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η-Einstein Solitons in a Bochner Flat Lorentzian K$\ddot{a}$hler Manifold

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Abstract *In the current work, we examine η-Einstein solitons in a Bochner flat Lorentzian Kähler manifolds. We have addressed various circumstances for Einstein solitons to be steady, shrinking or expanding in terms of isotropic pressure, the cosmological constant, energy density and gravitational constant in different perfect fluids such as dark fluid, stiff matter, dust fluid and radiation fluid.*

1 Introduction

The concept of an Einstein soliton was introduced for the first time in 2016 by G. Catino and L. Mazzieri [4], which may be viewed as a self-similar solution to the Einstein flow

$$\frac{\partial g}{\partial t} = -2\Big(S - \frac{r}{2}g\Big) \tag{1.1}$$

where g, S and r are the Riemannian metric, Ricci tensor and scalar curvature respectively. Ricci solitons and Einstein solitons are self-similar solution of Ricci flow and Einstein flow respectively.
The η-Einstein soliton $[1]$ on a Riemannian manifold (M^n, g) is given by,

$$L_\xi g + 2S + (2a - r)g + 2b\eta \otimes \eta = 0, \tag{1.2}$$

where L_ξ denotes the Lie derivative along the direction of the vector field ξ, S is the Ricci tensor, r is the scalar curvature and a, b are real constants. The η-Einstein soliton is called shrinking if $a < 0$, steady if $a = 0$ and expanding if $a > 0$. In particular, if $b = 0$ in equation (1.2), then η-Einstein soliton reduces to the Einstein soliton (g, ξ, a).
The Lie derivative of $g(\chi_1, \chi_2)$ with respect to ξ is given by

$$(L_\xi g)(\chi_1, \chi_2) = g(\nabla_{\chi_1}\xi, \chi_2) + g(\chi_1, \nabla_{\chi_2}\xi), \tag{1.3}$$

thus from equation (1.2) and (1.3), we get

$$S(\chi_1, \chi_2) = -\Big(a - \frac{r}{2}\Big)g(\chi_1, \chi_2) - b\eta(\chi_1)\eta(\chi_2) - \frac{1}{2}[g(\nabla_{\chi_1}\xi, \chi_2) + g(\chi_1, \nabla_{\chi_2}\xi)]. \tag{1.4}$$

Einstein solitons and η-Einstein solitons has been studied by many authors in different ways(see ref. [6], [7], [10], [11],[12], [13], [14]). Recently in 2023 B. B. Chaturvedi et al. [5] have studied Novel theorems for a Bochner Flat Lorentzian Kȧhler Space-time Manifold with η-Ricci-Yamabe Solitons.

The exploration of solitons in the context of space-time has motivated us to devise a research project focused on examining η-Einstein solitons within a Lorentzian Kähler manifold that exhibits Bochner flatness.

2 Preliminaries

An $n-$(even) dimensional semi-Riemannian manifold (M^n, g) equipped with a Lorentzian metric g is said to be a Lorentzian Kähler manifold if the following conditions hold [9]:

$$J^2 = -I, \ \ g(J\chi_1, J\chi_2) = g(\chi_1, \chi_2), and \ \ (\nabla_{\chi_1} J)\chi_2 = 0, \tag{2.1}$$

where, J is a tensor field of type (1,1) such that $J(\chi_1) = \chi_1$. In a Lorentzian Kähler manifold the following relations holds:

$$S(J\chi_1, J\chi_2) = S(\chi_1, \chi_2), \tag{2.2}$$

$$S(J\chi_1, \chi_2) = -S(\chi_1, J\chi_2), \tag{2.3}$$

$$g(J\chi_1, \chi_2) = -g(\chi_1, J\chi_2). \tag{2.4}$$

A four-dimensional Lorentzian Kähler manifold is called Lorentzian Kähler space-time manifold. Throughout this paper we consider this assumption.

