Palestine Journal of Mathematics

Vol 13(4)(2024) , 955–961

On Signless Laplacian Eigenvalues Of $\Gamma(\mathbb{Z}_n)$

Mohd Rashid, Muzibur Rahman Mozumder and Wasim Ahmed*

Communicated by Manoj Kumar Patel

MSC 2010 Classifications: Primary 05C25; Secondary 05C50, 05C12, 15A18.

Keywords and phrases: Signless Laplacian matrix; zero-divisor graph; ring of integer modulo n.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Muzibur Rahman Mozumder was very thankful to the DST-SERB (project MATRICS, File No. MTR/2022/000153) for their necessary support and facility.

Abstract For $n = (pq)^2$, Pirzada et al. in [7] showed the signless Laplacian spectrum of the graph $\Gamma(\mathbb{Z}_n)$. In this article, we generalised the finding of [7] and show the signless Laplacian eigenvalues of the graphs $\Gamma(\mathbb{Z}_n)$ for $n = p^{M_1}q^{M_2}$, where M_1 and M_2 are positive integers and p and q are primes $(p < q)$.

1 Notations and Introduction

In this paper, only simple, connected, undirected, and finite graphs are considered. Let V stand for the graph $G's$ vertex set and E for the graph $G's$ edge set. Then, we write $G = (V, E)$. The order of a graph G is it's number of vertices. The size of G is determined by the quantity of its edges. $deg(v)$ is used to denote the number of edges incident with $v \in V$. For each vertex v, a graph G is said to be *r-regular* if $deg(v) = r$. The collection of G vertices that are adjacent to a vertex v is referred to as its *neighbourhood* and is represented by the symbol $N_G(v)$. With A being any square matrix and its different eigenvalues being $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding multiplicities of $n_1, n_2, \ldots, n_k$, the *spectrum* of A is denoted by $\sigma(A)$ and is defined by

$$\sigma(A) = \left\{ \begin{array}{cccc} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ n_1 & n_2 & \cdots & n_k \end{array} \right\}.$$

The square n-dimensional *adjacency matrix* $A(G)$ of G is given by

$$A(G) = (a_{ij}) = \begin{cases} 1, & v_iv_j \in E(G), \\ 0, & otherwise. \end{cases}$$

The matrices $L(G) = Deg(G) - A(G)$ and $SL(G) = Deg(G) + A(G)$ are the *Laplacian and signless Laplacian* of graph G, respectively, where $Deg(G)$ be the diagonal matrix of vertex degrees given by $Deg(G) = diag(x_1,$
$x_2, \ldots, x_n)$, where $x_i = deg(v_i)$. The spectrum of signless Laplacian matrix and Laplacian matrix is respectively known to be signless Laplacian spectrum and Laplacian spectrum of the graph G. The details of adjacency and Laplacian spectrum can be found in [5, 8, 11].

Consider a commutative ring R with the identity $1 \neq 0$. On commutative rings, firstly, Beck in the year 1988 introduced the concept of zero-divisor graphs. The vertices of the graph are all elements of rings in the definition of Beck. Beck's definition was later amended by Anderson and Livingston in the year 1999, in which they define only the non-zero zero-divisors of the rings are the vertices of the graphs. For further information on the zero-divisor graph see [1, 2, 4].
In Section 2, we discuss several basic definitions, lemma's and theorems, which are employed

to support the key conclusions. In Section 3, we investigate signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ of order M, where $n = p^{M_1}q^{M_2}$. Also deduce the many corollaries from these findings as well. As a result of the application of the main result, we draw several significant conclusions.

2 Preliminaries

We start with definitions and a few well-known findings that will be used to support the main conclusions.

Definition 2.1. Let $G(V, E)$ be a graph of order m having vertex set $\{v_1, v_2, \ldots, v_m\}$ and $H_k(V_k, E_k)$ be disjoint graphs of order m_k, $1 \leq k \leq m$. The generalized join graph $G[H_1, H_2, \ldots, H_m]$ is formed by taking the graphs $H_1, H_2, \ldots, H_m$ and joined each vertex of H_k to every vertex of H_l whenever k and l are adjacent in G.

