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RADICAL ANTI-INVARIANT LIGHTLIKE SUBMANIFOLDS OF KAEHLAR STATISTICAL MANIFOLD

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Abstract In this paper, we introduce Kaehlar statistical manifold and investigated Radical anti-invariant lightlike submanifolds of Kaehlar statistical manifold. One example related to these concepts is also presented.

1 Introduction

The theory of submanifolds of a semi-Riemannian manifold is one of the most interesting topics in differential geometry. Lightlike submanifolds were introduced and studied by Duggal and Bejancu [1, 2, 3]. Kilic and Sahin [4] introduced radical anti-invariant lightlike submanifolds of a semi-Riemannian product manifold. Ahmad [5] discussed submanifolds in a Riemannian manifold with a golden structure. Qayyoom [6] investigated hypersurfaces immersed in a golden Riemannian manifold with a golden structure. Acet et. al. [7] discussed lightlike submanifolds of a para-Sasakian manifold admitting semi-metric connection.

A statistical manifold is a modern and fascinating subfield of manifolds that originated from the study of geometric structures on sets of certain probability distributions. It is a differentiable manifold where each point represents a probability distribution. The set of all probability measures is a statistical manifold with infinite dimensions where each point in parameter space is connected to a probability density function. Many ideas from the Euclidean space can be generalised to the statistical manifold. Additionally, these manifolds are geometrically described as Riemannian manifolds with a particular affine connection. Effron [8] first emphasized the role of differential geometry in statistics in 1975. Aydin [9] obtained generalized Weingarten inequalities for submanifolds of statistical manifolds of constant curvature. Rao [10] was the one to relate geometry with statistics resulting in the formation of the statistical manifold. He used Fisher information matrix to introduce the concept of Riemannian metric. Although various researchers worked in this direction in the subsequent years, yet an appreciable amount of work was done by Amari [11, 12, 13] and Simon [14] when they introduced statistical manifold on the basis of information geometry which is the study of probability and information from the view point of differential geometry having applications in the fields of statistics and applied mathematics. Then, Vos [15] developed certain fundamental equations and structural formulae for the statistical manifold. Thereafter, Kurose [16] developed the concept of holomorphic statistical manifold which was further elaborated by Furuhata et.al [17, 18, 19, 20]. Balgeshir [21, 22] introduced submanifolds of Sasakian statistical manifolds and semi-Riemannian statistical manifolds, Siddiqui et. al. [23] discusses the trans-Sasakian manifolds. Bahadir [24] discussed lightlike geometry of indefinite Sasakian statistical manifolds, Ahmad et. al. [25] introduced lightlike submanifolds of an indefinite LP-Sasakian statistical manifolds, Prasad [26] dicussed conformal anti-invariant of Kaehlar maniolds and Kaur [27] introduces distributions in

CR-lightlike submanifolds of Kaehler statistical manifold.

Almost Hermitian statistical manifolds have been the subject of a significant amount of effort in recent years. Keeping in focus the above facts, in this paper we introduce radical anti-invariant lightlike submanifolds of Kaehlar Statistical manifold.

In section 2, we define statistical manifolds from differential geometry point of view. Further, Kaehlar statistical manifold is defined and some results are given for further use.

In section 3, we consider radical anti-invariant lightlike submanifolds of Kaehlar statistical manifold. We characterize integrability and self adjoint of some distributions and also given one examples on radical anti-invariant lightlike submanifolds.

2 Basic concepts

A submanifold $\mathbb{O}^m$ immersed in a semi-Riemannian manifold $(\widetilde{\mathbb{O}}^{m+n}, \widetilde{\mathfrak{q}})$ is called a lightlike submanifold if it is a lightlike manifold with respect to the metric $\mathfrak{q}$ induced from $\widetilde{\mathfrak{q}}$ and the radical distribution $Rad\ T\mathbb{O}$ is of rank r, where $1 \leq r \leq m$. Let $S(T\mathbb{O})$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad\ T\mathbb{O}$ in $T\mathbb{O}$, that is

$$T\mathbb{O} = Rad\ T\mathbb{O} \perp S(T\mathbb{O}).$$

Consider a screen transversal vector bundle $S(T\mathbb{O}^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad\ T\mathbb{O}$ in $T\mathbb{O}^{\perp}$. Since for any local basis $\{\zeta_i\}$ of $Rad\ T\mathbb{O}$, there exists a local null frame $\{J_i\}$ of section with values in the orthogonal complement of $S(T\mathbb{O}^{\perp})$ in $[S(T\mathbb{O})]^{\perp}$ such that $\widetilde{\mathfrak{q}}(\zeta_i, J_j) = \delta_{ij}$, it follows that there exist a lightlike transversal vector bundle $ltr(T\mathbb{O})$ locally spanned by $\{J_i\}$ [[3], pg-144]. Let $tr(T\mathbb{O})$ be a complementary (but not orthogonal) vector bundle to $T\mathbb{O}$ in $T\widetilde{\mathbb{O}}|_{\mathbb{O}}$. Then

$$tr(T\mathbb{O}) = ltr(T\mathbb{O}) \perp S(T\mathbb{O}^{\perp}),$$

$$T\widetilde{\mathbb{O}}|_{\mathbb{O}} = S(T\mathbb{O}) \perp [Rad\ (T\mathbb{O}) \oplus ltr(T\mathbb{O})] \perp S(T\mathbb{O}^{\perp}).$$

Following are four subcases of a lightlike submanifold $(\mathbb{O}, \mathfrak{q}, S(T\mathbb{O}), S(T\mathbb{O}^{\perp})$.

Case 1: r-lightlike if $r < min\{m, n\}$,

Case 2: Co-isotropic if $r = n < m$; $S(T\mathbb{O}^{\perp}) = 0$,

Case 3: Isotropic if $r = m < n$; $S(T\mathbb{O}) = 0$,

Case 4: Totally lightlike if $r = m = n$; $S(T\mathbb{O}) = 0 = S(T\mathbb{O}^{\perp})$.