The concept of energy momentum tensor plays an important role in the general theory of relativity. Also, the nature of perfect fluid space-time depends on the different conditions of the energy momentum tensor in the study of perfect fluid space-time [17]. The energy momentum tensor is used to characterise the matter content of spacetime; matter is viewed as a fluid with density, pressure, and kinematical and dynamical properties like vorticity, shear, and expansion. Since the matter content of the universe is assumed to behave like a perfect fluid in the standard cosmological models, so it is often used in general relativity to model idealized distributions of matter, such as the interior of a star or an isotropic universe. A perfect fluid has no shear stress, viscosity or heat conduction and it is distinguished by energy momentum tensor T of the form [8]:

$$T(\chi_1, \chi_2) = pg(\chi_1, \chi_2) + (\sigma + p)\eta(\chi_1)\eta(\chi_2), \tag{2.5}$$

where p and σ are the isotropic pressure and energy density respectively. $\eta(\chi_1) = g(\chi_1, \xi)$ is a $1 - form$ such that $\eta(\xi) = -1$ and $g(\xi, \xi) = -1$.

Einstein field equation with cosmological constant for a perfect fluid space-time [8] is given by:

$$S(\chi_1, \chi_2) + \Big(\lambda - \frac{r}{2}\Big)g(\chi_1, \chi_2) = KT(\chi_1, \chi_2), \tag{2.6}$$

where λ is a cosmological constant and K is the gravitational constant such that $K \neq 0$. Using equation (2.5) and equation (2.6), we can write

$$S(\chi_1, \chi_2) = \Big(- \lambda + \frac{r}{2} + Kp\Big)g(\chi_1, \chi_2) + K(\sigma + p)\eta(\chi_1)\eta(\chi_2). \tag{2.7}$$

Now contracting equation (2.7) and using $g(\xi, \xi) = -1$, we attain

$$r = 4\lambda + K(\sigma - 3p). \tag{2.8}$$

Einstein field equation without cosmological constant for perfect fluid space-time is defined by:

$$S(\chi_1, \chi_2) - \frac{r}{2}g(\chi_1, \chi_2) = KT(\chi_1, \chi_2), \tag{2.9}$$

using equation (2.5) in equation (2.9), we attain

$$S(\chi_1, \chi_2) = \Big(\frac{r}{2} + Kp\Big)g(\chi_1, \chi_2) + K(\sigma + p)\eta(\chi_1)\eta(\chi_2). \tag{2.10}$$

Contracting equation (2.10) and using $g(\xi, \xi) = -1$, we get

$$r = K(\sigma - 3p). \tag{2.11}$$

Consider an orthonormal frame field [2] $\{E_i\}_{1\leq i\leq 4}$, that is $g(E_i, E_j) = \epsilon_{ij}\delta_{ij}$, $i, j \in \{1, 2, 3, 4\}$ with $\epsilon_{11} = -1$, $\epsilon_{ii} = -1$, $i \in \{2, 3, 4\}$, $\epsilon_{ij} = 0$, $i, j \in \{1, 2, 3, 4\}$, $i \neq j$.
Let $\xi = \sum_{i=1}^{n} \xi^i E_i$ then, we can write

$$-1 = g(\xi, \xi) = \sum_{1\leq i,j\leq 4} \xi^i \xi^j g(E_i, E_j) = \sum_{i=1}^{4} \epsilon_{ii}\{\xi^i\}^2 \tag{2.12}$$

and

$$\eta(E_i) = g(E_i, \xi) = \sum_{j=1}^{4} \xi^i g(E_i, E_j) = \epsilon_{ii}\xi^i \tag{2.13}$$

3 Bochner Flat Space

S. Bochner [3] in 1949 introduced and defined the Bochner curvature tensor as follows:

$$\begin{aligned} B(\chi_2, \chi_3, \chi_4, \chi_5) =& R(\chi_2, \chi_3, \chi_4, \chi_5) - \frac{1}{2(n+2)}\Big\{S(\chi_2, \chi_5)g(\chi_3, \chi_4) - S(\chi_2, \chi_4)g(\chi_3, \chi_5) \\ &+ g(\chi_2, \chi_5)S(\chi_3, \chi_4) - g(\chi_2, \chi_4)S(\chi_3, \chi_5) + S(J\chi_2, \chi_5)g(J\chi_3, \chi_4) \\ &- S(J\chi_2, \chi_4)g(J\chi_3, \chi_5) + S(J\chi_3, \chi_4)g(J\chi_2, \chi_5) - g(J\chi_2, \chi_4)S(J\chi_3, \chi_5) \\ &- 2S(J\chi_2, \chi_3)g(J\chi_4, \chi_5) - 2g(J\chi_2, \chi_3)S(J\chi_4, \chi_5)\Big\} \\ &+ \frac{r}{(2n+2)(2n+4)}\Big\{g(\chi_3, \chi_4)g(\chi_2, \chi_5) - g(\chi_2, \chi_4)g(\chi_3, \chi_5) \\ &+ g(J\chi_3, \chi_4)g(J\chi_2, \chi_5) - g(J\chi_2, \chi_4)g(J\chi_3, \chi_5) - 2g(J\chi_2, \chi_3)g(J\chi_4, \chi_5)\Big\}, \end{aligned} \tag{3.1}$$

