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Combinatorial Proof of Mock Theta Conjectures

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Abstract In 1988, D. Hickerson gives a proof of the mock theta conjectures using Hecke-type identities discovered by G.E. Andrews [11]. In 2008, A. Folsom provide a short proof for the same by realizing each side of the identities as the holomorphic projection of a harmonic weak Maass form, in respons to a remark made by Bringmann - Ono - Rhoades [10]. Both of these approaches involve proving the identities individually, relying on work of Andrews–Garvan. Recently, N. Andersen give a unified proof of the mock theta conjectures by realizing them as an equality between two nonholomorphic vector-valued modular forms which transform according to the Weil representation. Here we give a combinatorial proof for the mock theta conjectures in response of a open problem stated by M.P.Chaudhary [7].

1 Introduction

In his last letter to Hardy, dated three months before his death in early 1920, (see [7, Pages 33-34], [13, Pages 354-355] and [15, Pages 127-131]), Ramanujan gave a list of 17 functions which he called "mock theta functions". He separated these functions into three groups, which were described as four of third order, ten of fifth order, and three of seventh order. Further, the fifth order mock theta functions he divided into two groups. The mock theta functions are functions of a complex variable q, defined by q-series convergent for $|q| < 1$. He stated that they have certain asymptotic properties as q approaches a root of unity, similar to the properties of theta functions, but he conjectured that they are not, in fact, theta functions. He also stated some identities relating some of the functions to each other.

We start by defining some q-products and q-series identities after we give $R_{(1,0)}(0,q)$, $R_{(2,0)}(3,q)$ and $R_{(2,1)}(3,q)$ in q-series and q-products forms using two different methods , then we give simpler identities for showing the conjectures of mock theta functions.

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ the set of positive integers, the set of integers and the set of complex numbers respectively. We also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}.$$

The q-shifted factorial $(a;q)_n$ is defined (for $|q| < 1$) by

$$(a;q)_n := \begin{cases} 1 & (n = 0), \\ \prod_{k=0}^{n-1} (1 - aq^k) & (n \in \mathbb{N}), \end{cases} \tag{1}$$

where $a, q \in \mathbb{C}$ and it is assumed *tacitly* that $a \neq q^{-m} \quad (m \in \mathbb{N}_0)$. We also write

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n) = \prod_{n=1}^{\infty}(1-aq^{n-1}), \qquad (a,q\in\mathbb{C};\ |q|<1). \tag{2}$$

It should be noted that, when $a \neq 0$ and $|q| \geqq 1$, the infinite product in the equation (2) diverges. So, whenever $(a;q)_\infty$ is involved in a given formula, the constraint $|q|<1$ will be *tacitly* assumed to be satisfied. The following notations are also frequently used in our investigation:

$$(a_1,a_2,a_3\ldots a_k;q)_n = (a_1;q)_n(a_2;q)_n(a_3;q)_n\ldots(a_k;q)_n$$

and

$$(a_1,a_2,a_3\ldots a_k;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty\ldots(a_k;q)_\infty.$$

Ramanujan (see [14, 15, 6, page 13]) defined the general theta function $\mathfrak{f}(a,b)$ as follows:

$$\mathfrak{f}(a,b) = 1+\sum_{n=1}^{\infty}(ab)^{\frac{n(n-1)}{2}}\,(a^n+b^n) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}}\,b^{\frac{n(n-1)}{2}} = \mathfrak{f}(b,a), \qquad (|ab|<1), \tag{3}$$

where a and b are two complex numbers. The three most important special cases of $\mathfrak{f}(a,b)$ are defined as:

$$\phi(q) = \mathfrak{f}(q,q) = 1+2\sum_{n=1}^{\infty}q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{4}$$

$$\psi(q) = \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{5}$$

and

$$f(-q) = \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{+\infty}(-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty. \tag{6}$$

The last equality (6) is known as *Euler's Pentagonal Number Theorem*. Remarkably, the following q-series identity:

