (C,C)-controlled g-frame $\{\Lambda_j\}_{j\in J}$ if and only if $\{\Gamma_j\}_{j\in J}$ is a (TC, DT^{-1})-controlled g-dual of $\{\Lambda_j\}_{j\in J}$.*

Proof. As $\{\Lambda_j\}_{j\in J}$ is a (C,C)- controlled g-frame and $T \in GL^+(H)$, therefore, $\{\Lambda_j\}_{j\in J}$ is also a (TC,TC)-controlled g-frame. If $\{\Gamma_j\}_{j\in J}$ is a (C,D)-controlled g-dual of (C,C)-controlled g-frame $\{\Lambda_j\}_{j\in J}$, then $\{\Gamma_j\}_{j\in J}$ is a (D,D)-controlled g-Bessel sequence and there exists $\Xi \in GL(H)$ such that for all $x \in H, x = \sum_{j\in J} C\Lambda_j^*\Gamma_j D\Xi x$. Thus,

$$x = TT^{-1}x = T\sum_{j\in J} C\Lambda_j^*\Gamma_j D\Xi(T^{-1}x) = \sum_{j\in J} TC\Lambda_j^*\Gamma_j DT^{-1}(\Xi x).$$

Also, it can be readily seen that $\{\Gamma_j\}_{j\in J}$ is a (DT^{-1}, DT^{-1})-controlled g-Bessel sequence and hence $\{\Gamma_j\}_{j\in J}$is a (TC, DT^{-1})-controlled g-dual of $\{\Lambda_j\}_{j\in J}$.
Conversely, let $\{\Gamma_j\}_{j\in J}$ be a (TC, DT^{-1})-controlled g-dual of $\{\Lambda_j\}_{j\in J}$, where $\Xi \in GL(H)$. Then, $\{\Gamma_j\}_{j\in J}$ is a (DT^{-1}, DT^{-1})-controlled g-Bessel sequence and for any $x \in H$,

$$\begin{aligned} x &= T^{-1}Tx \\ &= T^{-1}\sum_{j\in J} TC\Lambda_j^*\Gamma_j DT^{-1}\Xi Tx \\ &= \sum_{j\in J} C\Lambda_j^*\Gamma_j DT^{-1}\Xi Tx \\ &= \sum_{j\in J} C\Lambda_j^*\Gamma_j DKx,\end{aligned}$$

where $K = T^{-1}\Xi T$ is invertible. It can be easily verified that $\{\Gamma_j\}_{j\in J}$ is a (D,D)-controlled g-Bessel sequence and therefore $\{\Gamma_j\}_{j\in J}$ is a (C,D)-controlled g-dual of $\{\Lambda_j\}_{j\in J}$. □

Next example illustrates the existence of $\{\Gamma_j\}_{j\in J}$ and $\{\Lambda_j\}_{j\in J}$ that satisfy the above conditions.

Example 4.10. Let $H = \mathbf{R}^3$ and $\{e_j\}_{j\in J}$ be an orthonormal basis of H. Let $H_1 = span\{e_2, e_3\}$, $H_2 = span\{e_1, e_3\}$ and $H_3 = span\{e_1, e_2\}$. For $1 \le j \le 3$, define $\Lambda_j : H \to H_j$ as $\Lambda_1 x = \langle x, e_2\rangle e_3$; $\Lambda_2 x = \langle x, e_3\rangle e_1$; $\Lambda_3 x = \langle x, e_1\rangle e_2$; and $C, D, T : H \to H$ as $Cx = \frac{1}{2}\langle x, e_1\rangle e_1 + \frac{1}{3}\langle x, e_2\rangle e_2 + \frac{1}{4}\langle x, e_3\rangle e_3$, $Dx = 2\langle x, e_1\rangle e_1 + 3\langle x, e_2\rangle e_2 + \frac{4}{5}\langle x, e_3\rangle e_3$ and $Tx = 5\langle x, e_1\rangle e_1 + 6\langle x, e_2\rangle e_2 + 4\langle x, e_3\rangle e_3$. Then $C, D, T \in GL^+(H)$. Also, $\frac{1}{4}\|x\|^2 \le \sum_{j\in J}\|\Lambda_j Cx\|^2 \le \|x\|^2$ and $\|x\|^2 \le \sum_{j\in J}\|\Lambda_j TCx\|^2 \le 5\|x\|^2$. So, $\{\Lambda_j\}_{j\in J}$ is a (C,C)-controlled g-frame and (TC,TC)-controlled g-frame. Take $\{\Gamma_j\}_{j\in J} = \{\Lambda_j\}_{j\in J}$, then $\sum_{j\in J}\|\Gamma_j Dx\|^2 \le 9\|x\|^2$ and $\sum_{j\in J}\|\Gamma_j DT^{-1}x\|^2 \le \|x\|^2$. So, $\{\Gamma_j\}_{j\in J}$ is both a (D,D)-controlled and (DT^{-1}, DT^{-1})-controlled g-Bessel sequence. Define $\Xi : H \to H$ as $\Xi(x) = \langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2 + 5\langle x, e_3\rangle e_3$. Then $\Xi \in GL(H)$ and $x = \sum_{j\in J} C\Lambda_j^*\Gamma_j D(\Xi x)$. Thus, $\{\Gamma_j\}_{j\in J}$ is a (C,D)-controlled g-dual of (C,C)-controlled g-frame $\{\Lambda_j\}_{j\in J}$. Also, $x = \sum_{j\in J} TC\Lambda_j^*\Gamma_j DT^{-1}(\Xi x)$. Therefore, $\{\Gamma_j\}_{j\in J}$ is a (TC, DT^{-1})-controlled g-dual of $\{\Lambda_j\}_{j\in J}$.