The Gauss and Weingarten equations are

$$\widetilde{\nabla}_G H = \nabla_G H + h(G, H),\ \forall\ G,\ H \in \Gamma(T\mathbb{O}), \tag{2.1}$$

$$\widetilde{\nabla}_G J = -A_P G + \nabla^t_P G,\ \ \forall\ P \in \Gamma(tr(T\mathbb{O})), \tag{2.2}$$

where $\{\nabla_G H,\ A_P G\}$ and $\{h(G, H),\ \nabla^t_G P\}$ belongs to $\Gamma(T\mathbb{O})$ and $\Gamma(tr(T\mathbb{O}))$, respectively, ∇ and ∇^t are linear connections on $\mathbb{O}$ and on the vector bundle $tr(T\mathbb{O})$, respectively. Moreover, we have

$$\widetilde{\nabla}_G H = \nabla_G H + h^l(G, H) + h^s(G, H), \tag{2.3}$$

$$\widetilde{\nabla}_G J = -A_J G + \nabla^l_G J + D^s(G, J), \tag{2.4}$$

$$\widetilde{\nabla}_G W = -A_W G + \nabla^s_G W + D^l(G, W) \tag{2.5}$$

for every $G,\ \ H \in \Gamma(T\mathbb{O}),\ \ J \in \Gamma(ltr(T\mathbb{O}))$ and $W \in \Gamma(S(T\mathbb{O}^{\perp}))$. Then, by using (2.1), (2.3)-(2.5) and the fact that $\widetilde{\nabla}$ is a metric connection, we get

$$\widetilde{\mathfrak{q}}(h^s(G, H), L) + \widetilde{\mathfrak{q}}(H, D^l(G, L)) = \mathfrak{q}(A_L G, H). \tag{2.6}$$

[21] In general, the induced connection ∇ on $\mathbb{O}$ is not a metric connection, by using (2.3), we have

$$(\nabla_G \mathfrak{q})(H, L) = \widetilde{\mathfrak{q}}(h^l(G, H), L) + \widetilde{\mathfrak{q}}(h^l(G, L), H) \tag{2.7}$$

for any $G,\ H \in \Gamma(T\mathbb{O})$, where $\{\nabla_G H, A_J G, A_W G\} \in \Gamma(T\mathbb{O})$, $\{h^l(G,H), \nabla^l_G J\} \in \Gamma(ltr(T\mathbb{O}))$ and $\{h^s(G,H), \nabla^s_G J\} \in \Gamma(S(T\mathbb{O}^\perp))$. If we set $\mathfrak{B}^l(G,H) = \widetilde{\mathfrak{q}}(h^l(G,H),\zeta)$, $\mathfrak{B}^s(G,H) = \widetilde{\mathfrak{q}}(h^s(G,H),\zeta)$, $\tau^l(G) = \widetilde{\mathfrak{q}}(\nabla^l_G J,\zeta)$ and $\tau^s(G) = \widetilde{\mathfrak{q}}(\nabla^s_G J,\zeta)$, then equation (2.3), (2.4) and (2.5) become

$$\widetilde{\nabla}_G H = \nabla_G H + \mathfrak{B}^l(G,H)J + \mathfrak{B}^s(G,H)J, \tag{2.8}$$

$$\widetilde{\nabla}_G J = -A_J G + \tau^l(G)J + \mathbb{E}^s(G,J), \tag{2.9}$$

$$\widetilde{\nabla}_G W = -A_W G + \tau^s(G)W + \mathbb{E}^l(G,W) \tag{2.10}$$

respectively.

Definition 2.1. [17, 24] Assume that $\widetilde{\mathbb{O}}$ be a differentiable manifold. $\widetilde{\Omega}$ to be an affine connection with the torsion tensor $T^{\widetilde{\Omega}}$ and $\widetilde{\mathfrak{q}}$ be a semi-Riemannian metric on $\widetilde{\mathbb{O}}$. Then the pair $(\widetilde{\Omega},\widetilde{\mathfrak{q}})$ is called statistical structure on $\widetilde{\mathbb{O}}$ if

(1) $(\widetilde{\Omega}_G\widetilde{\mathfrak{q}})(H,L) - (\widetilde{\Omega}_H\widetilde{\mathfrak{q}})(G,L) = \widetilde{\mathfrak{q}}(T^{\widetilde{\Omega}}(G,H),L)$ for all $G,\ H,\ L \in \Gamma(T\widetilde{\mathbb{O}})$
and
(2) $T^{\widetilde{\Omega}} = 0$.

Definition 2.2. [17, 24] Assume that $(\widetilde{\mathbb{O}},\widetilde{\mathfrak{q}})$ to be a semi-Riemannian manifold. $\widetilde{\Omega}$ and $\widetilde{\Omega}^*$ are two affine connections on $\widetilde{\mathbb{O}}$ are said to be dual with respect to the metric $\widetilde{\mathfrak{q}}$, if

$$L\widetilde{\mathfrak{q}}(G,H) = \widetilde{\mathfrak{q}}(\widetilde{\Omega}_L G,H) + \widetilde{\mathfrak{q}}(G,\widetilde{\Omega}^*_L H) \tag{2.11}$$

for all $G,\ H,\ L \in \Gamma(T\widetilde{\mathbb{O}})$.
A statistical manifold is denoted by $(\widetilde{\mathbb{O}},\widetilde{\mathfrak{q}},\widetilde{\Omega},\widetilde{\Omega}^*)$. If $\widetilde{\nabla}$ is Levi-Civita connection of $\widetilde{\mathfrak{q}}$, then

$$\widetilde{\nabla} = \frac{1}{2}(\widetilde{\Omega}+\widetilde{\Omega}^*). \tag{2.12}$$

In (2.12), by choosing $\widetilde{\Omega}^* = \widetilde{\Omega}$, Levi-Civita connection can be obtained.