Euler's phi function $\phi(n)$. denotes the number of positively divisible integers less than or equal to n that are relatively prime to n. If $n = {q_1}^{m_1}{q_2}^{m_2} \cdots {q_k}^{m_k}$, where $m_1, m_2, \ldots, m_k$ are positive integers and $q_1, q_2, \ldots, q_k$ are different primes, then we can say that n is in prime decomposition.

Lemma 2.2. [6] *If $n = {q_1}^{m_1}{q_2}^{m_2} \cdots {q_t}^{m_t}$ is a prime decomposition of n, then $\tau(n) = (m_1 + 1)(m_2 + 1) \cdots (m_t + 1)$.*

Theorem 2.3. [6] *The phi function ϕ of Euler satisfies the following conditions:*
(1) $\phi(xy) = \phi(x)\phi(y)$, whenever x and y are relatively prime.
(2) Sum of $\phi(x) = n$, whenever x divides n i.e., $\sum\limits_{x|n} \phi(x) = n$.

(3) For any prime q, $\sum\limits_{i=1}^{k} \phi(q^k) = q^k - 1$.

For $1 \leq r \leq k$, consider the sets

$$A_{x_r} = \{s \in \mathbb{Z}_n : (s, n) = x_r\},$$

where (s, n) denotes the gcd of s and n. Also we see that $A_{x_r} \cap A_{x_s} = \phi$, when $r \neq s$. This implies that the sets $A_{x_1}, A_{x_2}, \ldots, A_{x_k}$ are pairwise disjoint and partitions the vertex set of $\Gamma(\mathbb{Z}_n)$ as

$$V(\Gamma(\mathbb{Z}_n)) = A_{x_1} \cup A_{x_2} \cup \cdots \cup A_{x_k}.$$

By the definition of A_{x_r}, a vertex of A_{x_r} is adjacent to a vertex of A_{x_s} in $\Gamma(\mathbb{Z}_n)$ if and only if $n|x_r x_s$, for $r, s \in \{1, 2, \ldots, k\}$. The next lemma shows the cardinality of A_{x_r}.

Lemma 2.4. [11, Proposition 2.1] *Let x_r be the divisor of n, then $|A_{x_r}| = \phi(\frac{n}{x_r})$, $1 \leq r \leq k$.*

Lemma 2.5. [5, Corollary 2.5] *Let x_r be the proper divisor of the positive integer n, the following hold:*
(i) For $r \in \{1, 2, \ldots, k\}$, the induced subgraph $\Gamma(A_{x_r})$ of $\Gamma(\mathbb{Z}_n)$ on the vertex set A_{x_r} is either the complete graph $K_{\phi(\frac{n}{x_r})}$ or its complement graph $\overline{K}_{\phi(\frac{n}{x_r})}$. Indeed, $\Gamma(A_{x_r})$ is $K_{\phi(\frac{n}{x_r})}$ if and only if $n|x_r^2$.
(ii) For $r, s \in \{1, 2, \ldots, k\}$ with $r \neq s$, a vertex of A_{x_r} is adjacent to either all or none of the vertices of A_{x_s} in $\Gamma(\mathbb{Z}_n)$.

The above lemma shows that the induced subgraphs $\Gamma(A_{x_r})$ of $\Gamma(\mathbb{Z}_n)$ are either cliques or null graphs. The next lemma says that $\Gamma(\mathbb{Z}_n)$ is a generalized join of certain complete graphs and complements of complete graphs.

Lemma 2.6. [5, Lemma 2.7] *Let $\Gamma(A_{x_r})$ be the induced subgraph of $\Gamma(\mathbb{Z}_n)$ on the vertex set A_{x_r} for $1 \leq r \leq k$. Then $\Gamma(\mathbb{Z}_n) = \delta_n[\Gamma(A_{x_1}), \Gamma(A_{x_2}), \ldots, \Gamma(A_{x_k})]$.*

In term of the signless Laplacian spectrum of component H_r's the next result gives the signless Laplacian spectrum of $G[H_1, H_2, \ldots, H_n]$ and the eigenvalues of an auxiliary matrix.