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty} = \frac{1}{\chi(-q)}.$$

provides the analytic equivalent form of Euler's famous theorem. Ramanujan also defined the following function

$$\chi(q) = (-q;q^2)_\infty.$$

We also recall the Rogers-Ramanujan continued fraction $R(q)$ given by

$$R(q) := q^{\frac{1}{5}}\frac{H(q)}{G(q)} = q^{\frac{1}{5}}\frac{\mathfrak{f}(-q,-q^4)}{\mathfrak{f}(-q^2,-q^3)} = q^{\frac{1}{5}}\frac{(q;q^5)_\infty\,(q^4;q^5)_\infty}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty}$$
$$= \frac{q^{\frac{1}{5}}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+} \qquad (|q|<1). \tag{7}$$

Here $G(q)$ and $H(q)$, which are associated with the widely-investigated Roger-Ramanujan identities, are defined as follows:

$$G(q) := \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{f(-q^5)}{\mathfrak{f}(-q,-q^4)}$$
$$= \frac{1}{(q;q^5)_\infty\,(q^4;q^5)_\infty} = \frac{(q^2;q^5)_\infty\,(q^3;q^5)_\infty\,(q^5;q^5)_\infty}{(q;q)_\infty} \tag{8}$$

and

$$H(q) := \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n} = \frac{f(-q^5)}{\mathfrak{f}(-q^2,-q^3)} = \frac{1}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty}$$
$$= \frac{(q;q^5)_\infty\,(q^4;q^5)_\infty\,(q^5;q^5)_\infty}{(q;q)_\infty}, \tag{9}$$

where the functions $\mathfrak{f}(a,b)$ and $f(-q)$ are given by the equations (3) and (6), respectively.

Remark. In (8), the left side can be interpreted as the generating series of partitions of n such that two adjacent parts differ by at least 2 and the right side is the the generating function of partitions of n into parts congruent to 1 or 4 modulo 5. In (9), the left-hand side is the generating series of partitions of n such that two adjacent parts differ by at least 2 and the smallest part is at minus 2 and the right side is the generating function of partitions such that each part is congruent to 2 or 3 modulo 5. Then, the number of partitions of n such that two adjacent parts differ by at least 2 is equal to the number of partitions of n such that each part is congruent to 1 or 4 modulo 5 and the number of partitions of n such that two adjacent parts differ by at least 2 and the smallest part is at least 2 is equal to the number of partitions of n such that each part is congruent to 2 or 3 modulo 5.

2 Fifth Order Mock Theta Functions

Watson [17] (see also [7]) define the following two identies

$$\chi_0(q) = \sum_{n\geq 0}^{\infty} \frac{q^n}{(q^{n+1};q)_n} \tag{10}$$

and

$$\chi_1(q) = \sum_{n\geq 0}^{\infty} \frac{q^n}{(q^{n+1};q)_{n+1}}, \tag{11}$$

where all symbols and notations are having their usual meaning.
Combinatorial interpretations: $q\chi_1(q)$ is the generating function for partitions in which no part is as large as twice the smallest part and $\chi_0(q)$ is the generating function for partitions with unique smallest part and the largest part at most twice the smallest part.

Zwegers [16] (see also [7]) has found the following two identities for fifth-order mock theta functions $\chi_0(q)$ and $\chi_1(q)$ as follows:

$$\chi_0(q) = 2 - \frac{1}{(q)_\infty^2}\left(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\right)(-1)^{k+l+m} q^{\frac{1}{2}k^2+\frac{1}{2}l^2+\frac{1}{2}m^2+2km+2kl+2lm+1/2(k+l+m)} \tag{12}$$

and

$$\chi_1(q) = \frac{1}{(q)_\infty^2}\left(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\right)(-1)^{k+l+m} q^{\frac{1}{2}k^2+\frac{1}{2}l^2+\frac{1}{2}m^2+2km+2kl+2lm+1/2(k+l+m)}. \tag{13}$$

The mock theta conjectures related the functions $\chi_0(q)$ and $\chi_1(q)$ to the differences of rank generating functions [10, page 2] (see also [7])

$$R_{b,c}(d;q) = \sum_{n\geqslant 0} \left[N(b,5,5n+d) - N(c,5,5n+d)\right] q^n. \tag{14}$$

In equation (14), $N(b,t,r)$ denotes the number of partitions of r with rank congruent to b mod t and in equation (23) $N(m,n)$ denote the number of partitions of n with rank m. The rank of a given partition is defined by Dyson [12] as the number of parts of the partition subtracted from the largest part of the partition. For example, the partition $1+1+1+1+2+4$ of 10 has rank equal to $4-6=-2$.