Lemma 2.3. *[18, 24] For statistical manifold $(\widetilde{\mathbb{O}},\widetilde{\mathfrak{q}},\widetilde{\Omega},\widetilde{\Omega}^*)$, we set*

$$\widetilde{\mathbb{F}} = \widetilde{\Omega} - \widetilde{\nabla}. \tag{2.13}$$

Then, we have

$$\widetilde{\mathbb{F}}(G,H) = \widetilde{\mathbb{F}}(H,G), \quad \widetilde{\mathfrak{q}}(\widetilde{\mathbb{F}}((G,H),L) = \widetilde{\mathfrak{q}}(\widetilde{\mathbb{F}}((G,L),H) \tag{2.14}$$

for all $G,\ H,\ L \in \Gamma(T\widetilde{\mathbb{O}})$.
Conversely, for a Riemannian metric $\widetilde{\mathfrak{q}}$, if $\widetilde{\mathbb{F}}$ satisfies (2.14), the pair $(\widetilde{\Omega} = \widetilde{\nabla}+\widetilde{\mathbb{F}},\widetilde{\mathfrak{q}})$ is statistical structure on $\widetilde{\mathbb{O}}$.

Definition 2.4. [27] Let $(\widetilde{\mathbb{O}},\psi,\widetilde{\mathfrak{q}})$ be almost Hermitian manifold with an almost complex structure ψ and Hermitian metric $\widetilde{\mathfrak{q}}$ such that for all $G,\ H \in \Gamma(T\widetilde{\mathbb{O}})$,

$$\psi^2 = -I,\ \widetilde{\mathfrak{q}}(\psi G,\psi H) = \widetilde{\mathfrak{q}}(G,H). \tag{2.15}$$

Let ∇ be the Levi-Civita connection of $\widetilde{\mathbb{O}}$ with respect to metric $\widetilde{\mathfrak{q}}$, then the covariant derivative of ψ is defined by

$$(\nabla_G\psi)H = \nabla_G\psi H - \psi\nabla_G H. \tag{2.16}$$

Almost Hermitian manifold $\widetilde{\mathbb{O}}$ is called Kaehler manifold if ψ is parallel with respect to ∇,i.e.

$$(\nabla_G\psi)H = 0. \tag{2.17}$$

Definition 2.5. Let $(\widetilde{\mathfrak{q}},\psi)$ be Kaehlar structure on $\widetilde{\mathbb{O}}$. A quadruplet $(\widetilde{\Omega} = \widetilde{\nabla}+\widetilde{\mathbb{F}},\widetilde{\mathfrak{q}},\psi)$ is called Kaehlar statistical structure on $\widetilde{\mathbb{O}}$ if $(\widetilde{\Omega},\widetilde{\mathfrak{q}})$ is a statistical structure on $\widetilde{\mathbb{O}}$ and the formula

$$\widetilde{\mathbb{F}}(G,\psi H) = -\psi\widetilde{\mathbb{F}}(G,H) \tag{2.18}$$

holds for any $G,\ H \in \Gamma(T\widetilde{\mathbb{O}})$, then $(\widetilde{\mathbb{O}},\widetilde{\Omega},\widetilde{\mathfrak{q}},\psi)$ is said to Kaehlar statistical manifold.

Let $(\mathbb{O}, q)$ be a lightlike submanifold of a statistical manifold $(\widetilde{\mathbb{O}}, \widetilde{\mathfrak{q}}, \widetilde{\Omega}, \widetilde{\Omega}^*)$, then Gauss and Weingarten formulas with respect to dual connections are given by

$$\widetilde{\Omega}_G H = \Omega_G H + \mathfrak{B}^l(G, H)J + \mathfrak{B}^s(G, H)J, \tag{2.19}$$

$$\widetilde{\Omega}_G J = -A_J G + \tau^l(G)J + \mathbb{E}^s(G, J), \tag{2.20}$$

$$\widetilde{\Omega}_G W = -A_W G + \tau^s(G)W + \mathbb{E}^l(G, W), \tag{2.21}$$

$$\widetilde{\Omega}^*_G H = \Omega^*_G H + \mathfrak{B}^{l^*}(G, H)J + \mathfrak{B}^{s^*}(G, H)J, \tag{2.22}$$

$$\widetilde{\Omega}^*_G J = -A^*_J G + \tau^{l^*}(G)J + \mathbb{E}^{s^*}(G, J), \tag{2.23}$$

$$\widetilde{\Omega}^*_G W = -A^*_W G + \tau^{s^*}(G)W + \mathbb{E}^{l^*}(G, W) \tag{2.24}$$

for all $G, H \in \Gamma(T\mathbb{O})$, $J \in \Gamma(ltrT\mathbb{O})$ and $W \in \Gamma(S(T\mathbb{O}^\perp))$.
Here, Ω ,Ω^*, $\mathfrak{B}$, $\mathfrak{B}^{l^*}$, $\mathfrak{B}^s$, $\mathfrak{B}^{s^*}$, A_J, and A^*_J are called the induced connections on $\mathbb{O}$, the second fundamental forms and the Weingarten mappings with respect to $\widetilde{\Omega}$ and $\widetilde{\Omega}^*$, respectively.

3 Radical anti-invariant lightlike submanifolds of Kaehlar statistical manifold

Definition 3.1. Let $\mathbb{O}$ be a lightlike submanifold of a semi-Riemannian Kaehlar statistical manifold $(\widetilde{\mathbb{O}}, \widetilde{\mathfrak{q}})$. We say that $\mathbb{O}$ is a radical anti-invariant lightlike submanifold if $\psi(Rad(T\mathbb{O})) = ltr(T\mathbb{O})$. Moreover, we say that a radical anti-invariant submanifold is proper if there exists a subbundle $D^{'} \subset S(T\mathbb{O})$ such that $D^{'}$ is anti-invariant with respect to ψ, i.e. $\psi(D^{'}) \subset S(T\mathbb{O}^\perp)$ and $D^{'} \neq S(T\mathbb{O})$.
Now, we denote the orthogonal complementary to D' in $S(T\mathbb{O})$ by D_0. Thus we have the decompositions

$$T\mathbb{O} = D_0 \oplus D, \quad S(T\mathbb{O}) = D_0 \oplus D^{'}, \quad D = Rad(T\mathbb{O}) \oplus D^{'}. \tag{3.1}$$