Theorem 2.7. [10, Theorem 2.1] *Let K be a graph with $V(K) = \{v_1, v_2, \ldots, v_n\}$, and H_r's be k_r-regular graphs of order h_r with signless Laplacian eigenvalues $\lambda_{r1} \geq \lambda_{r2} \geq \cdots \geq \lambda_{rh_r}$, where $r = 1, 2, \ldots, n$. If $G = K[H_1, H_2, \ldots, H_n]$, then signless Laplacian spectrum of G can be computed as follows.*

$$\sigma_{SL}(G) = \left(\bigcup_{r=1}^{n} \Big(N_r + \big(\sigma_{SL}(H_r) \setminus \{2k_r\}\big)\Big)\right) \bigcup \sigma(Y(K)),$$

where

$$N_r = \begin{cases} \sum\limits_{v_s \in N_K(v_r)} h_s, & N_K(v_r) \neq \phi, \\ 0, & otherwise \end{cases}$$

and

$$Y(K) = (y_{st})_{n\times n} = \begin{cases} 2k_s + N_s, & s = t, \\ \sqrt{h_s h_t}, & st \in E(K), \\ 0 & otherwise. \end{cases} \tag{2.1}$$

The number N_r and the matrix $Y(K)$ are only dependent on the graph K.

3 Main results

We state and prove the first result of this paper.

Theorem 3.1. *The signless Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^{M_1}q^{M_2}})$ of order M, where $M_1 = 2m \leq 2n = M_2$ consists of the eigenvalues*

$$\left\{\begin{array}{ccccc} p^r - 1 & q^s - 1 & pq^s - 1 & \cdots & p^m q^k - 1 \\ \phi(p^{M_1 - r} q^{M_2}) - 1 & \phi(p^{M_1} q^{M_2 - s}) - 1 & \phi(p^{M_1 - 1} q^{M_2 - s}) - 1 & \cdots & \phi(p^m q^{M_2 - k}) - 1 \end{array}\right.$$

$$\left.\begin{array}{ccccc} p^m q^l - 3 & \cdots & p^{M_1} q^k - 3 & p^{M_1} q^t - 3 \\ \phi(p^m q^{M_2 - l}) - 1 & \cdots & \phi(q^{M_2 - k}) - 1 & \phi(q^{M_2 - t}) - 1 \end{array}\right\}$$

where $r = 1, 2, \ldots, m, \ldots, M_1$, $s = 1, 2, \ldots, M_2$, $k = 1, 2, \ldots, n-1$, $l = n, \ldots, 2n$ and $t = n, \ldots, 2n-1$. The eigenvalues of the matrix (2.1) are remaining signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$.

Proof. Let $u = p^{M_1}q^{M_2}$, where p and q are distinct primes and $2 \leq M_1 = 2m \leq 2n = M_2$. The proper divisors of u are
$\Big\{p, p^2, \ldots, p^m, \ldots, p^{M_1}, q, q^2, \ldots, q^n, \ldots, q^{M_2}, pq, pq^2, \ldots, pq^n, \ldots, pq^{M_2}, \ldots, p^m q,$
$p^m q^2, \ldots, p^m q^{n-1}, p^m q^n, \ldots,, p^m q^{M_2}, \ldots, p^{M_1} q, p^{M_1} q^2, \ldots, p^{M_1} q^{n-1}, p^{M_1} q^n,$
$\ldots, p^{M_1} q^{M_2 - 1}\Big\}.$
The following adjacency relations will be obtain by the definition of δ_n given in [3].

$$\begin{aligned} p^r &\sim\ p^s q^{M_2},\ \ r + s \geq M_1,\ \text{for } r = 1, 2, \ldots, M_1, \\ q^r &\sim p^{M_1} q^s,\ \ r + s \geq M_2,\ \text{for } r = 1, 2, \ldots, M_2, \\ pq^r &\sim p^k q^s,\ \ r + s \geq M_2,\ \text{for } r = 1, 2, \ldots, M_2 \text{ and } k \geq 2m - 1, \\ &\vdots \\ p^m q^r &\sim p^k q^s,\ \ r + s \geq M_2,\ \text{for } r = 1, 2, \ldots, M_2 \text{ and } k \geq m, \\ &\vdots \\ p^{M_1} q^r &\sim p^k q^s,\ \ r + s \geq M_2,\ \text{for } r = 1, 2, \ldots, M_2 - 1 \text{ and } k \geq 0. \end{aligned}$$

