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A Study of $q-k-$ Abelian Rings

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Abstract This paper embodies the algebraic structures of $q-k-abelian$ rings are investigated using the concept of non-zero k-potent elements in a ring. A k-potent element a in a ring R is called $quater-k-central$ or $(q-k-central)$ if $aR(1-a^{k-1})Ra = 0$ and the ring R is called $q-k-abelian$ if all the elements of R are $q-k-central$. We have presented various characterizations of $q-k-abelian$ rings and some associated concepts. It is prove that an element x in a $q-k-abelian$ ring R is Von Neumann regular, then x is strongly regular. Moreover, we have established that the property $a[a^{k-1}, R][R, a^{k-1}]a = 0$ for all k-potents $a \in R$ can be used to describe $q-k-abelian$ rings, where $[x, y] := xy - yx$ is the additive commutator of a ring.

1 Introduction

In this article R is a ring consisting identity unless otherwise specified. We refer to the set of all idempotents, nilpotents, and units in R, respectively, as $I(R), N(R)$ and $U(R)$. Also, we denote $\mathbb{T}_n(S)$, as the upper triangular matrix ring over any ring S. If for some natural number k, $a^k = a$, then an element a is said to be k-potent. We denote $K(R)$ as set of all k-potent elements of R. According to [2], an element $e \in I(R)$ is called left (right) semicentral if $ere = re(ere = er)$ for each $r \in R$ and e is called central if $re = er$ for each $r \in R$. Whenever every idempotent e is central in a ring R, then the ring is said to be abelian. For all idempotent e in a ring R, either $er = ere$ or $re = ere$ for all r in R, then the ring R was considered to be semiabelian according to Chen [3] (2007). Following that, the concept of semicentral idempotents and semiabelian rings was extensively researched by numerous researchers and extended in a variety of ways (refer to [2], [4], [9], [6], [14], [16], [15] and [17]). A k-potent $a \in R$ is called left (right) semicentral k-potent if $(a-1)Ra = 0$ $(aR(1-a) = 0)$ or $ara = ar$ $(ara = ar)$ for all $r \in R$.

Recently, the concept of semicentral idempotents in a ring was further extended by T. Y. Lam. According to T. Y. Lam [12], if $eR(1-e)Re = 0$ for all $e \in I(R)$ then e is referred to be quarter-central, and if all of the idempotent elements in a ring R are q-central, then the ring is said to be quarter-abelian (simply, a q-abelian ring). Though $\mathbb{T}_2(S)$(the set of all $n \times n$ upper triangular matrices over S) is q-abelian if and only if S is abelian, they demonstrated that for $n \geq 3$, $\mathbb{T}_n(S)$ is not q-abelian ring. They gave equivalent definitions of $q-abelian$ rings based on the concept of additive commutators of the ring $[x, y] := xy - yx$. Moreover, they studied and discussed the notion of q-central idempotents in relation to the idea of regular, unit-regular as well as strongly regular elements in arbitrary rings and various associated concepts. It is observed that abelian rings sits inside $q-abelian$ rings. In [[11], Ex. 12.8C] we get, when for all a in R, $a^k = a$ for some $k \in \mathbb{N}$ then R is a commutative ring. But in this article, we are only interested in those k-potent elements $a \in R$ which are quarter central k-potent.

In this article, we further generalise as well as extend the concept of q-abelian rings by introducing the concept of $q-k-abelian$ rings using non-zero k-potent elements of the ring and we have established various properties associated with this concepts.

2 Preliminaries

In this section, we have presented some basic results which are needed in the following sections.

Following [7] for a ring R, define a binary operation $\circ$ on R as $a \circ b = a + b - ab$ for each $a, b \in R$. Then it can be easily proved that $(R, \circ)$ is a monoid. An element $x \in R$ is said to be quasi-regular if x has an inverse i.e., there exists $x' \in R$ such that $x' \circ x = x \circ x' = 0$. $q(R)$ denotes the set of all quasi-regular elements in R. It is proved that $N(R) \subseteq q(R)$.

The following lemma is derived from [7].

Lemma 2.1. *The subsequent claims are identical for a ring R and $a \in K(R)$:*

(1) a^{k-1} is central.

(2) $ra^{k-1} = a^{k-1}ra^{k-1}$ whenever r is in $K(R)$.

(3) $ra^{k-1} = a^{k-1}ra^{k-1}$ whenever r is Von Neumann regular element.

(4) $ra^{k-1} = a^{k-1}ra^{k-1}$ whenever r is in $N(R)$.

(5) $ra^{k-1} = a^{k-1}ra^{k-1}$ whenever $r \in R$ and $r^2 = 0$.

(6) $ra^{k-1} = a^{k-1}ra^{k-1}$ whenever r is in $q(R)$.

Proposition 2.2. *The subsequent claims are identical for a ring R and $a \in K(R)$:*

(1) $ar = ara^{k-1}$ for all $r \in R$ or, $aR(1 - a^{k-1}) = 0$.