where, $B(\chi_2, \chi_3, \chi_4, \chi_5) = g(B(\chi_2, \chi_3)\chi_4, \chi_5)$, $R(\chi_2, \chi_3, \chi_4, \chi_5) = g(R(\chi_2, \chi_3)\chi_4, \chi_5)$, S and r are the Ricci tensor and scalar curvature of the manifold respectively.

If $B(\chi_2, \chi_3, \chi_4, \chi_5) = 0$, then equation (3.1) becomes

$$\begin{aligned} &R(\chi_2, \chi_3, \chi_4, \chi_5) - \frac{1}{12}\Big\{S(\chi_2, \chi_5)g(\chi_3, \chi_4) - S(\chi_2, \chi_4)g(\chi_3, \chi_5) \\ &+ g(\chi_2, \chi_5)S(\chi_3, \chi_4) - g(\chi_2, \chi_4)S(\chi_3, \chi_5) + S(J\chi_2, \chi_5)g(J\chi_3, \chi_4) \\ &- S(J\chi_2, \chi_4)g(J\chi_3, \chi_5) + S(J\chi_3, \chi_4)g(J\chi_2, \chi_5) - g(J\chi_2, \chi_4)S(J\chi_3, \chi_5) \\ &- 2S(J\chi_2, \chi_3)g(J\chi_4, \chi_5) - 2g(J\chi_2, \chi_3)S(J\chi_4, \chi_5)\Big\} \\ &+ \frac{r}{(10)(12)}\Big\{g(\chi_3, \chi_4)g(\chi_2, \chi_5) - g(\chi_2, \chi_4)g(\chi_3, \chi_5) \\ &+ g(J\chi_3, \chi_4)g(J\chi_2, \chi_5) - g(J\chi_2, \chi_4)g(J\chi_3, \chi_5) - 2g(J\chi_2, \chi_3)g(J\chi_4, \chi_5)\Big\} = 0. \end{aligned} \tag{3.2}$$

Contracting χ_2 and χ_5 in equation (3.2) and using equation (2.1), (2.2), (2.3) and (2.4), we get

$$S(\chi_3, \chi_4) = \frac{r}{10}g(\chi_3, \chi_4), \tag{3.3}$$

from equation (1.4) and (3.3), we get

$$\frac{r}{10}g(\chi_3, \chi_4) = -\Big(a - \frac{r}{2}\Big)g(\chi_3, \chi_4) - b\eta(\chi_3)\eta(\chi_4) - \frac{1}{2}[g(\nabla_{\chi_3}\xi, \chi_4) + g(\chi_3, \nabla_{\chi_4}\xi)]. \tag{3.4}$$

Multiplying (3.4) by ϵ_{ii} and taking $\chi_3 = \chi_4 = E_i$ and utilising equation (2.12) and (2.13), we get

$$4a - b = \frac{8}{5}r + 4div\xi, \tag{3.5}$$

using equation (2.8) in equation (3.5), we get

$$4a - b = \frac{8}{5}[4\lambda + K(\sigma - 3p)] + 4div\xi, \tag{3.6}$$

Now taking $\chi_3 = \chi_4 = \xi$ in equation (3.4) and using $g(\xi,\xi) = -1$ and $\eta(\xi) = -1$, we get

$$a - b = \frac{2}{5}r, \tag{3.7}$$

using equation (2.8) in equation (3.7), we get

$$a - b = \frac{2}{5}[4\lambda + K(\sigma - 3p)], \tag{3.8}$$

From equation (3.6) and (3.8), we get

$$a = \frac{2}{5}[4\lambda + K(\sigma - 3p)] + \frac{4}{3}div\xi \ \text{ and } \ b = \frac{4}{3}div\xi. \tag{3.9}$$

Since, for Einstein soliton, $b = 0$, therefore, from equation (3.9), we get $a = \frac{2}{5}[4\lambda + K(\sigma - 3p)]$. We know that soliton will be steady if $a = 0$, therefore, from equation (3.9), we get $p = \frac{4}{3}\frac{\lambda}{K} + \frac{\sigma}{3}$. Again, soliton will be shrinking if $a < 0$, therefore, from equation (3.9), we get $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\sigma}{3}$. Also, soliton will be expanding if $a > 0$, therefore, from equation (3.9), we get $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\sigma}{3}$.