Similarly, if we denote the orthogonal complementary to $\psi(D^{'})$ in $S(T\mathbb{O}^\perp)$ by E, we have

$$S(T\mathbb{O}^\perp) = \psi(D^{'}) \perp E.$$

Since $S(T\mathbb{O})$ is non-degenerate, for any $G \in \Gamma(D_0)$, we have $\widetilde{\mathfrak{q}}(\psi G, L) = \widetilde{\mathfrak{q}}(G, \psi L) = 0, \forall \ L \in \Gamma(D^{'})$ and

$$\widetilde{\mathfrak{q}}(\psi G, J) = \widetilde{\mathfrak{q}}(G, \psi J) = 0, \forall \ J \in \Gamma(tr(T\mathbb{O})),$$

due to $\psi J \in \Gamma(Rad(T\mathbb{O}))$. Similarly, we get

$$\widetilde{\mathfrak{q}}(\psi G, \zeta) = \widetilde{\mathfrak{q}}(G, \psi\zeta) = 0, \ \ \forall \ \zeta \in \Gamma(Rad(T\mathbb{O}))$$

and

$$\widetilde{\mathfrak{q}}(\psi G, W) = \widetilde{\mathfrak{q}}(G, \psi W) = 0, \ \ \forall \ W \in \Gamma(S(T\mathbb{O}^\perp)).$$

Hence we conclude that D_0 is an invariant distribution with respect to ψ.
Similarly, it is easy to check that, E is an invariant distribution with respect to ψ.

Let $\mathbb{O}$ be a radical anti-invariant lightlike submanifold of Kaehlar statistical manifold $\widetilde{\mathbb{O}}$. Then, for any $G \in \Gamma(T\mathbb{O})$, we can write

$$\psi G = fG + wG, \tag{3.2}$$

where $fG \in \Gamma(D_0)$ and $wG \in \Gamma(tr(T\mathbb{O}))$.
Similarly, for any $H \in \Gamma(tr(T\mathbb{O}))$, we can write

$$\psi H = BH + CH, \tag{3.3}$$

where $BH \in \Gamma(\mathbb{O})$ and $CH \in \Gamma(E)$.

Now, we denote the projections on $D_0, D^{'}, Rad(T\mathbb{O})$ in $T\mathbb{O}$ by $S_1,\ S_2,\ S_3$ respectively. Then, for any $G \in \Gamma(T\mathbb{O})$

$$\psi G = \psi S_1 G + \psi S_2 G + \psi S_3 G, \tag{3.4}$$

where $\psi S_1 G \in \Gamma(D_0),\ \ \psi S_2 G \in \Gamma(S(T\mathbb{O}^{\perp}))$ and $\psi S_3 G \in \Gamma(ltr(T\mathbb{O})$. Hence we have $\psi S_1 G = kG,\ \psi S_2 G = wS_2 G,\ \psi S_3 G = wS_3 G$. Similarly, we denote the projections on $S(T\mathbb{O}^{\perp})$ and $ltr(T\mathbb{O})$ in $tr(T\mathbb{O})$ by F_1 and F_2, respectively. Then, we obtain

$$\psi H = BF_1 H + CF_2 H + \psi F_2 H \tag{3.5}$$

for $H \in \Gamma(tr(T\mathbb{O}))$, where $BF_1 H \in \Gamma(D^{'}),\ CF_1 H \in \Gamma(E)$ and $BF_2 H \in \Gamma(Rad(T\mathbb{O}))$.

Theorem 3.2. *Suppose* $(\mathbb{O}, \mathfrak{q}, \Omega, \Omega^{*})$ *be a radical anti-invariant lightlike submanifold of Kaehlar statistical manifold* $(\widetilde{\mathbb{O}}, \widetilde{\mathfrak{q}}, \widetilde{\Omega}, \widetilde{\Omega}^{*})$*. Then the induced connections* $\Omega,\ \Omega^{*}$ *are the metric connections iff*

$$A_{\psi\zeta} G,\ \ A^{*}_{\psi\zeta} G \in \Gamma(D^{'}) \tag{3.6}$$

and

$$\mathbb{E}^{s}(G, \psi\zeta),\ \ \mathbb{E}^{s^{*}}(G, \psi\zeta) \in \Gamma(E) \tag{3.7}$$

for any $G \in \Gamma(T\mathbb{O})$ and $\zeta \in \Gamma(Rad(T\mathbb{O}))$.

Proof. From (2.12), (2.16) and (2.17), we have $\widetilde{\mathbf{\Omega}}_G \psi H = \psi \widetilde{\mathbf{\Omega}}_G H$ for any $G,\ H \in \Gamma(T\mathbb{O})$. Then using (3.2) and (3.3), we have

$$-A_{\psi\zeta} G + \tau^{l}(G)\psi\zeta + \mathbb{E}^{s}(G, \psi\zeta) = \psi \mathbf{\Omega}_G \zeta + \psi \mathbb{E}^{l}(G, \zeta) + B\mathbb{E}^{s}(G, \zeta) + C\mathbb{E}^{s}(G, \zeta)$$

for any $G \in \Gamma(T\mathbb{O})$ and $\zeta \in \Gamma(Rad(T\mathbb{O}))$. Applying ψ to this equation and using (3.2) and (3.3), we have

$$-kA_{\psi\zeta} G + \psi\tau^{l}(G)\psi\zeta + B\mathbb{E}^{s}(G, \psi\zeta) = -\mathbf{\Omega}_G \zeta$$