(2) $ar = ara^{k-1}$ whenever r is in $K(R)$.

(3) $ar = ara^{k-1}$ whenever r is Von neumann regular element.

(4) $ar = ara^{k-1}$ whenever r is in $N(R)$.

(5) $ar = ara^{k-1}$ whenever r is in R and $r^2 = 0$.

(6) $ar = ara^{k-1}$ whenever r is in $q(R)$.

Proof. We can clearly see, $(1) \implies ((2), (3), (4), (5), (6))$ are obvious. For $(2) \implies (1)$. Let us consider $r = a^{k-1} - a^{k-1}x + a^{k-1}xa^{k-1}$ which is a k-potent element for any $x \in R$. So, $ara^{k-1} = ar \implies a(a^{k-1} - a^{k-1}x + a^{k-1}xa^{k-1})a^{k-1} = a(a^{k-1} - a^{k-1}x + a^{k-1}xa^{k-1}) \implies a = a - ax + axa^{k-1} \implies ax = axa^{k-1}$. $(3) \implies (2)$ is also clear, as every k-potent is regular. For $(5) \implies (1)$. Let us consider $y = a^{k-1}xa^{k-1} - a^{k-1}x$ then $y^2 = 0$. Now $aya^{k-1} = ay \implies a(a^{k-1}xa^{k-1} - a^{k-1}x)a^{k-1} = a(a^{k-1}xa^{k-1} - a^{k-1}x) \implies 0 = axa^{k-1} - ax \implies ax = axa^{k-1}$ for all $x \in R$. Again we notice that $(6) \implies (4)$ as $N(R) \subseteq q(R)$. Also, $(4) \implies (5)$ is clear. □

We have the next proposition, which is similar to Proposition 2.2.

Proposition 2.3. *The subsequent claims are identical for a ring R and $a \in K(R)$:*

(1) $ra = a^{k-1}ra$ for all r is in R or, $aR(1 - a^{k-1}) = 0$.

(2) $ra = a^{k-1}ra$ whenever r is in $K(R)$.

(3) $ra = a^{k-1}ra$ where r is an element which is Von neumann regular.

(4) $ra = a^{k-1}ra$ whenever r is in $N(R)$.

(5) $ra = a^{k-1}ra$ whenever r is in R and $r^2 = 0$.

(6) $ra = a^{k-1}ra$ whenever r is in $q(R)$.

3 $q-k-$Abelian Rings

We begin with the definition of $q-k-central$ elements and various characteristics of $q-k-abelian$ rings using k-potent elements in a ring R. Moreover we discuss the relationship between regular elements and $q-k-central$ elements of R.

Definition 3.1. A k-potent element a in R is called $quater-k-central$ or $(q-k-central)$ if $aR(1-a^{k-1})Ra=0$ or $aRa'Ra=0$ with the complimentary k-potent $a'=1-a^{k-1}$. The set of all $q-k-central$ elements of R which is represented by $q-K(R)$.

Definition 3.2. A ring R is said to be $q-k-$abelian if all the k- potents of the ring are $q-k-central$ or $K(R)=q-K(R)$.

Lemma 3.3. *A ring R is* $q-k-abelian \iff axya=axa^{k-1}ya$ *for any* $a\in K(R)$*;* $x,y\in R$.

Proof. Let $x,y\in R$ and $a\in K(R)$. Let R be a $q-k-abelian$ ring then by Definition 3.1 we get, $aR(1-a^{k-1})Ra=0$. Thus, $ax(1-a^{k-1})ya=0 \implies axya=axa^{k-1}ya$ for all $x,y\in R$. The converse part is clear. □

Theorem 3.4. *If* a *is* $q-k-central$*, the subsequent claims are identical for a ring* R *and* $a\in K(R)$ *:*

(1) $ar(1-a^{k-1})sa=0$ *for all* $r,s\in R$ *or* $a\in q-k(R)$*;*

(2) $ar(1-a^{k-1})sa=0$ *whenever* $r,s\in U(R)$*;*

(3) $ar(1-a^{k-1})sa=0$ *whenever* $r,s\in I(R)$*;*

(4) $ar(1-a^{k-1})sa=0$ *whenever* $r^2=s^2=0$*,* $k\geq 2$*;*

(5) $ar(1-a^{k-1})sa=0$ *whenever* $r\in a^{k-1}Ra'$ *and* $s\in a'Ra^{k-1}$*. Where* $a'=1-a^{k-1}$.