We note the number of partitions of n by $p(n)$. We have

$$p(n) = \sum_{m=-\infty}^{+\infty} N(m,n).$$

The ten fifth order mock theta functions founded by Ramanujan in his lost notebook divided into two groups, and each groups has five mock theta functions. *The mock theta conjectures* are ten identities each involving one of the fifth-order mock theta functions. Andrews and Garvan [3] show that these identities in Ramanujan's first group of five 5th order functions are equivalent to each other; they call these "First Mock Theta Conjecture". Similarly, the five 5th order identities for the second group are equivalent; they call these "Second Mock Theta Conjecture". They present combinatorial interpretations of these conjectures in terms of the ranks of partitions.

Folsom [10, Page 4144] (see also [7]) has stated as

$$\chi_0(q) - 1 = R_{1,0}(0;q) \tag{15}$$

and

$$\chi_1(q) = R_{2,1}(3;q) + R_{2,0}(3;q). \tag{16}$$

Let recall Ramanujan's celebrated congruences for the partition function $p(n)$,

$$p(5n+4) \equiv 0 \pmod{5}, \tag{17}$$

$$p(7n+5) \equiv 0 \pmod{7} \tag{18}$$

and

$$p(11n+6) \equiv 0 \pmod{11}. \tag{19}$$

In attempting to find combinatorial interpretations for (17)–(19), Dyson conjectured that

$$N(k,5,5n+4) = \frac{p(5n+4)}{5},\ 0 \leq k \leq 4 \tag{20}$$

and

$$N(k,7,7n+5) = \frac{p(7n+5)}{7},\ 0 \leq k \leq 6. \tag{21}$$

Thus, if (20) and (21) are true the partitions counted by $p(5n+4)$ and $p(7n+5)$ fall into five and seven equinumerous classes respectively. Hence providing a partial answer to Dyson's query. Furthermore, he conjectured that the generating function for $N(m,n)$ is given by

$$\sum_{m=-\infty}^{+\infty}\sum_{n=0}^{\infty} N(m,n)z^m q^n = \sum_{n=0}^{\infty}\frac{q^{n^2}}{(zq;q)_n(z^{-1}q;q)_n},\ |q|<1,\ \ |q|<|z|<\frac{1}{|q|}. \tag{22}$$

We note that

$$N(m,q,n) = \sum_{r=-\infty}^{\infty} N(m+rq,n). \tag{23}$$

We will use the following result (see [12]):

$$\sum_{m=0}^{\infty}\delta_m\sum_{r=0}^{m}\alpha_r = \left[\sum_{r=0}^{\infty}\alpha_r\right]\left[\sum_{m=0}^{\infty}\delta_m\right] - \sum_{r=0}^{\infty}\alpha_{r+1}\sum_{m=0}^{r}\delta_m. \tag{24}$$

Theorem 2.1. *For $|q|<1$, we have*

$$R_{1,0}(0,q) = \sum_{n=0}^{\infty} q^{5n^2}\left[\frac{1}{(q^{1/5};q)_n^2} - \frac{1}{(q;q)_n^2}\right], \tag{25}$$

$$R_{2,0}(3,q) = q^{\frac{6}{5}}\sum_{n=0}^{\infty} q^{5n^2+6n}\left[\frac{1}{(q^{7/5};q)_n^2} - \frac{1}{(q^{3/5};q)_n^2}\right], \tag{26}$$