Thus, $\mathbf{\Omega}_G \zeta \in \Gamma(Rad(T\mathbb{O}))$ if and only if $A_{\psi\zeta} G \in \Gamma(D^{'})$ and $\mathbb{E}^{s}(G, \psi\zeta) \in \Gamma(E)$. Similarily, from (2.12), (2.16) and (2.17), we have $\widetilde{\mathbf{\Omega}}^{*}_G \psi H = \psi \widetilde{\mathbf{\Omega}}^{*}_G H$ for any $G,\ H \in \Gamma(T\mathbb{O})$. Then using (3.2) and (3.3), we have

$$-A^{*}_{\psi\zeta} G + \tau^{l^{*}}(G)\psi\zeta + \mathbb{E}^{s^{*}}(G, \psi\zeta) = \psi \mathbf{\Omega}^{*}_G \zeta + \psi \mathbb{E}^{l^{*}}(G, \zeta) + B\mathbb{E}^{s^{*}}(G, \zeta) + C\mathbb{E}^{s^{*}}(G, \zeta)$$

for any $G \in \Gamma(T\mathbb{O})$ and $\zeta \in \Gamma(Rad(T\mathbb{O}))$. Applying ψ to this equation and using (3.2) and (3.3), we have

$$-kA^{*}_{\psi\zeta} G + \psi\tau^{l^{*}}(G)\psi\zeta + B\mathbb{E}^{s^{*}}(G, \psi\zeta) = -\mathbf{\Omega}^{*}_G \zeta$$

Thus, $\mathbf{\Omega}^{*}_G \zeta \in \Gamma(Rad(T\mathbb{O}))$ if and only if $A_{\psi\zeta} G \in \Gamma(D^{'})$ and $\mathbb{E}^{s}(G, \psi\zeta) \in \Gamma(E)$. This way we get our result.

Now, using (3.3), (3.4) and (3.5) and taking the tangential and transversal parts, we get

$$(\mathbf{\Omega}_G \psi S_1) H = A_{\psi S_2 H} G + A_{\psi S_2 H} G + \psi \mathfrak{B}^{l}(G, H) + \psi \mathfrak{B}^{s}(G, H), \tag{3.8}$$

$$\psi S_3 \mathbf{\Omega}_G H = \mathfrak{B}^{l}(G, \psi S_1 H) + \mathbb{E}^{l}(G, \psi S_2 H) + \tau^{l}(G)\psi S_3 H, \tag{3.9}$$

$$\mathfrak{B}^{s}(G, \psi S_1 H) + \mathbb{E}^{s}(G, \psi S_3 H) + \tau^{s}(G)\psi S_2 H = \psi S_2 \mathbf{\Omega}_G H + C\mathfrak{B}^{s}(G, H) \tag{3.10}$$

and

$$(\mathbf{\Omega}^{*}_G \psi S_1) H = A^{*}_{\psi S_2 H} G + A^{*}_{\psi S_2 H} G + \psi \mathfrak{B}^{l^{*}}(G, H) + \psi \mathfrak{B}^{s^{*}}(G, H), \tag{3.11}$$

$$\psi S_3 \mathbf{\Omega}^{*}_G H = \mathfrak{B}^{l^{*}}(G, \psi S_1 H) + \mathbb{E}^{l^{*}}(G, \psi S_2 H) + \tau^{l^{*}}(G)\psi S_3 H, \tag{3.12}$$

$$\mathfrak{B}^{s^{*}}(G, \psi S_1 H) + \mathbb{E}^{s^{*}}(G, \psi S_3 H) + \tau^{s^{*}}(G)\psi S_2 H = \psi S_2 \mathbf{\Omega}^{*}_G H + C\mathfrak{B}^{s^{*}}(G, H) \tag{3.13}$$

for any $G,\ H \in \Gamma(T\mathbb{O})$.

Theorem 3.3. *Suppose* $(\mathbb{O}, \mathfrak{q}, \Omega, \Omega^{*})$ *be a radical anti-invariant lightlike submanifold of Kaehlar statistical manifold* $(\widetilde{\mathbb{O}}, \widetilde{\mathfrak{q}}, \widetilde{\Omega}, \widetilde{\Omega}^{*})$*. Then, the distribution* D_0 *is integrable if and only if for any* $G \in \Gamma(D_0)$

$$\mathfrak{B}(G, kH) = \mathfrak{B}(H, kG) \tag{3.14}$$

and

$$\mathfrak{B}^*(G, kH) = \mathfrak{B}^*(H, kG). \tag{3.15}$$

Proof. From (3.9), for $G,\ H \in \Gamma(D_0)$, we obtain

$$\mathfrak{B}^l(G, \psi S_1 H) = \psi S_3 \Omega_G H.$$

From equation (3.4), $\psi S_1 = k$, then above equation will be

$$\mathfrak{B}^l(G, kH) = \psi S_3 \Omega_G H. \tag{3.16}$$

Interchanging G and H, we get

$$\mathfrak{B}^l(H, kG) = \psi S_3 \Omega_H G. \tag{3.17}$$

From equation (3.16) and (3.17),

$$\mathfrak{B}^l(G, kH) - \mathfrak{B}^l(H, kG) = \psi S_3 [G, H]. \tag{3.18}$$

Now, from (3.10), we get

$$\mathfrak{B}^s(G, \psi S_1 H) = \psi S_2 \Omega_G H + C\mathfrak{B}^s(G, H).$$

From equation (3.4), $\psi S_1 = k,\ \ \psi S_2 = wS_2$, then above equation will be

$$\mathfrak{B}^s(G, kH) = wS_2 \Omega_G H + C\mathfrak{B}^s(G, H). \tag{3.19}$$

Interchanging the role of G and H, we get

$$\mathfrak{B}^s(H, kG) = wS_2 \Omega_H G + C\mathfrak{B}^s(H, G). \tag{3.20}$$

From equation (3.19) and (3.20), we get

$$\mathfrak{B}^s(G, kH) - \mathfrak{B}^s(H, kG) = kS_2 [G, H], \tag{3.21}$$

this way we get equation (3.14).
From (3.12), for $G,\ H \in \Gamma(D_0)$, we obtain

$$\mathfrak{B}^{l^*}(G, \psi S_1 H) = \psi S_3 \Omega^*_G H.$$

From equation (3.4), $\psi S_1 = k$, then above equation will be

$$\mathfrak{B}^{l^*}(G, kH) = \psi S_3 \Omega^*_G H. \tag{3.22}$$

Interchanging G and H, we get

$$\mathfrak{B}^{l^*}(H, kG) = \psi S_3 \Omega^*_H G. \tag{3.23}$$

From equation (3.22) and (3.23),

$$\mathfrak{B}^{l^*}(G, kH) - \mathfrak{B}^{l^*}(H, kG) = \psi S_3 [G, H]. \tag{3.24}$$