Proof. It is clear that $(1) \implies (2),(3),(4)$ and (5).
$(3) \implies (1)$ Let $x,y\in R$. Let us consider $r=a^{k-1}-a^{k-1}x+a^{k-1}xa^{k-1}$ and $s=a^{k-1}-ya^{k-1}+a^{k-1}ya^{k-1}$ Then we see $r^2=(a^{k-1}-a^{k-1}x+a^{k-1}xa^{k-1})=r$ and $s^2=(a^{k-1}-ya^{k-1}+a^{k-1}ya^{k-1})=s$. So, $r,s\in I(R)$. Now let, $a'=1-a^{k-1}$. By assumption $ara'sa=0$ $\implies ar(1-a^{k-1})sa=0 \implies arsa=ara^{k-1}sa \implies a(a^{k-1}-a^{k-1}x+a^{k-1}xa^{k-1})(a^{k-1}-ya^{k-1}+a^{k-1}ya^{k-1})a=a(a^{k-1}-a^{k-1}x+a^{k-1}xa^{k-1})a^{k-1}(a^{k-1}-ya^{k-1}+a^{k-1}ya^{k-1})a$ $\implies axya=axa^{k-1}ya \implies ax(1-a^{k-1})ya=0 \implies axa'ya=0\,\forall x,y\in R$.
$(2) \implies (4)$ Let $r,s\in R$ such that $r^2=s^2=0$. Then $1+r,1+s\in U(R)$. So by assumption $a(1+r)a'(1+s)a=0 \implies (a+ar)(1-a^{k-1})(a+sa)=0 \implies arsa-ara^{k-1}sa=0$ $\implies ara'sa=0$.
$(4) \implies (1)$ Let, $r,s\in R$ we see $(a^{k-1}ra')^2=a^{k-1}ra'aa^{k-2}ra'=0$ and $(a'sa^{k-1})^2=a'sa^{k-2}aa'sa^{k-1}=0$. So, by assumption $a(a^{k-1}ra')a'(a'sa^{k-1})a=0 \implies ara'sa=0$.
$(5) \implies (1)$ Let, $r,s\in R$ then $a^{k-1}ra'$ is in $a^{k-1}Ra'$ and $a'sa^{k-1}$ is in $a'Ra^{k-1}$. So, by assumption $a(a^{k-1}ra')a'(a'sa^{k-1})=0 \implies ara'a'a'sa=0 \implies ara'sa=0$. Hence the proof. □

Remark 3.5. For any ring R if a is left semicentral/right semicentral k-potent then a is $q-k-central$. As, $0=(1-a)Ra=(1+a+a^2+...+a^{k-2})(1-a)Ra=(1-a^{k-1})Ra=aR(1-a^{k-1})Ra$. Similarly for right semicentral k-potent.

The following result is a modified Ánh-Birkenmeier-Van Wyk Theorem [[10], Lemma 3.4].

Theorem 3.6. *The subsequent claims are identical for the ring* R *and* $a\in K(R)$ *with complementary* k*-potent* $a'=1-a^{k-1}$*:*

(1) $a\in q-K(R)$.

(2) The map $\psi:R\to a^{k-1}Ra^{k-1}$ *defined by* $\psi(r)=a^{k-1}ra^{k-1}$ *for* $k\geq 2$ *is a ring homomorphism that sends unity to unity. Where* a^{k-1} *is the unity element in* $a^{k-1}Ra^{k-1}$.

(3) aRa' *is a right ideal in R.*

(4) $a'Ra$ *is a left ideal in R.*

Proof. (1) $\iff$ (2) Let $a \in q-K(R)$ then $\psi(1)=a^{k-1}.1.a^{k-1}=a^{k-1}$
Let, $r_1, r_2 \in R$ then

$$\begin{aligned}\psi(r_1+r_2) &= a^{k-1}(r_1+r_2)a^{k-1}\\ &= a^{k-1}r_1a^{k-1}+a^{k-1}r_2a^{k-1}\\ &= \psi(r_1)+\psi(r_2).\end{aligned}$$

Agian,

$$\begin{aligned}\psi(r_1r_2) &= a^{k-1}r_1r_2a^{k-1}\\ &= a^{k-2}(ar_1r_2a)a^{k-2}\\ &= a^{k-2}(ar_1a^{k-1}r_2a)a^{k-2} \iff a\in q-K(R)\\ &= a^{k-1}r_1a^{k-1}.a^{k-1}r_2a^{k-1} \iff a\in q-K(R)\\ &= \psi(r_1)\psi(r_2) \iff a\in q-K(R).\end{aligned}$$

Therefore, ψ is a ring homomorphism that sends unity to unity if and only if $a \in q-K(R)$.

(1) $\iff$ (3) Let, $a \in q-K(R)$ then for any $r, s \in R$
$ara'sa=0 \implies ara'sa^{k-1}=0 \implies ara's(1-a')=0 \implies (ara')s=a(ra's)a' \in aRa'$.
So, aRa' is a right ideal.
conversely, let aRa' is a right ideal.
So, we have $(ara')s \in aRa'$ for any $r, s \in R$. So, $ara's=ar'a'$ for some $r' \in R$ and hence $ara'sa=(ara's)a=ar'a'a=0$. So, $a \in q-K(R)$.
Similarly, we can prove (1) $\iff$ (4). □

Remark 3.7. It is observed that if both $a, b \in K(R)$ so that $a, b \in q-K(R)$ then aRb is not an ideal in R by following example.