Hence, we arrive at the following conclusion:

Theorem 3.1. *In a Bochner flat Lorentzian Kähler manifold satisfying Einstein field equation with cosmological constant, the Einstein soliton (g, ξ, a) is:*
(i) steady: if $p = \frac{4}{3}\frac{\lambda}{K} + \frac{\sigma}{3}$,
(ii) shrinking: if $p > \frac{4}{3}\frac{\lambda}{K} + \frac{\sigma}{3}$,
(iii) or expanding: if $p < \frac{4}{3}\frac{\lambda}{K} + \frac{\sigma}{3}$.

Using equation (2.11) in equation (3.5), we have

$$4a - b = \frac{8}{5}K(\sigma - 3p) + 4div\xi, \tag{3.10}$$

using equation (2.11) in equation (3.7), we have

$$a - b = \frac{2}{5}K(\sigma - 3p), \tag{3.11}$$

thus, from equation (3.10) and (3.11), we obtain

$$a = \frac{2}{5}K(\sigma - 3p) + \frac{4}{3}div\xi \ \text{ and } \ b = \frac{4}{3}div\xi, \tag{3.12}$$

Since, for Einstein soliton $b = 0$, therefore, from equation (3.12), we get $a = \frac{2}{5}K(\sigma - 3p)$. We know that soliton will be steady if $a = 0$, therefore, from equation (3.12), we get $p = \frac{\sigma}{3}$. Again, soliton will be shrinking if $a < 0$, therefore, from equation (3.12), we get $p > \frac{\sigma}{3}$. Also, soliton will be expanding if $a > 0$, therefore, from equation (3.12), we get $p < \frac{\sigma}{3}$.

Hence, we arrive at the following conclusion:

Theorem 3.2. *In a Bochner flat Lorentzian Kähler manifold satisfying Einstein field equation without cosmological constant, the Einstein soliton (g, ξ, a) is:*
(i) steady: if $p = \frac{\sigma}{3}$,
(ii) shrinking: if $p > \frac{\sigma}{3}$,
(iii) or expanding: if $p < \frac{\sigma}{3}$.

4 Behaviour of η-Einstein Soliton in a Dark Fluid

Perfect fluid is referred to as dark fluid if $p = -\sigma$, and the energy momentum tensor of a dark fluid is

$$T(\chi_1, \chi_2) = pg(\chi_1, \chi_2), \tag{4.1}$$

applying equation (4.1) in equation (2.6), we have

$$S(\chi_1, \chi_2) = \Big(\frac{r}{2} + Kp - \lambda\Big) g(\chi_1, \chi_2), \tag{4.2}$$

contracting equation (4.2) and using $g(\xi, \xi) = -1$, we obtain

$$r = 4(\lambda - Kp). \tag{4.3}$$

Now, using equation (3.5) and (4.3), we obtain

$$4a - b = \frac{32}{5}(\lambda - Kp) + 4div\xi, \tag{4.4}$$

Now, using equation (3.7) and (4.3), we obtain

$$a - b = \frac{8}{5}(\lambda - Kp). \tag{4.5}$$

Thus, from equation (4.4) and (4.5), we get

$$a = \frac{8}{5}(\lambda - Kp) + \frac{4}{3}div\xi \text{ and } b = \frac{4}{3}div\xi. \tag{4.6}$$

Since, for Einstein soliton $b = 0$, therefore, from equation (4.6), we get $a = \frac{8}{5}(\lambda - Kp)$.
We know that soliton will be steady if $a = 0$, therefore, from equation (4.6), we get $p = \frac{\lambda}{K}$.
Again, soliton will be shrinking if $a < 0$, therefore, from equation (4.6), we get $p > \frac{\lambda}{K}$.
Also, soliton will be expanding if $a > 0$, therefore, from equation (4.6), we get $p < \frac{\lambda}{K}$.