Now, from (3.13), we get

$$\mathfrak{B}^{s^*}(G, \psi S_1 H) = \psi S_2 \Omega^*_G H + C\mathfrak{B}^{s^*}(G, H).$$

From equation (3.4), $\psi S_1 = k,\ \ \psi S_2 = wS_2$, then above equation will be

$$\mathfrak{B}^{s^*}(G, kH) = wS_2 \Omega^*_G H + C\mathfrak{B}^{s^*}(G, H). \tag{3.25}$$

Interchanging the role of G and H, we get

$$\mathfrak{B}^{s^*}(H, kG) = wS_2 \Omega^*_H G + C\mathfrak{B}^{s^*}(H, G). \tag{3.26}$$

From equation (3.25) and (3.26), we get

$$\mathfrak{B}^{s^*}(G, kH) - \mathfrak{B}^{s^*}(H, kG) = kS_2 [G, H]. \tag{3.27}$$

This way we get equation (3.15).

Theorem 3.4. *Suppose* $(\mathbb{O}, \mathfrak{q}, \Omega, \Omega^*)$ *be a radical anti-invariant lightlike submanifold of Kaehlar statistical manifold* $(\widetilde{\mathbb{O}}, \widetilde{\mathfrak{q}}, \widetilde{\Omega}, \widetilde{\Omega}^*)$*, then*
(i) *the distribution* D *is integrable if and only if*
$A_{wG}H = A_{wH}G$ *and* $A^*_{wG}H = A^*_{wH}G$*, for any* $G,\ H \in \Gamma(D)$.
(ii) *the distribution* D *defines a totally geodesic foliation in* $\mathbb{O}$ *if and only if* $A_{wH}G,\ A^*_{wH}G \in \Gamma(D)\ ,\ G \in \Gamma(T\mathbb{O}),\ H\ \in \Gamma(D)$.

Proof. **(i).** For any $G,\ H \in \Gamma(D)$, from equation (2.12), (2.17) and (2.19), we obtain

$$\psi(\widetilde{\Omega}_G H) = \widetilde{\Omega}_G \psi H,$$

$$\psi(\Omega_G H) + C\mathfrak{B}^l(G,H) + C\mathfrak{B}^s(G,H) = -A_{\psi H}G + \tau^l(G)\psi H + \mathbb{E}^s(G,\psi H).$$

On taking tangential parts of the above equation, we have

$$\psi\Omega_G H + B\mathfrak{B}^l(G,H) + C\mathfrak{B}^s(G,H) = -A_{wH}G. \tag{3.28}$$

Interchanging the role of G and H, we get

$$\psi\Omega_H G + B\mathfrak{B}^l(H,G) + C\mathfrak{B}^s(H,G) = -A_{wG}H. \tag{3.29}$$

From (3.30) and (3.31), we get

$$\psi[G,H] = A_{wH}G - A_{wG}H.$$

Thus, $[G,H] \in \Gamma(D)$ if and only if $A_{wH}G = A_{wG}H$.
Now, from equation (2.12), (2.17) and (2.22), we obtain

$$\psi(\widetilde{\Omega}^*_G H) = \widetilde{\Omega}^*_G \psi H,$$

$$\psi(\Omega^*{}_G H) + B\mathfrak{B}^{l^*}(G,H) + C\mathfrak{B}^{s^*}(G,H) = -A^*_{\psi H}G + \tau^{l^*}(G)\psi H + \mathbb{E}^{s^*}(G,\psi H).$$

On taking tangential parts of the above equation, we have

$$\psi\Omega^*{}_G H + B\mathfrak{B}^{l^*}(G,H) + C\mathfrak{B}^{s^*}(G,H) = -A^*_{wH}G. \tag{3.30}$$

Interchanging the role of G and H, we get

$$\psi\Omega^*{}_H G + B\mathfrak{B}^{l^*}(H,G) + C\mathfrak{B}^{s^*}(H,G) = -A^*_{wG}H. \tag{3.31}$$

From (3.30) and (3.31), we get

$$\psi[G,H] = A^*_{wH}G - A^*_{wG}H.$$

Thus, $[G,H] \in \Gamma(D)$ if and only if $A^*_{wH}G = A^*_{wG}H$.
(ii). Now we will show that $\mathfrak{q}(\Omega_G H, \psi L) = \mathfrak{q}(\Omega^*_G H, \psi L) = 0$
for any $G \in \Gamma(T\mathbb{O}),\ H \in \Gamma(D)$ and $L \in \Gamma(D_0)$.
Since,

$$\begin{aligned} \mathfrak{q}(\Omega_G H, \psi L) &= \widetilde{\mathfrak{q}}(\widetilde{\Omega}_G H, \psi L), \\ &= \widetilde{\mathfrak{q}}(\psi\widetilde{\Omega}_G H, L), \\ &= \widetilde{\mathfrak{q}}(\widetilde{\Omega}_G \psi H, L), \\ \mathfrak{q}(\Omega_G H, \psi L) &= -\mathfrak{q}(A_{\psi H}G, L). \end{aligned}$$

Similarily, we get

$$\mathfrak{q}(\Omega^*_G H, \psi L) = -\mathfrak{q}(A^*_{\psi H}G, L).$$

Theorem 3.5. *Suppose* $(\mathbb{O}, \mathfrak{q}, \Omega, \Omega^*)$ *be a radical anti-invariant lightlike submanifold of Kaehlar statistical manifold* $(\widetilde{\mathbb{O}}, \widetilde{\mathfrak{q}}, \widetilde{\Omega}, \widetilde{\Omega}^*)$*. Then the distribution* D_0 *defines a totally geodesic foliation in* $\mathbb{O}$ *if and only if*
$\mathfrak{B}(G,H),\ \mathfrak{B}^*(G,H) \in \Gamma(E)$ *for any* $G,\ H \in \Gamma(D_0)$.