Example 3.8. Let, $R=\mathbb{T}_2(S)$ and $a=b=\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}$ then, $a, b \in q-K(R)$ for $k=3$, as $1-a^2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}^2=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. Therefore,
$\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}t & s\\ 0 & p\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}t' & s'\\ 0 & p'\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ for any $t, s, p, t', s', p' \in S$.
But $aRb=\{\begin{pmatrix}t & 0\\ 0 & 0\end{pmatrix} : t \in S\}$ is not an ideal.

Corollary 3.9. *Suppose R is a ring and $a \in q-K(R)$ then the map ψ defined in Theorem 3.6(2) takes*

(1) unit elements of R to unit elements of $a^{k-1}Ra^{k-1}$;

(2) nilpotent elements of R to nilpotent elements of $a^{k-1}Ra^{k-1}$;

(3) idempotent elements of R to idempotent elements of $a^{k-1}Ra^{k-1}$;

(4) right (left) semicental idempotents of R to right (left) semicentral idempotents of $a^{k-1}Ra^{k-1}$;

(5) $q-k-central$ elements of R to $q-k-central$ elements of $a^{k-1}Ra^{k-1}$.

Proof. As ψ is a unital ring homomorphsim so (1) to (4) are clear.
For (5) let, $g \in q-K(R)$ for any $r, s \in R$ we have
$(a^{k-1}ga^{k-1})(a^{k-1}ra^{k-1})(1-(a^{k-1}ga^{k-1})^{k-1})(a^{k-1}sa^{k-1})(a^{k-1}ga^{k-1})$
$=\psi(g)\psi(r)(\psi(1)-\psi(g)^{k-1})\psi(s)\psi(g)=\psi(g)\psi(r)(\psi(1)-\psi(g^{k-1}))\psi(s)\psi(g)$
$=\psi(gr(1-g^{k-1})sg)=\psi(0)=0.$ □

Remark 3.10. It is observed that from Corollary 3.9(5), when R is $q-k-abelian$ ring, consequently each ring of the type $a^{k-1}Ra^{k-1} \subseteq R$ is also $q-k-abelian$.

Corollary 3.11. *For any $a \in q{-}K(R)$ if $r \in N(R)$ then $\{ra^{k-1}, a^{k-1}r, a^{k-1}ra^{k-1}, ara, rar, ra^{k-1}r\} \in N(R)$.*

Proof. Let $r^n = 0$ then we see

$$\begin{aligned}(ra^{k-1})^{n+1} &= ra^{k-1}.ra^{k-1}...ra^{k-1}\\ &= ra^{k-2}(ara^{k-1}ra)a^{k-2}....ra^{k-1}\\ &= ra^{k-2}(ar^2a)a^{k-2}....ra^{k-1}\end{aligned}$$

In this way we will reach a certain stage where we have r^n and thus $(ra^{k-1})^{n+1} = 0$. Similarly we can check the others. □

Remark 3.12. It is observed from Theorem 3.6 that R is $q-k-abelian$ if and only if for all $aR(1-a^{k-1})$ is a right ideal or $(1-a^{k-1})Ra$ is a left ideal for every $a \in K(R)$. If $k=2$ then $q-k-abelian$ and $q-abelian$ rings coincide. In general $q-abelian$ rings sit inside $q-k-abelian$ rings. Further, it is apparent that a ring is $q-k-abelian$ if it is abelian. As, for any $a \in K(R)$ we have, $a^{k-1} \in I(R)$ also, $1-a^{k-1} \in I(R)$. But $q-k-abelain$ ring need not be abelian by the following example.

Example 3.13. Let us consider the abelian ring $S = \mathbb{Z}_6$ and thus $R = \mathbb{T}_2(S)$ is a $q-k-abelian$ ring for $k=2$ by [[12], Theorem 3.5].
Now we show that $R = \mathbb{T}_2(S)$ is not semi abelian ring and hence not abelian.
we take $A = \begin{pmatrix}3 & 0\\ 0 & 4\end{pmatrix} \in K(\mathbb{T}_2(S))$ but, $\begin{pmatrix}3 & 0\\ 0 & 4\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}4 & 0\\ 0 & 3\end{pmatrix} = \begin{pmatrix}0 & 3\\ 0 & 0\end{pmatrix} \neq \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$.
Also, $\begin{pmatrix}4 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}3 & 0\\ 0 & 4\end{pmatrix} = \begin{pmatrix}0 & 4\\ 0 & 0\end{pmatrix} \neq \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$. So, A is not semicentral.