Also, by Lemma 2.4, the cardinalities of A_{x_r} for $k = 1, 2, \dots, M_2 - 1$, $s = 1, 2, \dots, M_2$ and $r = 1, 2, \dots, M_1$, are

$|A_{p^r}| = \phi(p^{M_1-r}q^{M_2}), |A_{q^s}| = \phi(p^{M_1}q^{M_2-s}), |A_{pq^s}| = \phi(p^{M_1-1}q^{M_2-s}),$
$\dots, |A_{p^m q^s}| = \phi(p^m q^{M_2-s}), \dots, |A_{p^{M_1-1}q^s}| = \phi(pq^{M_2-s}), |A_{p^{M_1}q^k}| = \phi(q^{M_2-k})$.
The induced subgraphs $\Gamma(A_{x_{p^r}})$ from Lemma 2.5 are

$$H_r = \begin{cases} \Gamma(A_{x_{p^r}}) = \overline{K}_{\phi(p^{M_1-r}q^{M_2})}, & 1 \le r \le M_1, \\ \Gamma(A_{x_{q^s}}) = \overline{K}_{\phi(p^{M_1}q^{M_2-s})}, & 1 \le s \le M_2, \\ \Gamma(A_{x_{p^r q^s}}) = \overline{K}_{\phi(p^{M_1-r}q^{M_2-s})}, & 1 \le r \le m-1 \text{ and } 1 \le s \le M_2 \\ \text{ or } m \le r \le M_1 \text{ and } 1 \le s \le n-1, \\ \Gamma(A_{x_{p^r q^s}}) = K_{\phi(p^{M_1-r}q^{M_2-s})}, & m \le r \le M_1 \text{ and } n \le s \le M_2. \end{cases} \tag{3.1}$$

Using Theorems 2.3 and 2.7, the values of $N_r{}'s$ are

$$\begin{aligned} N_1 =& \phi(p) = p - 1, \\ N_2 =& \phi(p) + \phi(p^2) = p^2 - 1, \\ &\vdots \\ N_m =& \phi(p^m) + \phi(p^{m-1}) + \cdots + \phi(p) = p^m - 1, \\ &\vdots \\ N_{M_1} =& \phi(p^{M_1}) + \phi(p^{M_1-1}) + \cdots + \phi(p) = p^{M_1} - 1, \end{aligned}$$

so, we can say that,

$$N_r = p^r - 1, \text{ for } r = 1, 2, \dots, M_1.$$

For $r \ge n$ and $s \ge n$, $\Gamma(A_{p^r q^s})$ is adjacent to itself as a vertex of δ_n, Therefore, to obtain N_r's, we add and subtract the cardinalities of such $\Gamma(A_{p^r q^s})$'s. The other N_r's, as calculated above, are given by

$$\begin{aligned} N_r =& q^s - 1, \text{ for } r = M_1 + 1, \dots, M_1 + M_2 \text{ and } s = 1, 2, \dots, n, \dots, M_1, \\ N_r =& pq^s - 1, \text{ for } r = M_1 + M_2 + 1, \dots, M_1 + 2M_2 \text{ and } s = 1, 2, \dots, n, \dots, M_1, \\ &\vdots \\ N_r =& p^m q^s - 1, \text{ for } r = M_1 + mM_2 + 1, \dots, M_1 + mM_2 + n - 1 \text{ and } s = 1, 2, \dots, n-1, \\ N_r =& p^m q^s - 1 - \phi(p^m q^s), \text{ for } r = M_1 + mM_2, \dots, M_1 + (m+1)M_2 \text{ and } s = n, \dots, M_2, \\ &\vdots \\ N_r =& p^{M_1} q^s - 1, \text{ for } r = M_1 + M_1M_2 + 1, \dots, M_1 + M_1M_2 + n - 1 \text{ and } s = 1, 2, \dots, n-1, \\ N_r =& p^{M_1} q^s - 1 - \phi(q^{M_2-s}), \text{ for } r = M_1 + M_1M_2 + n, \dots, M_1 + M_1M_2 + M_2 - 1 \\ & \text{and } s = n, \dots, M_2 - 1. \end{aligned}$$