We conclude this paper by proving that (C,D)-controlled g-duality property of g-frames is preserved under invertible operators.

Theorem 4.11. *Let $C, D, U, V \in GL^+(H)$. Suppose that $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are (C,C) and (D,D)-controlled g-frames for H with respect to $\{H_j\}_{j\in J}$. Then $\{\Gamma_j\}_{j\in J}$ is a (C,D)-controlled g-dual of $\{\Lambda_j\}_{j\in J}$ if and only if $\{\Gamma_j U\}_{j\in J}$ is a (C,D)-controlled g-dual of $\{\Lambda_j V\}_{j\in J}$.*

Proof. Let $\{\Gamma_j\}_{j\in J}$ be a (C,D)-controlled g-dual of $\{\Lambda_j\}_{j\in J}$, therefore there exists $\Xi \in GL(H)$ such that for all $x \in H$, $x = \sum_{j\in J} C\Lambda_j^*\Gamma_j D\Xi x$. Thus,

$$\begin{aligned} x &= \sum_{j\in J} VC\Lambda_j^*\Gamma_j D\Xi V^{-1}x \\ &= \sum_{j\in J} C(\Lambda_j V)^*\Gamma_j UU^{-1}D\Xi V^{-1}x \\ &= \sum_{j\in J} C(\Lambda_j V)^*(\Gamma_j U)(U^{-1}D\Xi V^{-1})x \\ &= \sum_{j\in J} C(\Lambda_j V)^*(\Gamma_j U)D\Xi_1 x, \end{aligned}$$

where $\Xi_1 = (U^{-1}\Xi V^{-1}) \in GL(H)$.
Conversely, let $\{\Gamma_j U\}_{j\in J}$ be a (C,D)-controlled g-dual of $\{\Lambda_j V\}_{j\in J}$. Then there exists $\Xi \in GL(H)$ such that for all $x \in H$,

$$x = \sum_{j\in J} C(\Lambda_j V)^*(\Gamma_j U)D\Xi x.$$

Thus,

$$\begin{aligned} x &= V^{-1}\sum_{j\in J} C(\Lambda_j V)^*(\Gamma_j U)D\Xi V x \\ &= \sum_{j\in J} C\Lambda_j^*\Gamma_j DU\Xi V x \\ &= \sum_{j\in J} C\Lambda_j\Gamma_j D\Xi_2 x, \end{aligned}$$

where $\Xi_2 = U\Xi V \in GL(H)$, which completes the proof. □

5 Conclusion remarks

The study of g-duals has great significance in the area of frame theory. The aim of this paper is to obtain (C,D)-dual of a (C,C)-controlled g-frames which are the generalization of the dual of a g-frame. Also, some new properties of controlled g-frames and their g-duals have been discussed. Therefore, the results of this work can be a subject of future research work.

References

[1] P. Balazs, Peter; J. P. Antoine and A. Gryboś, *Weighted and controlled frames: mutual relationship and first numerical properties*, International Journal of Wavelets, Multiresolution and Information Processing, **8, no. 1**, 109–132, (2010).