Proposition 3.14. *For a ring R and $r \in N(R)$, if $a \in q-K(rR)$ then $a = 0$. It also holds if rR is replaced by Rr.*

Proof. We consider a nilpotent element r of index m. Let $a \in q-K(rR)$ then $a = rs$ for some $s \in R$ then we have $a = a^k = a(a^{k-2})a = a((rs)^{k-2})a = ar(s(rs)^{k-3})a = ara^{k-1}(s(rs)^{k-2})a = ara(a^{k-2}(s(rs)^{k-2}))a = ar^2s(a^{k-2}(s(rs)^{k-2}))a = \ldots.$ In this way after a finite number of steps we obtain, r^m which is 0 and thus $a = 0$. □

Following results are extension of Wei and Li's [[17], Theorems (2.4), (2.8) and (2.9)].

Proposition 3.15. *(1) For any ring R and $a \in q-K(R)$ such that $RaR = R$ then $a^{k-1} = 1$.*

(2) $q-k-abelian$ ring is Dedekind finite. But the converse is not true.

Proof. (1) $aR(1-a^{k-1})Ra = 0 \implies RaR(1-a^{k-1})RaR = 0 \implies R(1-a^{k-1})R = 0$. As, $1 \in R$. So, $a^{k-1} = 1$.

(2) Let R be $q-k-abelian$ ring and $xy = 1$. We consider $a = yx$ then $a^2 = yxyx = yx = a$ so, $a \in I(R) \subseteq K(R)$. So, $a^k = a$ for all $k \geq 2$. Now, $xay \in RaR$ also, $xay = xyxy = 1 \in R$. So, $RaR = R$. Therefore by part (1) $a^{k-1} = 1 \implies (yx)^{k-1} = 1 \implies (yxy...xyx)_{(k-1)copies} = 1 \implies yx = 1$.

For the converse part we consider $\mathbb{T}_3(R)$ upper triangular matrix ring which is Dedekind finite. But $A = diag(1,0,1) = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \notin q-K(R)$ for $k=2$.

□

From Proposition 3.15(1), for any $a \in K(R)$ we obtain $a^{k-1} \in I(R)$ so any simple ring R has only trivial $q-k-central$ elements. Thus as a consequences we obtain the subsequent results.

Proposition 3.16. *For a ring R and M is a maximal ideal of R. If $a \in q-K(R)$ then a or $a' = 1 - a^{k-1} \in M$*

Proof. We have $\frac{R}{M}$ is a simple ring if M is a maximal ideal of R. Therefore $\frac{R}{M}$ has only trivial k-potent elements and consequently $q-k-central$ elements. □

Proposition 3.17. *For a ring R and $a \in q-K(R)$. If $M \subseteq R$ is a maximal right ideal, or a maximal left ideal then $a \in M$ or $a' = 1 - a^{k-1} \in M$.*

Proof. Considering that, $a, a' \notin M$ which is maximal right ideal. As $1 \in R$, so there exists $m, m' \in M$ and $r, s \in R$ such that $1 = ar + m = a's + m'$. Now, $a = 1.a = (a's + m')a = a'sa + m'a = (ar + m)a'sa + m'a = ara'sa + ma'sa + m'a = m(a'sa) + m'a \in M$ which is a contradiction. So, $a \in M$ or $a' = 1 - a^{k-1} \in M$. Similarly for maximal left ideal. □

Proposition 3.18. *For a ring R and $a \in q-K(R)$ and $ar(1-a^{k-1}) \neq 0$ for some $r \in R$.*

(1) If aR is minimal left ideal then $a^2 = 0$.

(2) If aR is minimal right ideal then $a^2 = 0$.

Proof. (1) Let $a' = 1 - a^{k-1}$. We have $ara' \neq 0$ for some r is in R and consider aR is minimal left ideal. Then $0 \subset ara'R \subseteq aR \implies aR = ara'R \implies aR \subseteq aRa'R \implies aRa \subseteq aRa'Ra = 0$. Since, $1 \in R$ we have $a^2 = 0$. Similarly we can prove (2). □

We note that for any 2×2 upper triangular matrix $\begin{pmatrix} t & s \\ 0 & p \end{pmatrix}$ to be k-potent we must have t, p also k-potent. This is because, $\begin{pmatrix} t & s \\ 0 & p \end{pmatrix}^k = \begin{pmatrix} t & s \\ 0 & p \end{pmatrix} \implies t^k = t$ and $p^k = p$.