Proof. For any $G,\ H \in \Gamma(D_0),\ L \in \Gamma(D')$ and $J \in \Gamma(ltr(T\mathbb{O}))$, then

$$\mathfrak{q}(\Omega_G \psi H, L) = \widetilde{\mathfrak{q}}(\widetilde{\Omega}_G \psi H, L),$$
$$= \widetilde{\mathfrak{q}}(\psi \widetilde{\Omega}_G H, L),$$
$$= \widetilde{\mathfrak{q}}(\widetilde{\Omega}_G H, \psi L),$$

on using equation (2.19), we obtain

$$\mathfrak{q}(\Omega_G \psi H, L) = \widetilde{\mathfrak{q}}(\mathfrak{B}^s(G, H), \psi L).$$

Now,

$$\mathfrak{q}(\Omega_G \psi H, J) = \widetilde{\mathfrak{q}}(\widetilde{\Omega}_G \psi H, J),$$
$$= \widetilde{\mathfrak{q}}(\psi \widetilde{\Omega}_G H, J),$$
$$= \widetilde{\mathfrak{q}}(\widetilde{\Omega}_G H, \psi J),$$

on using equation (2.19), we obtain

$$\mathfrak{q}(\Omega_G \psi H, J) = \widetilde{\mathfrak{q}}(\mathfrak{B}^l(G, H), \psi J).$$

Similarily, on using equation (2.22), we have

$$\mathfrak{q}(\Omega^*_G \psi H, L) = \widetilde{\mathfrak{q}}(\mathfrak{B}^{s^*}(G, H), \psi L)$$

and

$$\mathfrak{q}(\Omega^*_G \psi H, J) = \widetilde{\mathfrak{q}}(\mathfrak{B}^{l^*}(G, H), \psi J).$$

These results shows that $\mathfrak{B}(G, H),\ \mathfrak{B}^*(G, H) \in \Gamma(E)$,

Theorem 3.6. *Suppose* $(\mathbb{O}, \mathfrak{q}, \Omega, \Omega^*)$ *be a radical anti-invariant lightlike submanifold of Kaehlar statistical manifold* $(\widetilde{\mathbb{O}}, \widetilde{\mathfrak{q}}, \widetilde{\Omega}, \widetilde{\Omega}^*)$*. Then* $\mathbb{O}$ *is locally Kaehlar statistical if and only if* k *is parallel with respect to the induced connections* Ω *and* Ω^**.*

Proof. Let us have $\mathbb{O}$ as locally Kaehlar statistical manifold then the leaves of the distributions D_0 and D are totally geodesic in $\mathbb{O}$.
Now, for any $Z \in \Gamma(T\mathbb{O})$ and $G \in \Gamma(D_0)$, $\Omega_Z kG$ and $\Omega^*_Z kG \in \Gamma(D_0)$. Moreover, for any $G \in \Gamma(D_0),\ \psi S_2 G = 0,\ \psi S_3 G = 0$. From (3.8) and (3.11), we get

$$(\Omega_Z k)G = \psi \mathfrak{B}^l(Z, G) + \psi \mathfrak{B}^s(Z, G), \quad (\Omega^*_Z k)G = \psi \mathfrak{B}^{l^*}(Z, G) + \psi \mathfrak{B}^{s^*}(Z, G).$$

Since $(\Omega_Z k)G \in \Gamma(D_0),\ \ (\Omega^*_Z k)G$, we have

$$(\Omega_Z k)G = (\Omega^*_Z k)G = 0.$$

For $H \in \Gamma(D)$ and $Z \in \Gamma(T\mathbb{O})$, from (3.8) and (3.11) we get

$$(\Omega_Z k)H = A_{\psi S_2 H} Z + A_{\psi S_3 H} Z + \psi \mathfrak{B}^l(Z, H) + \psi \mathfrak{B}^s(Z, H)$$

and

$$(\Omega^*_Z k)H = A^*_{\psi S_2 H} Z + A^*_{\psi S_3 H} Z + \psi \mathfrak{B}^{l^*}(Z, H) + \psi \mathfrak{B}^{s^*}(Z, H).$$

Since $kH = 0$, for $H \in \Gamma(D)$, we get

$$-k\Omega_Z H = A_{\psi S_2 H} Z + A_{\psi S_3 H} Z, \quad -k\Omega^*_Z H = A^*_{\psi S_2 H} Z + A^*_{\psi S_3 H} Z.$$

We get $k\Omega_Z H = k\Omega^*_Z H = 0$ by Theorem (3.4), Thus we obtain

$$(\Omega_Z k)H = (\Omega^*_Z k)H = 0.$$

Conversely, let $\Omega k = \Omega^* k = 0$, then we have

$$k\Omega_G H = \Omega_G kH, \quad k\Omega^*_G H = \Omega^*_G kH$$

for any $G,\ H \in \Gamma(D_0)$ and

$$k\Omega_Z L = \Omega_Z kL, \quad k\Omega^*_Z L = \Omega^*_Z kL$$

for any $Z,\ L \in \Gamma(D)$. Then it shows that $\Omega_G kH,\ \Omega^*_G kH \in \Gamma(D_0)$ and $\Omega_Z L,\ \Omega^*_Z L \in \Gamma(D)$, respectively. Thus the leaves of the distributions D_0 and D are totally geodesic in $\mathbb{O}$.