$$R_{2,1}(3,q) = q^{\frac{6}{5}}\sum_{n=0}^{\infty} q^{5n^2+6n}\left[\frac{1}{(q^{3/5};q)_n^2} - \frac{1}{(q^{2/5};q)_n^2}\right], \tag{27}$$

$$\chi_0(q) + \chi_1(q) = 2 \tag{28}$$

and

$$\chi_0(q)\chi_1(q) = \sum_{n=0}^{\infty}\frac{q^n}{(q^{n+1};q)_n}\left(q^{n^2}\sum_{r=0}^{n}\frac{q^r}{(q^{r+1};q)_{r+1}} + \frac{1}{1-q^{n+2}}\sum_{r=0}^{n}\frac{q^{r^2+r}}{(q^{r+1};q)_{r+1}}\right). \tag{29}$$

Further, if g_n *is a function such that* $|g_n(q)| < 1$*, then we have*

$$\sum_{n=0}^{\infty} \frac{g_n(q)}{(q;q)_n} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} g_n(q). \tag{30}$$

Proof. We have

$$R_{1,0}(0,q) = \sum_{n=0}^{\infty} [N(1,5,5n) - N(0,5,5n)]\, q^n,$$

where

$$N(1,5,5n) = \sum_{r=-\infty}^{\infty} N(1+5r,5n) \tag{31}$$

and

$$N(0,5,5n) = \sum_{r=-\infty}^{\infty} N(5r,5n). \tag{32}$$

Let us assume that $Q = q^{\frac{1}{5}}$. The identities (22), (23) and (32) with $z = 1$ yield

$$\begin{aligned}
\sum_{n=0}^{\infty} N(0,5,5n)q^n &= \sum_{n=0}^{\infty} \sum_{r=-\infty}^{\infty} N(5r,5n)q^n \\
&= \sum_{n\equiv 0 \pmod 5}^{\infty} \sum_{k\equiv 0 \pmod 5} N(r,n)q^{\frac{n}{5}} \\
&= \sum_{n\equiv 0 \pmod 5}^{\infty} \sum_{k\equiv 0 \pmod 5} N(r,n)Q^n \\
&= \sum_{n\equiv 0 \pmod 5}^{\infty} \frac{Q^{n^2}}{\prod_{k\equiv 5 \pmod 5}^{n} (1-Q^k)^2} \\
&= \sum_{n=0}^{\infty} \frac{Q^{25n^2}}{\prod_{k\equiv 5 \pmod 5}^{5n} (1-Q^k)^2} \\
&= \sum_{n=0}^{\infty} \frac{Q^{25n^2}}{\prod_{k=0}^{n-1} (1-Q^{5k+5})^2} \\
&= \sum_{n=0}^{\infty} \frac{Q^{25n^2}}{(Q^5;Q^5)_n^2} \\
&= \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q)_n^2}.
\end{aligned}$$

Identities (22), (23) and (31) with $z=1$ give

$$\begin{aligned}
\sum_{n=0}^{\infty} N(1,5,5n)q^{\frac{n}{5}} &= \sum_{n\equiv 0 \pmod 5}^{\infty} \sum_{k\equiv 1 \pmod 5} N(r,n)q^{\frac{n}{5}} \\
&= \sum_{n\equiv 0 \pmod 5}^{\infty} \sum_{k\equiv 1 \pmod 5} N(r,n)Q^{n} \\
&= \sum_{n\equiv 0 \pmod 5}^{\infty} \frac{Q^{n^2}}{\prod_{k\equiv 0 \pmod 5}^{n-1}(1-Q^{k+1})^2} \\
&= \sum_{n=0}^{\infty} \frac{Q^{25n^2}}{\prod_{k\equiv 0 \pmod 5}^{5n-1}(1-Q^{k+1})^2} \\
&= \sum_{n=0}^{\infty} \frac{Q^{25n^2}}{\prod_{k=0}^{n-1}(1-Q^{5k+1})^2} \\
&= \sum_{n=0}^{\infty} \frac{Q^{25n^2}}{(Q;Q^5)_n^2} \\
&= \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^{\frac{1}{5}};q)_n^2}.
\end{aligned}$$