Proposition 3.19. *Let S be any arbitrary ring and $R = \mathbb{T}_2(S)$. Consider $A = \begin{pmatrix} t & s \\ 0 & p \end{pmatrix} \in K(R)$, where necessarily $t, p \in K(S)$. If $tS(1-t^{k-1}) = 0$ and $(1-p^{k-1})Sp = 0$ then $T \in q-K(R)$.*

Proof. Let $A' = 1 - A^{k-1} = \begin{pmatrix} 1-t^{k-1} & * \\ 0 & 1-p^{k-1} \end{pmatrix}$ then for any $X = \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}$, $Y = \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix}$ we have

$$\begin{aligned} AXA' &= \begin{pmatrix} t & s \\ 0 & p \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \begin{pmatrix} 1-t^{k-1} & * \\ 0 & 1-p^{k-1} \end{pmatrix} \\ &= \begin{pmatrix} ta_1(1-t^{k-1}) & * \\ 0 & pa_3(1-p^{k-1}) \end{pmatrix} \\ &= \begin{pmatrix} t & s \\ 0 & p \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \begin{pmatrix} 1-t^{k-1} & * \\ 0 & 1-p^{k-1} \end{pmatrix} \\ &= \begin{pmatrix} ta_1(1-t^{k-1}) & * \\ 0 & pa_3(1-p^{k-1}) \end{pmatrix} \\ &= \begin{pmatrix} 0 & * \\ 0 & pa_3(1-p^{k-1}) \end{pmatrix} (since,\ tS(1-t^{k-1}) = 0) \end{aligned}$$

Again

$$A'YA = \begin{pmatrix} 1-t^{k-1} & * \\ 0 & 1-p^{k-1} \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} \begin{pmatrix} t & s \\ 0 & p \end{pmatrix}$$
$$= \begin{pmatrix} (1-t^{k-1})b_1 t & * \\ 0 & (1-p^{k-1})b_3 p \end{pmatrix}$$
$$= \begin{pmatrix} (1-t^{k-1})b_1 t & * \\ 0 & 0 \end{pmatrix} (Since,\ (1-p^{k-1})Sp = 0)$$

Now, $AXA'YA = AXA'.A'YA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Therefore, $A \in q - K(R)$.

□

Proposition 3.20. *Let* $R = \mathbb{T}_2(S)$. *If* R *is* $q - k - abelian$ *then* S *is* $q - k - abelian$.

Proof. Let $a \in K(S)$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \in K(R)$ then the complimentary k-potent $A' = \begin{pmatrix} 0 & 0 \\ 0 & 1-a^{k-1} \end{pmatrix} \in K(R)$. We consider $X = sE_{12}, Y = rE_{22}$. Then, $AXA'YA = 0 \implies (E_{11} + aE_{22})sE_{12}A'rE_{22}(E_{11} + aE_{22}) = 0 \implies as(1 - a^{k-1})ra = 0$, as $s, r \in S$ are arbitrary. So, $a \in q - K(S)$. Therefore, S is $q - k - abelian$.

□

Proposition 3.21. *Let* S *be a ring. If* S *is abelian and* $aSb = 0$ *for all* $a, b \in K(S)$, *where both* a, b *non trivial then* $R = \mathbb{T}_2(S)$ *is* $q - k - abelian$.

Proof. Let S be an $abelian$ ring. We consider $A = \begin{pmatrix} a & s \\ 0 & b \end{pmatrix} \in K(\mathbb{T}_2(S))$, where necessarily $a, b \in K(S)$ and the complimentary k-potent of A is $A' = \begin{pmatrix} a' & s' \\ 0 & b' \end{pmatrix}$ where, $a' = 1 - a^{k-1}$, $b' = 1 - b^{k-1}$ and for some $s' \in S$.
Now, we show $\mathbb{T}_2(S)$ is $q - k - abelian$, for this we have to prove $AXA'YA = 0$ for any $X, Y \in \mathbb{T}_2(S)$.
Let $X = \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}$ and $Y = \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} \in \mathbb{T}_2(S)$.
Now, $AXA' = \begin{pmatrix} a & s \\ 0 & b \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \begin{pmatrix} a' & s' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} aa_1a' & aa_1s' + (aa_2 + sa_3)b' \\ 0 & ba_3b' \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$,
as S is abelian so $aa_1a' = aa'a_1 = 0, ba_3b' = bb'a_3 = 0$.
Again, $YA = \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} \begin{pmatrix} a & s \\ 0 & b \end{pmatrix} = \begin{pmatrix} b_1a & b_1s + b_1b \\ 0 & b_3b \end{pmatrix}$
Therefore, $AXA'YA = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_1a & b_1s + b_1b \\ 0 & b_3b \end{pmatrix} = \begin{pmatrix} 0 & *b_3b \\ 0 & 0 \end{pmatrix}$
Here, $*b_3b = (aa_1s' + (aa_2 + sa_3)b')b_3b = aa_1s'b_3b + aa_2p'b_3b + sa_3b'b_3b = 0$, as $aSb = 0$, S is abelian and $b'b = 0$. Therefore, $AXA'YA = 0$. □

Remark 3.22. Abelian condition of S in the above proposition can not be ignored. For example we take the non abelian ring $S = \mathbb{H}/(\mathbb{Z}_{13})$, Quaternion ring with co-efficients from $\mathbb{Z}_{13}$. Consider, $t = 7+4i$ which is an idempotent and its complimentary idempotent for $t' = 1-t = -6-4i = 7-4i = \bar{t}$ thus $\begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix} \in K(\mathbb{T}_2(S))$ But, $\begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix} \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{t} & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix} =$

$$\begin{pmatrix} 0 & tj\overline{tt} \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

As, $tj\overline{tt} = 7j + 4k$. Therefore, $\mathbb{T}_2(S)$ is not $k - abelian$ for $k = 2$.