Now, by using (3.1) and Theorem 2.7, we have

$$N_r + \lambda_{rk}(H_r) = N_r + \lambda_{rk}(\overline{K}_{\phi(p^{M_1-r}q^{M_2})}) = N_r = p^r - 1, \text{ for } r = 1, 2, \dots, M_1.$$

Thus, for $r = 1, 2, \dots, M_1$, the signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_{p^{M_1}q^{M_2}})$ is $p^r - 1$ with multiplicity $\phi(p^{M_1-r}q^{M_2}) - 1$.
Using similar steps, we obtain

$$\left\{ \begin{matrix} q^s - 1 & pq^s - 1 & \cdots & p^m q^k - 1 \\ \phi(p^{M_1}q^{M_2-s}) - 1 & \phi(p^{M_1-1}q^{M_2-s}) - 1 & \cdots & \phi(p^m q^{M_2-k}) - 1 \end{matrix} \right\}$$

are also the signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_{p^{M_1}q^{M_2}})$. Now, by Equation (3.1) $H_r = K_{\phi(p^{M_1-r}q^{M_2-s})}$, for $m \leq r \leq M_1$ and $n \leq s \leq M_2$, so, we have

$$N_r + \lambda_{rk}(H_r) = p^m q^l - 1 - \phi(p^m q^l) + \phi(p^m q^l) - 2 = p^m q^l - 3.$$

Thus, $p^m q^l - 3$ is the signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_{p^{M_1}q^{M_2}})$ with multiplicity $\phi(p^m q^l) - 1$, where $l = n, \ldots, M_2$.
Similarly, we see that $p^{M_1}q^k - 3$ with multiplicity $\phi(q^{M_2-k}) - 1$ and $p^{M_1}q^t - 3$ with multiplicity $\phi(q^{M_2-t}) - 1$ are the signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_{p^{M_1}q^{M_2}})$ for $k = 1, 2, \ldots, n-1$ and $t = n, \ldots, M_2 - 1$. The remaining signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_{p^{M_1}q^{M_2}})$ are the roots of the characteristic polynomial of the matrix (2.1). □

Theorem 3.1 will get the following conclusion if we put $n = 0$.

Corollary 3.2. [7, Theorem 3.5] *If $n = p^{2m}$ for some positive integer $m \geq 2$, then the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ is*

$$\left\{ \begin{array}{ccccc} p-1 & p^2-1 & \cdots & p^{m-2}-1 & p^{m-1}-1 \\ \phi(p^{2m-1})-1 & \phi(p^{2m-2})-1 & \cdots & \phi(p^{m+2})-1 & \phi(p^{m+1})-1 \end{array}\right.$$

$$\left. \begin{array}{ccccc} p^m-3 & p^{m+1}-3 & \cdots & p^{2m-2}-3 & p^{2m-1}-3 \\ \phi(p^m)-1 & \phi(p^{m-1})-1 & \cdots & \phi(p^2)-1 & \phi(p)-1 \end{array} \right\}.$$

The other signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the zeros of the characteristic polynomial of the matrix (2.1).

Example 3.3. The signless Laplacian spectrum of $\Gamma(\mathbb{Z}_{36})$.

Proof. As $n = 2^2 3^2$. Now the proper divisor of 36 are $2, 3, 4, 6, 9, 12, 18$ and $\delta_{36} : 2 \sim 18 \sim 4 \sim 9 \sim 12 \sim 3, 12 \sim 18 \sim 6 \sim 12$. Applying Lemma 2.6 and increasing the divisor sequence to order the vertices,

$$\Gamma(\mathbb{Z}_{36}) = \delta_{36}[\Gamma(A_2), \Gamma(A_3), \Gamma(A_4), \Gamma(A_6), \Gamma(A_9), \Gamma(A_{12}), \Gamma(A_{18})].$$

By Lemma 2.5, we have

$$\Gamma(\mathbb{Z}_{36}) = G_7[\overline{K}_6, \overline{K}_4, \overline{K}_6, K_2, \overline{K}_2, K_2, K_1].$$