Now, estimations of the sums $\sum_{n=0}^{\infty} N(1,5,5n)q^n$ and $\sum_{n=0}^{\infty} N(0,5,5n)q^n$ give

$$R_{1,0}(0,q) = \sum_{n=0}^{\infty} q^{5n^2}\left[\frac{1}{(q^{\frac{1}{5}};q)_n^2} - \frac{1}{(q;q)_n^2}\right].$$

Hence, we get identity (25).
Similarly, we prove identities (26) and (27).
Identity (28) follows from identities (12) and (13).
Identity (29) can be proved by substituting $\alpha_r = \frac{q^{r^2+r}}{(q^{r+1};q)_r}$ and $\delta_n = \frac{q^n}{(q^{n+1};q)_{n+1}}$ in (24).
Now, it remain to prove the identity (30). We have (see [1, equation (3.1)])

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} x^n}{(q;q)_n} = (x;q)_\infty. \tag{33}$$

Then, replace x by $-q$ in (33), we obtain

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} = (-q;q)_\infty \tag{34}$$

or

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = (-q;q)_\infty (q^2;q^2)_\infty. \tag{35}$$

Moreover, for all functions $|g_n| < 1$ one has

$$\sum_{n=0}^{\infty} \frac{g_n(q)}{(q;q)_n} = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} g_n(q).$$

This prove identity (30). We thus have completed the proof of Theorem 2.1. □

Proposition 2.2. *We have*

$$R_{1,0}(0,q^5) = \frac{1}{(q^2;q^2)_\infty^2} \sum_{n=1}^{\infty} q^{25n^2}\left[(q^2,q^3,q^4,q^5;q^5)_n^2 - (q,q^2,q^3,q^4;q^5)_n^2\right] \tag{36}$$

and

$$R_{2,0}(3,q^5)+R_{2,1}(3,q^5)=q^6\sum_{n=0}^{\infty}q^{25n^2+30n}\left[\frac{1}{(q^7;q)_n^2}-\frac{1}{(q^2;q)_n^2}\right]. \tag{37}$$

Proof. Replacing q by q^5 in identity (25), we get

$$\begin{aligned}R_{1,0}(0,q^5)&=\sum_{n=0}^{\infty}q^{25n^2}\left[\frac{1}{(q;q^5)_n^2}-\frac{1}{(q^5;q^5)_n^2}\right]=\sum_{n=0}^{\infty}q^{25n^2}\frac{\left[(q^5;q^5)_n^2-(q;q^5)_n^2\right]}{(q,q^5;q^5)_n^2}\\&=\sum_{n=0}^{\infty}q^{25n^2}\frac{\left[(q^2,q^3,q^4,q^5;q^5)_n^2-(q,q^2,q^3,q^4;q^5)_n^2\right]}{(q;q)_n^2}\\&=\frac{1}{(q^2;q^2)_\infty^2}\sum_{n=0}^{\infty}q^{25n^2}\left[(q^2,q^3,q^4,q^5;q^5)_n^2-(q,q^2,q^3,q^4;q^5)_n^2\right].\end{aligned}$$

Noting that for $n=0$, $R_{1,0}(0,q^5)=0$, then

$$R_{1,0}(0,q^5)=\frac{1}{(q^2;q^2)_\infty^2}\sum_{n=1}^{\infty}q^{25n^2}\left[(q^2,q^3,q^4,q^5;q^5)_n^2-(q,q^2,q^3,q^4;q^5)_n^2\right]. \tag{38}$$

This prove identity (36). Replacing q by q^5 in identities (26) and (27), we get (37). We thus have completed the proof of Proposition 2.2. □

Let us recall the following lemma.