[2] A. Bhandari and S. Mukherjee, *Atomic subspaces for operators*, Indian Journal of Pure and Applied Mathematics, **51** , 1039–1052, (2020).

[3] A. Bhandari and S. Mukherjee, *Perturbations on K-fusion frames*, Journal of Applied Analysis, **27, no. 22**, 175–185, (2021).

[4] P. G. Casazza and G. Kutyniok, *Frames of subspaces*, Contemporary Mathematics, **345**, 87–114, (2004).

[5] O. Christensen, *An introduction to frames and Riesz bases*, Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, (2003).

[6] I. Daubechies, A. Grossmann, and Y. Meyer, *Painless nonorthogonal expansions*, Journal of Mathematical Physics, **27, no. 5**, 1271–1283, (1986).

[7] M. A. Dehghan and M. A. Hasankhani Fard, *G-dual frames in Hilbert spaces*, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, **75, no. 1**, 129–140, (2013).

[8] R. J. Duffin and A. C. Schaeffer, *A class of non-harmonic Fourier series*, Transactions of American Mathematical Society, **72** , 341–366, (1952).

[9] L. Găvruţa, *New results on frames for operators*, Analele University Oradea Fasc. Mathematica, **19, no. 2**, 55–61, (2012).

[10] L. Găvruţa, *Frames for operators*, Applied Computational Harmonic Analysis, **32, no. 1**, 139–144, (2012).

[11] S. Jahan, V. Kumar and C. Shekhar, *Cone associated with frames in Banach spaces*, Palestine Journal of Mathematics, **7, no. 2**, 641–649, (2018).

[12] S. K. Kaushik and V. Kumar, *Frames of subspaces for Banach spaces*, International Journal of Wavelets, Multiresolution and Information Processing, **8, no. 2**, 243–252, (2010).

[13] S. K. Kaushik and V. Kumar, *A note on fusion Banach frames*, Archivum Mathematicum (Brno), **46, no. 3**, 203–209, (2010).

[14] S. K. Kaushik and V. Kumar, *On fusion frames in Banach spaces*, Georgian Mathematical Journal, **18, no. 1**, 121–130, (2011).

[15] A. Khosvari and F. Takhteh, *Duality principles in g-Frames*, Palestine Journal Of Mathematics, **6, no. 2**, 403–411,(2017).

[16] V. Kumar and S. Malhotra, *Some Properties of Generalized Frames for Operators*, Poincare Journal of Analysis and Applications, **10, no. 3**, Special issue, 115–126, (2023).

[17] V. Kumar, S. Malhotra and N. Khanna, *Scalability of Generalized Frames for Operators*, Journal of Function Spaces, **2024**, 9pp., (2024).

[18] H. M. Liu, Y. L. Fu and Y. Tian, *Controlled g-frames and dual g-frames in Hilbert spaces*, Journal of Inequalities and Applications, Paper No. 64, 2023, 14 pp, (2023).

[19] H. K. Pathak and Mayur Puri Goswami, *Some Results on F-Fusion Banach Frame*, Palestine Journal of Mathematics ,**8, no. 1**, 285–293, (2019).

[20] A. Rahimi and A. Fereydooni, *Controlled g-frames and their g-multipliers in Hilbert spaces*, Analele Ştiinţifice University "Ovidius” Constanţa Seria Matematica, **21, no. 2**, 223–236, (2013).

[21] S. Ramesan and K. T. Ravindran, *Scalability and K-frames*, Palestine Journal of Mathematics, **12, no. 1**, 493–500, (2023).

[22] S. M. Ramezani and A. Nazari, *g-orthonormal bases, g-Riesz bases and g-dual of g-frames*,UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, **78, no. 1**, 91–98, (2016).

[23] S.M. Ramezani, *Controlled g-dual frames and their approximates in Hilbert spaces*, Hacettepe Journal of Mathematics and Statistics, **51, no. 2**, 421–429, (2022)

[24] W. Sun, *G-frames and g-Riesz bases*. Journal of Mathematical Analysis and Applications, **322, no. 1**, 437–452, (2006).

[25] W. Sun, *Stability of g-frames*, Journal of Mathematical Analysis and Applications, **326, no. 2**, 858–868, (2007).

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