Using Theorem 2.7, the values of N_r's are

$$(N_1, N_2, N_3, N_4, N_5, N_6, N_7) = (1, 2, 3, 3, 8, 9, 16).$$

Again, using Theorem 2.7, the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_{36})$ consists of the eigenvalues

$$\left\{ \begin{array}{ccccccc} 1 & 2 & 3 & 3 & 8 & 9 & 15 \\ 5 & 3 & 5 & 1 & 1 & 1 & 0 \end{array} \right\}.$$

Now, using Theorem 2.7, we can determine the matrix $Y(G_7)$.

$$Y(G_7) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & 2 & 0 & 0 & 0 & \sqrt{8} & 0 \\ 0 & 0 & 3 & 0 & \sqrt{12} & 0 & \sqrt{6} \\ 0 & 0 & 0 & 5 & 0 & 2 & \sqrt{2} \\ 0 & 0 & \sqrt{12} & 0 & 8 & 2 & 0 \\ 0 & \sqrt{8} & 0 & 2 & 2 & 11 & \sqrt{2} \\ \sqrt{6} & 0 & \sqrt{6} & \sqrt{2} & 0 & \sqrt{2} & 16 \end{bmatrix} \qquad (3.2)$$

From the matrix $Y(G_7)$ given in (3.2) we have calculated the remaining eigenvalues of $\Gamma(\mathbb{Z}_{36})$. The characteristic polynomial of matrix (3.2) is

$$-\lambda^7 + 46\lambda^6 - 774\lambda^5 + 5888\lambda^4 - 20345\lambda^3 + 29538\lambda^2 - 17088\lambda + 2880.$$

The approximated eigenvalues of matrix (3.2) are

$$\{17.567, 12.480, 9.010, 4.456, 1.324, 0.883, 0.280\}.$$

□

Theorem 3.4. *The signless Laplacian Spectrum of* $\Gamma(\mathbb{Z}_{p^{M_1}q^{M_2}})$ *of order* M*, where* $M_1 = 2m + 1 \leq 2n + 1 = M_2$ *consists of the eigenvalues*

$$\left\{\begin{array}{cccccc} p^r - 1 & q^s - 1 & pq^s - 1 & \cdots & p^{m+1}q^k - 3 & \cdots & p^{2m+1}q^k - 3 \\ \phi(p^{M_1-r}q^{M_2}) - 1 & \phi(p^{M_1}q^{M_2-s}) - 1 & \phi(p^{M_1-1}q^{M_2-s}) - 1 & \cdots & \phi(p^m q^{M_2-s}) - 1 & \cdots & \phi(q^{M_2-k}) - 1 \end{array}\right\},$$

where $r = 1, 2, \ldots, m, \ldots, M_1,\ \ s = 1, 2, \ldots, M_2,\ \ k = 1, 2, \ldots, 2n$*. The remaining signless Laplacian eigenvalues of* $\Gamma(\mathbb{Z}_{p^{M_1}q^{M_2}})$ *are the eigenvalues of the matrix* (2.1).

If M_1 and M_2 are odd, the preceding conclusion yields the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$, where $n = p^{M_1}q^{M_2}$. The proof of the above Theorem is similar to Theorem 3.1.

If we substitute $M_2 = 0$ in Theorem 3.4, we obtain the following result.

Corollary 3.5. [7, Theorem 3.5] *If* $n = p^{2m+1}$ *for some positive integer* $m \geq 2$*, then the signless Laplacian spectrum of* $\Gamma(\mathbb{Z}_n)$ *is*

$$\left\{\begin{array}{ccccccccc} p - 1 & p^2 - 1 & \cdots & p^{m-1} - 1 & p^m - 1 & p^{m+1} - 3 & p^{m+2} - 3 & \cdots & p^{2m-1} - 3 & p^{2m} - 3 \\ \phi(p^{2m}) - 1 & \phi(p^{2m-1}) - 1 & \cdots & \phi(p^{m+2}) - 1 & \phi(p^{m+1}) - 1 & \phi(p^m) - 1 & \phi(p^{m-1}) - 1 & \cdots & \phi(p^2) - 1 & \phi(p) - 1 \end{array}\right\}.$$

The roots of the characteristic polynomial of the matrix (2.1) *are the remaining signless Laplacian eigenvalues of* $\Gamma(\mathbb{Z}_n)$.

It gives the following result when we put $m = n = 1$ in Theorem 3.1.