Theorem 3.7. *Suppose* $(\mathbb{O}, \mathfrak{q}, \Omega, \Omega^*)$ *be a totally umbilical radical anti-invariant lightlike submanifold of Kaehlar statistical manifold* $(\widetilde{\mathbb{O}}, \widetilde{\mathfrak{q}}, \widetilde{\Omega}, \widetilde{\Omega}^*)$. *Then the radical distribution* $Rad(T\mathbb{O})$ *is integrable if and only if*

$$\mathfrak{B}^l(\zeta_1, kL) = \mathfrak{B}^l(\zeta_2, kL),$$

$$\mathfrak{B}^{l^*}(\zeta_1, kL) = \mathfrak{B}^{l^*}(\zeta_2, kL).$$

Proof. From (2.19) and (2.22), we have

$$\mathfrak{q}([\zeta_1, \zeta_2], L) = \widetilde{\mathfrak{q}}(\widetilde{\Omega}_{\zeta_1}\zeta_2, L) - \widetilde{\mathfrak{q}}(\widetilde{\Omega}_{\zeta_2}\zeta_1, L) = \widetilde{\mathfrak{q}}(\widetilde{\Omega}^*_{\zeta_1}\zeta_2, L) - \widetilde{\mathfrak{q}}(\widetilde{\Omega}^*_{\zeta_2}\zeta_1, L).$$

For all $\zeta_1, \zeta_2 \in Rad(T\mathbb{O}), \quad L \in \Gamma(D_0)$.
Using (2.15), we get

$$\mathfrak{q}([\zeta_1, \zeta_2], L) = -\widetilde{\mathfrak{q}}(\psi\zeta_2, \mathfrak{B}^l(\zeta_1, \psi L) + \widetilde{\mathfrak{q}}(\psi\zeta_1, \mathfrak{B}^l(\zeta_2, \psi L)$$

$$= -\widetilde{\mathfrak{q}}(\psi\zeta_2, \mathfrak{B}^{l^*}(\zeta_1, \psi L) + \widetilde{\mathfrak{q}}(\psi\zeta_1, \mathfrak{B}^{l^*}(\zeta_2, \psi L).$$

Since

$$\mathfrak{B}^l(\zeta_1, kL) = \mathfrak{B}^l(\zeta_2, kL) = 0,$$

$$\mathfrak{B}^{l^*}(\zeta_1, kL) = \mathfrak{B}^{l^*}(\zeta_2, kL) = 0.$$

Thus,

$$\mathfrak{q}([\zeta_1, \zeta_2], L) = 0.$$

Conversely,

$$\mathfrak{q}([\zeta_1, \zeta_2], L) = 0.$$

This implies that

$$\mathfrak{B}^l(\zeta_1, kL) = \mathfrak{B}^l(\zeta_2, kL) = 0,$$

$$\mathfrak{B}^{l^*}(\zeta_1, kL) = \mathfrak{B}^{l^*}(\zeta_2, kL) = 0.$$

This way we get our proof.

Example 3.8. Let $\widetilde{\mathbb{O}} = (R^8_4, \widetilde{\mathfrak{q}}, \psi)$ be a Kaehlar manifold. $\widetilde{\mathfrak{q}}$ is of the signature $(+,-,+,-,+,-,+,-)$ with respect to the canonical basis $\{\partial s_1, \partial s_2, \partial s_3, \partial s_4, \partial s_5, \partial s_6, \partial s_7, \partial s_8\}$. If $(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8)$ be the standard co-ordinate system of R^8_4 then by setting

$$\psi(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8) = (is_1, -is_2, is_3, -is_4, is_5, -is_6, is_7, -is_8)$$

i and $-i$ are the roots of Kaehlar structure. From definition (2.5), the triplet $(\widetilde{\Omega} = \widetilde{\nabla} + \widetilde{\mathbb{F}}, \widetilde{\mathfrak{q}}, \widetilde{\psi})$, where $\widetilde{\mathbb{F}}$ satisfies (2.18), defines Kaehlar statistical structure on $\widetilde{\mathbb{O}}$.
Consider submanifold $\mathbb{O}$ of R^8_4, given by the equation

$$s_1 = sinht_2, \ s_2 = cosht_2, \ s_3 = t_1,$$

$$s_4 = t_3 - \frac{1}{2}t_4, \ s_5 = t_2, \ s_6 = t_5,$$

$$s_7 = t_1, \quad s_8 = t_3 + \frac{1}{2}t_4.$$

Here, $T\mathbb{O}$ is spanned by

$$Z_1 = \partial s_3 + \partial s_7,$$

$$Z_2 = sinht_2\partial s_1 - cosht_2\partial s_2 + \partial s_5,$$

$$Z_3 = \partial s_4 + \partial s_8,$$

$$Z_4 = \frac{1}{2}\{-\partial s_4 + \partial s_8\},$$

$$Z_5 = \partial s_6,$$

we see that $\mathbb{O}$ is 1-lightlike with $Rad(T\mathbb{O})$ spanned by Z_1. $S(T\mathbb{O})$ spanned by $\{Z_2, Z_3, Z_4, Z_5\}$ and $ltr(T\mathbb{O})$ is spanned by

$$J = \frac{1}{2}\{\partial s_3 + \partial s_7\}.$$

Then it can be easily seen that $\psi(Z_1) = iJ$, which shows that $\psi(Rad(T\mathbb{O})) = ltr(T\mathbb{O})$. Now, we obtain screen transversal bundle $S(T\mathbb{N}^{\perp}) = span\{W\}$, where

$$W = isinht_2\partial s_1 + icosht_2\partial s_2 + i\partial s_5.$$

We can see that $D_0 = span\{Z_3, Z_4, Z_5\}$ and $\psi Z_2 = W$ which shows that D_0 is invariant and $D^{'} = span\{Z_2\}$ is anti-invariant. Thus, $\mathbb{O}$ is radical anti-invariant lightlike submanifold of Kaehlar statistical manifold $(\widetilde{\mathbb{O}}, \widetilde{\mathfrak{q}}, \widetilde{\Omega}, \widetilde{\Omega}^*)$.

4 Conclusion remarks

This paper aims is to introduces the concept of a Kaehlar statistical manifold and discusses Radical anti-invariant lightlike submanifolds of this manifold. Additionally, the mention of presenting an example related to these concepts suggests that the paper likely delves into specific mathematical structures and their properties. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

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