Theorem 3.23. *For any non zero ring S. Then $R = \mathbb{T}_n(S)$ is not $q - k - abelian$ ring for $n \geq 3$.*

Proof. Since, we have a corner ring of R which is isomorphic to $\mathbb{T}_3(S)$. For any $A \in K(R)$ we have the corner ring of the type $A^{k-1}RA^{k-1}$ is isomorphic to $\mathbb{T}_3(S)$. So, by Remark 3.10 it is enough to prove $\mathbb{T}_3(S)$ is not $q - k - abelian$.

Let $A = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$, $a \neq 0$ with the complimentary idempotent $A' = \begin{pmatrix} a' & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a' \end{pmatrix}$, where $a' = 1 - a^{k-1}$. Let us consider $X = E_{12}, Y = E_{23} \in R$, so we have, $AXA'YA = AE_{12}A'E_{23}A = a^2E_{13} \neq 0$. Therefore, A is not $q - k - central$ consequently $\mathbb{T}_3(S)$ is not $q - k - abelian$. So, R is not $q - k - abelian$. □

4 Some applications

In accordance with [13], an element x in a ring R is said to as π-regular if there exists y in R such that $x^n = x^n y x^n$; $n \geqslant 1$ and for $n = 1$, x is referred to as Von Neumann regular. If $x^n = yx^{n+1}$ then x is called strongly π-regular and x is said to be strongly regular if $n = 1$. A ring R is said to be Von Neumann regular, strongly regular, π-regular and strongly π-regular if every elements of R is Von Neumann regular, strongly regular, π-regular and strongly π-regular respectively. A ring R is referred to as a unit-regular, if for any $a \in R$ such that $a = aua$, where u is in $U(R)$. We have unit regular implies Von Neumann regular. It is well known that a ring R is said to be strongly regular if and only if it is reduced and Von Neumann regular [[11], Ex. 12.6A].

As an immediate consequence of Theorem 4.6, we obtain the subsequent remark.

Remark 4.1. Let us consider R be a ring and $x \in R$ is a regular element (Von Neumann) such that $l_R(x) = Rx^{n-1}$, $n \geq 3$ then we have R is not a $q - k - abelian$ ring by [[5], Theorem 2.4]. Here, $l_R(x) = \{a \in R : ax = 0\}$

Lemma 4.2. *Let us consider a $q - k - abelian$ ring R and x is in R. Then x is strongly regular whenever x is Von Neumann regular.*

Proof. For some $y \in R$ we get, $x = xyx$ if x is Von Neumann regular. Let $a \in K(R)$ and let $a = yx$, then $a^k = (yxyx...yx)_{(k)copies} = yxyx = yx = a$, $a^{k-1} = (yxyx...yx)_{(k-1)copies} = yx = a$ and $x = xa$. Since, $a = a^k = a^{k-1}aa^{k-1} = ayxa = aya^{k-1}xa$, by Lemma 3.3. Thus, $a = aya^{k-1}xa = ayaxa = ayyxxa = ayyxx = ay^2x^2$. Thus, we get $x = xa = xay^2x^2 = xy^2x^2$. In a similar way, we can prove that $x = x^2y^2x$. Hence x is strongly regular.

□

Corollary 4.3. *If x is an unit π-regular then there is a k-potent, $a \in K(R)$ so that ax and xa are Von Neumann regular.*

Proof. If x is an unit π-regular then there exists $n \geqslant 1$, such that $x^n = x^n u x^n$, where $u \in U(R)$, this implies that x^n is Von Neumann regular. So by Lemma 4.2, x^n is strongly regular. Let $a = x^n u$ then $a^k = (x^n u x^n u ... x^n u)_{(k)copies} = x^n u x^n u = x^n u = a$. Thus, $a^k = a$ and so a is a k-potent. Also, $x^n = ax^n$ and $a^{k-1} = (x^n u x^n u ... x^n u)_{(k-1)copies} = x^n u \implies x^n = a^{k-1}u^{-1} = a^{k-1}v$, for $v = u^{-1}$. Since, $(ax)(x^{n-1}u)(ax) = ax^n uax = aa^{k-1}vuax = a^k vuax = a^k 1ax = a^k x = ax$, as $a^{k-1} = x^n u = a$. This shows that ax is Von Neumann regular.

Similarly, it can be proved that xa is also Von Neumann regular by letting $a = ux^n$.