Lemma 2.3. *[1] We have*

$$\sum_{n=0}^{\infty}\frac{(A;q)_n z^n}{(q;q)_n}=\frac{(Az;q)_\infty}{(z;q)_\infty}, \tag{39}.$$

$$(A;q)_n=\frac{(A;q)_\infty}{(Aq^n;q)_\infty} \tag{40}.$$

and

$$(q^{n+1};q)_n=\frac{(q;q)_{2n-1}}{(q;q)_n}. \tag{41}.$$

Proposition 2.4. *We have*

$$\chi_0(q^5)-1=\sum_{n=1}^{\infty}q^{5n}\sum_{k=0}^{n-1}\frac{(q^{5-5n};q^5)_k q^{10nk}}{(q^5;q^5)_k} \tag{42}$$

and

$$\chi_1(q^5)=\sum_{n=0}^{\infty}\frac{q^{5n}}{1-q^{10n+5}}\sum_{k=0}^{n-1}\frac{(q^{5-5n};q^5)_k q^{10nk}}{(q^5;q^5)_k}. \tag{43}$$

Proof. Combining identities (10), (40) and (41) and applying little algebra, we obtain

$$\chi_0(q)=\sum_{n\geq 0}\frac{q^n}{(q^{n+1};q)_n}=\sum_{n=0}^{\infty}\frac{(q;q)_n q^n}{(q;q)_{2n-1}}=\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty q^n}{(q^{2n};q)_\infty}. \tag{44}$$

Further, using (39) and (44) we get

$$\chi_0(q)=\sum_{n=0}^{\infty}q^n\sum_{k=0}^{\infty}\frac{(q^{1-n};q)_k q^{2nk}}{(q;q)_k}. \tag{45}$$

We have

$$(q^{1-n};q)_k=\prod_{j=0}^{k-1}(1-q^{j+1-n}). \tag{46}$$

If $k \geq n$, then $(q^{1-n};q)_k = 0$. Hence

$$\chi_0(q) = \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n-1} \frac{(q^{1-n};q)_k q^{2nk}}{(q;q)_k}. \tag{47}$$

Moreover

$$\chi_0(q^5) - 1 = \sum_{n=1}^{\infty} q^{5n} \sum_{k=0}^{n-1} \frac{(q^{5-5n};q^5)_k q^{10nk}}{(q^5;q^5)_k}. \tag{48}$$

Therefore, we get identity (42).

To show (43), consider

$$\chi_1(q) = \sum_{n\geq 0}^{\infty} \frac{q^n}{(q^{n+1};q)_{n+1}} \tag{49}$$

and

$$(q^{n+1};q)_{n+1} = \prod_{k=0}^{n}(1-q^{n+k+1}) = \prod_{k=0}^{n-1}(1-q^{n+k+1})(1-q^{2n+1}) = (q^{n+1};q)_n(1-q^{2n+1}). \tag{50}$$

Equations (49) and (50) give

$$\chi_1(q) = \sum_{n\geq 0}^{\infty} \frac{q^n}{(1-q^{2n+1})(q^{n+1};q)_n}. \tag{51}$$

Moreover (51) and the proof of (42) give (43). We thus have completed our proof of Proposition 2.4. □

As a consequence, in order to prove the above conjectures it is sufficient to show the following tow identities :

$$\sum_{n=1}^{\infty} q^{5n} \sum_{k=0}^{n-1} \frac{(q^{5-5n};q^5)_k q^{10nk}}{(q^5;q^5)_k} = \sum_{n=1}^{\infty} \frac{q^{25n^2}}{(q;q)_n^2}\left[(q^2,q^3,q^4,q^5;q^5)_n^2 - (q,q^2,q^3,q^4;q^5)_n^2\right] \tag{52}$$

or

$$\sum_{n=1}^{\infty} q^{5n} \sum_{k=0}^{n-1} (q^{5-5n};q^5)_k q^{10nk} = \frac{(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty^2} \sum_{n=0}^{\infty} q^{25n^2}\left[(q^2,q^3,q^4,q^5;q^5)_n^2 - (q,q^2,q^3,q^4;q^5)_n^2\right]$$

and

$$\sum_{n=1}^{\infty} \frac{q^{5n}}{1-q^{10n+5}} \sum_{k=0}^{n-1} \frac{(q^{5-5n};q^5)_k q^{10nk}}{(q^5;q^5)_k} = q^6 \sum_{n=0}^{\infty} q^{25n^2+30n}\left[\frac{1}{(q^7;q)_n^2} - \frac{1}{(q^2;q)_n^2}\right]. \tag{53}$$