Hence, we arrive at the following conclusion:

Theorem 4.1. *In a Bochner flat Lorentzian Kähler manifold satisfying Einstein field equation with cosmological constant, the Einstein soliton (g, ξ, a, b) for dark fluid is:*
(i) steady: if $p = \frac{\lambda}{K}$,
(ii) shrinking: if $p > \frac{\lambda}{K}$,
(iii) or expanding: if $p < \frac{\lambda}{K}$.

applying equation (4.1) in equation (2.9), we have

$$S(\chi_1, \chi_2) = \Big(\frac{r}{2} + Kp\Big) g(\chi_1, \chi_2), \tag{4.7}$$

contracting equation (4.7) and using $g(\xi, \xi) = -1$, we get

$$r = -4Kp. \tag{4.8}$$

Now, using equation (3.5) and (4.8), we get

$$4a - b = -\frac{32}{5}Kp + 4div\xi, \tag{4.9}$$

Now, using equation (3.7) and (4.8), we get

$$a - b = -\frac{8}{5}Kp. \tag{4.10}$$

Thus, from equation (4.9) and (4.10), we get

$$a = -\frac{8}{5}Kp + \frac{4}{3}div\xi \text{ and } b = \frac{4}{3}div\xi. \tag{4.11}$$

Since, for Einstein soliton $b = 0$, therefore, from equation (4.11), we get $a = -\frac{8}{5}Kp$.
We know that soliton will be steady if $a = 0$, therefore, from equation (4.11), we get $p = 0$.
Again, soliton will be shrinking if $a < 0$, therefore, from equation (4.11), we get $K < 0$ and $p < 0$ or $K > 0$ and $p > 0$.
Also, soliton will be expanding if $a > 0$, therefore, from equation (4.11), we get $K < 0$ and $p > 0$ or $K > 0$ and $p < 0$.

Hence, we arrive at the following conclusion:

Theorem 4.2. *In a Bochner flat Lorentzian Kähler manifold satisfying Einstein field equation without cosmological constant, the Einstein soliton (g, ξ, a) for dark fluid is:*
(i) steady: if $p = 0$,
(ii) shrinking: if $K < 0$ and $p < 0$ or $K > 0$ and $p > 0$,
(iii) or expanding: if $K < 0$ and $p > 0$ or $K > 0$ and $p < 0$.

5 Behaviour of η-Einstein Soliton in a stiff matter

If $p = \sigma$, then perfect fluid is referred to as stiff matter, and the energy momentum tensor in this case become

$$T(\chi_1, \chi_2) = p[g(\chi_1, \chi_2) + 2\eta(\chi_1)\eta(\chi_2)], \tag{5.1}$$

applying equation (5.1) in equation (2.6), we have

$$S(\chi_1, \chi_2) = \Big(\frac{r}{2} + Kp - \lambda\Big)g(\chi_1, \chi_2) + 2Kp\eta(\chi_1)\eta(\chi_2), \tag{5.2}$$

contracting equation (5.2) and using $g(\xi, \xi) = -1$, we obtain

$$r = 2(2\lambda - Kp). \tag{5.3}$$

Now, using equation (3.5) and (5.3), we obtain

$$4a - b = \frac{16}{5}(2\lambda - Kp) + 4div\xi, \tag{5.4}$$

Now, using equation (3.7) and (5.3), we obtain

$$a - b = \frac{4}{5}(2\lambda - Kp). \tag{5.5}$$

Thus, from equation (5.4) and (5.5), we get

$$a = \frac{4}{5}(2\lambda - Kp) + \frac{4}{3}div\xi \text{ and } b = \frac{4}{3}div\xi. \tag{5.6}$$

Since, for Einstein soliton $b = 0$, therefore, from equation (5.6), we get $a = \frac{4}{5}(2\lambda - Kp)$.
We know that soliton will be steady if $a = 0$, therefore, from equation (5.6), we get $p = \frac{2\lambda}{K}$.
Again, soliton will be shrinking if $a < 0$, therefore, from equation (5.6), we get $p > \frac{2\lambda}{K}$.
Also, soliton will be expanding if $a > 0$, therefore, from equation (5.6), we get $p < \frac{2\lambda}{K}$.