□

In the following results we have tried to build a new criteria for a k-potent element to be $q - k - central$ in terms of additive commutators.

Proposition 4.4. *The subsequent claims are identical for a ring R and $a \in K(R)$:*

(1) $aR.[a^{k-1}, R] = 0$;

(2) $a.[a^{k-1}, R] = 0$;

(3) $ar - ara^{k-1} = 0$ *for all* $r \in R$.

Proof. (1) $\implies$ (2) trivial as $1 \in R$.
(2) $\implies$ (3) Assume, $a[a^{k-1}, R] = 0$ then $a(a^{k-1}r - ra^{k-1}) = 0$ for all $r \in R$. Therefore, $ar - ara^{k-1} = 0$ for all $r \in R$ as, $a^k = a$.
(3) $\implies$ (1) For all $r \in R$, we assume, $ar - ara^{k-1} = 0$.
Now, $aR[a^{k-1}, R] = ar(a^{k-1}s - sa^{k-1}) = (ara^{k-1})s - arsa^{k-1} = (ar)s - arsa^{k-1} = 0 = a(rs) - a(rs)a^{k-1} = 0$ for all $r, s \in R$. □

Similar to Proposition 4.4, we have the next proposition.

Proposition 4.5. *The subsequent claims are identical for a ring R and $a \in K(R)$:*

(1) $[R, a^{k-1}].aR = 0$;

(2) $[R, a^{k-1}].a = 0$;

(3) $ra - a^{k-1}ra = 0$ *for all* $r \in R$.

Based on the Propositions 4.4 and 4.5, we demonstrate the subsequent result.

Theorem 4.6. *For a ring R and $a \in K(R)$. Then* $a \in q - K(R) \iff a[a^{k-1}, R][R, a^{k-1}]a = 0$

Proof. Let $a \in K(R)$ then $a[a^{k-1}, R][R, a^{k-1}]a = 0$
$\iff a(a^{k-1}r - ra^{k-1})(sa^{k-1} - a^{k-1}s)a = 0$
$\iff (ar - ara^{k-1})(sa - a^{k-1}sa) = 0$
$\iff arsa - ara^{k-1}sa - ara^{k-1}sa + ara^{k-1}sa = 0$
$\iff arsa - ara^{k-1}sa = 0$ for all $r, s \in R$
$\iff a \in q - K(R)$. □

Corollary 4.7. *A ring R is $q - k - abelian$* $\iff a[a^{k-1}, R][R, a^{k-1}]a = 0$ *for all* $a \in K(R)$.

References

[1] A. Badawi, On Abelian π-regular Rings, *Comm. Algebra* **25** (4) (1997) 1009–1021.

[2] G. F. Birkenmeier, Idempotents and Completely Semiprime Ideals, *Comm. Alg.*, **11**(1983), 567–580.

[3] W. Chen (2007), On semiabelian π-regular rings, Int. J. Math. Math. Sci. 2007:63171.

[4] W. Chen and S. Y. Cui (2010), On π-regularity of general rings. Commun. Math. Research 26:313–320

[5] D. R. Goyal and D. Khurana, A characterisation of matrix rings, Bulletin of the Australian Mathematical Society 107.1 (2023): 95-101.

[6] J. Han, Y. Lee and S. Park (2014), Semicentral idempotents in a ring. J. Korean Math. Soc. 51:463–472

[7] H. Heatherly and R. P. Tucci, Central and Semicentral Idempotents, *Kyungpook Math. J.* **40**(2000), 255–258

[8] H. M. I Hoque and H. K. Saikia, A study on weakly tri normal and quasi tri normal rings, *Palestine Journal of Mathematics,* vol. **12** (2), (2023), 125–132

[9] P. Kanwar, A. Leroy and J. Matczuk, (2013). Idempotents in ring extensions. J. Algebra 389:128–136

[10] P. N. Ánh, G. F. Birkenmeier and L. V. Wyk, (2016). Idempotents and structures of rings. Linear Multilinear Algebra 64:2002–2029.

[11] Lam, Tsit-Yuen. Exercises in classical ring theory. Springer Science & Business Media, 2006.

[12] T. Y. Lam (2022): An introduction to q-central idempotents and q-abelian rings, Communications in Algebra, DOI: 10.1080/00927872.2022.2123921

[13] J. Lambek, Lectures on rings and modules, *AMS Chelsea Publishing.*

[14] Lomp, C., Matczuk, J. (2017). A note on semicentral idempotents. Commun. Algebra 45:2735–2737

[15] A. Sahan (2019), Elementary reduction of idempotent matrices over semiabelian rings. In Proceedings, XII International Algebraic Conference in Ukraine.

[16] J. Wei and L. Li (2010), Quasi-normal rings. Commun. Algebra 38:1855–1868.

[17] J. Wei and N. Li (2011), Some notes on semiabelian rings. Int. J. Math. Math. Sci. 2011:154636

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