Lemma 2.5. *For $k \in \mathbb{N}\setminus\{0\}$ and for all function $|g_n(q)| < 1$, we have*

$$\sum_{n=0}^{\infty} \frac{g_n(q)}{(q^k;q)_n} = \frac{1}{(q^2;q^2)_\infty}\frac{1}{1-q^k}\sum_{n=0}^{\infty} g_n(q). \tag{54}$$

Proof. In [3, equation (2.5)], we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1};q)_n} &= 1 + \sum_{n=1}^{\infty} \frac{q^n}{(q^{n+1};q)_n} = 1 + \sum_{n=0}^{\infty} \frac{q^{n+1}}{(q^{n+2};q)_{n+1}} \\ &= 1 + \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^{n+1};q)_{n+1}} = 1 + \sum_{n=0}^{\infty} \frac{q^{n+1}}{\frac{1}{q^n}(q^{n+1};q)_{n+1}}.\end{aligned} \tag{55}$$

This can also be expressed as

$$\sum_{n=0}^{\infty} \frac{q^{n+1}}{(q^{n+2};q)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{n+1}}{\frac{1}{q^n}(q^{n+1};q)_{n+1}}. \tag{56}$$

Further, by applying properties of the recurrence relations, we obtain the desired result. □

Proposition 2.6. *We have*

$$\sum_{n=0}^{\infty}\frac{q^{25n^2}}{(q^5;q^5)_n^2}=\frac{(-q^{25};q^{50})_\infty^2(q^{50};q^{50})_\infty}{2(q^{10};q^{10})_\infty^2}+\frac{1}{2(q^{10};q^{10})_\infty^2} \tag{57}$$

and

$$\sum_{n=0}^{\infty}\frac{q^{25n^2}}{(q;q^5)_n^2}=\frac{(-q^{25};q^{50})_\infty^2(q^{50};q^{50})_\infty}{2(q^{10};q^{10})_\infty^2}+\frac{1}{2(q^{10};q^{10})_\infty^2(1-q)^2}. \tag{58}$$

Proof. Combining identities (10) and (30) and applying little algebra, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{25n^2}}{(q^5;q^5)_n^2} &= \frac{1}{(q^{10};q^{10})_\infty^2}\sum_{n=0}^{\infty}q^{25n^2}\\ &= \frac{1}{(q^{10};q^{10})_\infty^2}\frac{1}{2}\left(1+2\sum_{n=1}^{\infty}q^{25n^2}\right)+\frac{1}{2(q^{10};q^{10})_\infty^2}\\ &= \frac{(-q^{25};q^{50})_\infty^2(q^{50};q^{50})_\infty}{2(q^{10};q^{10})_\infty^2}+\frac{1}{2(q^{10};q^{10})_\infty^2}.\end{aligned}$$

We thus have completed the proof of the Proposition 2.6. □

In order to show identity (58), we require a result, which is stated in the previous lemma.

Proposition 2.7. *We have*

$$R_{1,0}(0,q^5)=\frac{2q-q^2}{2(q^{10};q^{10})_\infty^2(1-q)^2}. \tag{59}$$

Proof. Combining identities (57) and (58) together, after simplification we obtain the desired result. □

Proposition 2.8. *We have*

$$\chi_0(q)=\frac{(-q^2;q^4)_\infty^2}{2(q^2;q^4)_\infty}+\frac{1}{2(q^2;q^2)_\infty}. \tag{60}$$