Hence, we arrive at the following conclusion:

Theorem 5.1. *In a Bochner flat Lorentzian Kähler manifold satisfying Einstein field equation with cosmological constant, the Einstein soliton (g, ξ, a) for stiff matter is:*
(i) steady: if $p = \frac{2\lambda}{K}$,
(ii) shrinking: if $p > \frac{2\lambda}{K}$,
(iii) or expanding: if $p < \frac{2\lambda}{K}$.

applying equation (5.1) in equation (2.9), we have

$$S(\chi_1, \chi_2) = \left(\frac{r}{2} + Kp\right) g(\chi_1, \chi_2) + 2Kp\eta(\chi_1)\eta(\chi_2), \tag{5.7}$$

contracting equation (5.7) and using $g(\xi, \xi) = -1$, we get

$$r = -2Kp. \tag{5.8}$$

Now, using equation (3.5) and (5.8), we get

$$4a - b = -\frac{16}{5}Kp + 4div\xi, \tag{5.9}$$

Now, using equation (3.7) and (5.8), we get

$$a - b = -\frac{4}{5}Kp. \tag{5.10}$$

Thus, from equation (5.9) and (5.10), we get

$$a = -\frac{4}{5}Kp + \frac{4}{3}div\xi \text{ and } b = \frac{4}{3}div\xi. \tag{5.11}$$

Since, for Einstein soliton $b = 0$, therefore, from equation (5.11), we get $a = -\frac{4}{5}Kp$.
We know that soliton will be steady if $a = 0$, therefore, from equation (5.11), we get $p = 0$.
Again, soliton will be shrinking if $a < 0$, therefore, from equation (5.11), we get $K < 0$ and $p < 0$ or $K > 0$ and $p > 0$.
Also, soliton will be expanding if $a > 0$, therefore, from equation (5.11), we get $K < 0$ and $p > 0$ or $K > 0$ and $p < 0$.

Hence, we arrive at the following conclusion:

Theorem 5.2. *In a Bochner flat Lorentzian Kähler manifold satisfying Einstein field equation without cosmological constant, the Einstein soliton (g, ξ, a) for stiff matter is:*
(i) steady: if $p = 0$,
(ii) shrinking: if $K < 0$ and $p < 0$ or $K > 0$ and $p > 0$,
(iii) or expanding: if $K < 0$ and $p > 0$ or $K > 0$ and $p < 0$.

6 Behaviour of η-Einstein Soliton in a dust fluid

The energy momentum tensor in a dust fluid is [16]

$$T(\chi_1, \chi_2) = \sigma\eta(\chi_1)\eta(\chi_2), \tag{6.1}$$

applying equation (6.1) in equation (2.6), we have

$$S(\chi_1, \chi_2) = -\left(\lambda - \frac{r}{2}\right) g(\chi_1, \chi_2) + K\sigma\eta(\chi_1)\eta(\chi_2), \tag{6.2}$$

contracting equation (6.2) and using $g(\xi, \xi) = -1$, we obtain

$$r = (4\lambda + K\sigma). \tag{6.3}$$

Now, using equation (3.5) and (6.3), we obtain

$$4a - b = \frac{8}{5}(4\lambda + K\sigma) + 4div\xi, \tag{6.4}$$

Now, using equation (3.7) and (6.3), we obtain

$$a - b = \frac{2}{5}(4\lambda + K\sigma). \tag{6.5}$$

Thus, from equation (6.4) and (6.5), we get

$$a = \frac{2}{5}(4\lambda + K\sigma) + \frac{4}{3}div\xi \text{ and } b = \frac{4}{3}div\xi. \tag{6.6}$$

Since, for Einstein soliton $b = 0$, *therefore, from equation (6.6), we get* $a = \frac{2}{5}(4\lambda + K\sigma)$.
We know that soliton will be steady if $a = 0$, *therefore, from equation (6.6), we get* $\sigma = -\frac{4\lambda}{K}$.
Again, soliton will be shrinking if $a < 0$, *therefore, from equation (6.6), we get* $\sigma < -\frac{4\lambda}{K}$.
Also, soliton will be expanding if $a > 0$, *therefore, from equation (6.6), we get* $\sigma > -\frac{4\lambda}{K}$.