Corollary 3.6. [7, Proposition 3.4] *If* $n = p^2q^2$ *then the signless Laplacian spectrum of* $\Gamma(\mathbb{Z}_n)$ *is*

$$\left\{\begin{array}{ccccccc} p - 1 & p^2 - 1 & q - 1 & q^2 - 1 & pq - 3 & pq^2 - 3 & p^2q - 3 \\ \phi(pq^2) - 1 & \phi(q^2) - 1 & \phi(p^2q) - 1 & \phi(p^2) - 1 & \phi(pq) - 1 & \phi(p) - 1 & \phi(q) - 1 \end{array}\right\}.$$

The roots of the characteristic polynomial of the matrix (2.1) *are the remaining signless Laplacian eigenvalues of* $\Gamma(\mathbb{Z}_n)$.

The following outcome is produced when $m = n = 0$ is put in Theorem 3.4.

Corollary 3.7. [7, Lemma 3.2] *The signless Laplacian spectrum of* $\Gamma(\mathbb{Z}_{pq})$ *is*

$$\left\{\begin{array}{cccc} 0 & q - 1 & p - 1 & p + q - 2 \\ 1 & p - 2 & q - 2 & 1 \end{array}\right\}.$$

Here the eigenvalues $q - 1$ and $p - 1$ is obtained from Theorem 3.4 and the other two eigenvalues is obtained from the matrix (2.1).

Conflict of interest

There is no conflict of interest among the authors.

References

[1] S. Akbari and A. Mohammadian: *On zero-divisor graphs of a commutative ring,* J. Algebra **274,** 847-855, (2004).

[2] D. F. Anderson and P. S. Livingston: *The zero-divisor graph of a commutative ring,* J. Algebra **217,** 434-447, (1999).

[3] M. Ashraf, M. R. Mozumder, M. Rashid and Nazim: *On A_α spectrum of the zero-divisor graph of the ring $\mathbb{Z}_n$,* Discrete Math. Algorithms Appl. (2023) DOI: 10.1142/S1793830923500362.

[4] I. Beck: *Coloring of a commutative rings,* J. Algebra **116,** 208-226, (1988).

[5] S. Chattopadhyay, K. L. Patra, B. K. Sahoo: *Laplacian eigenvalues of the zero-divisor graph of the ring $\mathbb{Z}_n$,* Linear Algebra Appl. **584,** 267-286, (2020).

[6] T. Koshy: *Elementary number theory with application,* 2nd ed., academic press: cambridge, UK, (1985).

[7] S. Pirzada, B. A. Rather, R. U. Shaban and Merajuddin: *On signless Laplacian spectrum of zero-divisor graphs of the ring $\mathbb{Z}n$,* Korean J. Math. **29,** 13-24, (2021). https://doi.org/10.2478/ausm-2021-002821.

[8] B. A. Rather, S. Pirzada, T. A. Naikoo and Y. Shang: *On Laplacian eigenvalues of the zero-divisor graph associated to the ring of integers modulo n,* Mathematics, **9,** 482, (2021). http://doi.org/10.3390/math905048222.

[9] N. U. Rehman, Nazim, M. Nazim: *Exploring normalized distance Laplacian eigenvalues of the zero-divisor graph of ring Zn,* Rendiconti del Circolo Matematico di Palermo Series 2, (2023), https://doi.org/10.1007/s12215-023-00927-y.

[10] B. F. Wu, Y. Y. Lou and C. X. He: *Signless Laplacian and normalized Laplacian on the H-join operation of graphs,* Discrete Math. Algorithm. Appl. **06,** 13 pages, (2014) , DOI:http://dx.doi.org/10.1142/S1793830914500463.

[11] M. Young: *Adjacency matrices of zero-divisor graphs of integer modulo n,* Involve **8,** 753-761, (2015).

Author information

Mohd Rashid, Department of Mathematics, Aligarh Muslim University, India.
E-mail: `rashidaraz253@gmail.com`

Muzibur Rahman Mozumder, Department of Mathematics, Aligarh Muslim University, India.
E-mail: `muzibamu81@gmail.com`

Wasim Ahmed*, Department of Mathematics, Aligarh Muslim University, India.
E-mail: `wasim10041994@gmail.com`

Received: 2023-09-08
Accepted: 2024-02-19