Proof. To show identity (60), apply $R_{1,0}(0,q^5)$ into (56), after little algebra, we have

$$\chi_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^{n+1};q)_n}=\sum_{n=0}^{\infty}\frac{q^{n^2+n+n^2-n}}{(q;q)_n}=\frac{1}{(q^2;q^2)_\infty}\frac{1}{2}\left(1+2\sum_{n=1}^{\infty}q^{2n^2}+1\right)$$

$$=\frac{(-q^2;q^4)_\infty^2(q^4;q^4)_\infty}{2(q^2;q^2)_\infty}+\frac{1}{2(q^2;q^2)_\infty}=\frac{(-q^2;q^4)_\infty^2}{2(q^2;q^4)_\infty}+\frac{1}{2(q^2;q^2)_\infty}.$$

We thus have completed the proof of the Proposition 2.8. □

Theorem 2.9. *We have*

$$R_{1,0}(0)=3\sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n}-\frac{(q;q)_\infty}{(q,q^4;q^5)_\infty^3}-2 \tag{61}$$

and

$$R_{2,1}(3)+R_{2,0}(3)=\frac{-3}{q}+\frac{3}{q}\sum_{n=0}^{\infty}\frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}+\frac{(q;q)_\infty}{(q^2,q^3;q^5)_\infty^3}. \tag{62}$$

Proof. To show identity (61), we use [3, equations (3.11) and (4.6)]. Then, we have

$$R_{(a,b)}=R_{(a,c)}+R_{(c,b)}.$$

Hence

$$\begin{aligned} R_{0,2}(0)+2R_{1,2}(0) &= R_{0,1}+R_{1,2}(0)+2R_{1,2}(0) \\ &= 3R_{1,2}(0)+R_{0,1}(0) \\ &= A(q)-1. \end{aligned}$$

Moreover, one has

$$\begin{aligned} R_{1,0}(0) &= -A(q)+1+3R_{1,2}(0) \\ &= -\frac{G(q)^2(q^5;q^5)}{H(q)}+1+3\varphi(q) \\ &= -\frac{(q;q)_\infty}{(q,q^4;q^5)_\infty}+1-3+3\sum_0^\infty \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n} \\ &= 3\sum_{n=0}^\infty \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n}-\frac{(q;q)_\infty}{(q,q^4;q^5)_\infty^3}-2. \end{aligned}$$

To show identity (62), we use [3, equations (3.12), (4.5) and (4.7)]. Then

$$\begin{aligned} R_{2,1}(3)+R_{2,0}(3) &= R_{2,0}(3)+R_{0,1}(3)+R_{2,0}(3)-R_{0,2}(3)+R_{0,2}(3) \\ &= D(q)+3R_{2,0}(3)-1 \\ &= \frac{H(q)^2(q^5;q^5)_\infty}{G(q)}+3R_{2,0}(3)-1 \\ &= \frac{H(q)^2(q^5;q^5)_\infty}{G(q)}+3\psi(q)-1 \\ &= \frac{-3}{q}+\frac{3}{q}\sum_{n=0}^\infty \frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}+\frac{(q;q)_\infty}{(q^2,q^3;q^5)_\infty^3}. \end{aligned}$$

We thus have completed the proof of the Theorem 2.9. □

3 Final comments

In this section, we present a few problems which need to be further addressed. There is a need for better understanding the conjecture. To do so it suffices to prove the following identities:

$$3\sum_{n=0}^\infty \frac{q^{5n^2}}{(q;q^5)_{n+1}(q^4;q^5)_n}=\frac{1}{(q^2,q^3;q^5)_\infty}.+\frac{(q;q)_\infty}{(q,q^4;q^5)_\infty^3}+1 \tag{63}$$

and

$$3\sum_{n=0}^\infty \frac{q^{5n^2}}{(q^2;q^5)_{n+1}(q^3;q^5)_n}=\frac{1}{(q,q^4;q^5)_\infty}-\frac{q(q;q)_\infty}{(q^2,q^3;q^5)_\infty^3}+3. \tag{64}$$

4 Concluding remarks

This paper aims to give a combinatorial proof for the mock theta conjectures in response to an open problem stated by M.P.Chaudhary [7]. Therefore, the results of this work are useful, significant and so it is interesting and capable of developing its study in the future.

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