Hence, we arrive at the following conclusion:

Theorem 6.1. *In a Bochner flat Lorentzian Kähler manifold satisfying Einstein field equation with cosmological constant, the Einstein soliton* (g, ξ, a) *for dust fluid is:*
(i) steady: if $\sigma = -\frac{4\lambda}{K}$,
(ii) shrinking: if $\sigma < -\frac{4\lambda}{K}$,
(iii) or expanding: if $\sigma > -\frac{4\lambda}{K}$.

applying equation (6.1) in equation (2.9), we have

$$S(\chi_1, \chi_2) = \frac{r}{2}g(\chi_1, \chi_2) + K\sigma\eta(\chi_1)\eta(\chi_2), \tag{6.7}$$

contracting equation (6.7) and using $g(\xi, \xi) = -1$, *we obtain*

$$r = K\sigma. \tag{6.8}$$

Now, using equation (3.5) and (6.8), we get

$$4a - b = \frac{8}{5}K\sigma + 4div\xi, \tag{6.9}$$

Now, using equation (3.7)and (6.8), we get

$$a - b = \frac{2}{5}K\sigma. \tag{6.10}$$

Thus, from equation (6.9) and (6.10), we get

$$a = \frac{2}{5}K\sigma + \frac{4}{3}div\xi \text{ and } b = \frac{4}{3}div\xi. \tag{6.11}$$

Since, for Einstein soliton $b = 0$, *therefore, from equation (6.11), we get* $a = \frac{2}{5}K\sigma$.
We know that soliton will be steady if $a = 0$, *therefore, from equation (6.11), we get* $\sigma = 0$.
Again, soliton will be shrinking if $a < 0$, *therefore, from equation (6.11), we get* $K < 0$ *and* $\sigma > 0$ *or* $K > 0$ *and* $\sigma < 0$.
Also, soliton will be expanding if $a > 0$, *therefore, from equation (6.11), we get* $K < 0$ *and* $\sigma < 0$ *or* $K > 0$ *and* $\sigma > 0$.

Hence, we arrive at the following conclusion:

Theorem 6.2. *In a Bochner flat Lorentzian Kähler manifold satisfying Einstein field equation without cosmological constant, the Einstein soliton* (g, ξ, a) *for dust fluid is:*
(i) steady: if $\sigma = 0$,
(ii) shrinking: if $K < 0$ *and* $\sigma > 0$ *or* $K > 0$ *and* $\sigma < 0$,
(iii) or expanding: if $K < 0$ *and* $\sigma < 0$ *or* $K > 0$ *and* $\sigma > 0$.

7 Behaviour of η-Einstein Soliton in a radiation fluid

Perfect fluid is known as radiation fluid if $\sigma = 3p$. The energy momentum tensor for radiation is [15]

$$T(\chi_1, \chi_2) = p[g(\chi_1, \chi_2) + 4\eta(\chi_1)\eta(\chi_2)], \tag{7.1}$$

applying equation (7.1) in equation (2.6), we obtain

$$S(\chi_1, \chi_2) = \Big(\frac{r}{2} + Kp - \lambda\Big)g(\chi_1, \chi_2) + 4Kp\eta(\chi_1)\eta(\chi_2), \tag{7.2}$$

contracting equation (7.2) and using $g(\xi, \xi) = -1$, we obtain

$$r = 4\lambda. \tag{7.3}$$

Now, using equation (3.5) and (7.3), we obtain

$$4a - b = \frac{32}{5}\lambda + 4div\xi, \tag{7.4}$$

Now, using equation (3.7) and (7.3), we obtain

$$a - b = \frac{8}{5}\lambda. \tag{7.5}$$

Thus, from equation (7.4) and (7.5), we get

$$a = \frac{8}{5}\lambda + \frac{4}{3}div\xi \text{ and } b = \frac{4}{3}div\xi. \tag{7.6}$$

Since, for Einstein soliton $b = 0$, therefore, from equation (7.6), we get $a = \frac{8}{5}\lambda$.
We know that soliton will be steady if $a = 0$, therefore, from equation (7.6), we get $\lambda = 0$.
Again, soliton will be shrinking if $a < 0$, therefore, from equation (7.6), we get $\lambda < 0$.
Also, soliton will be expanding if $a > 0$, therefore, from equation (7.6), we get $\lambda > 0$.

Hence, we arrive at the following conclusion:

Theorem 7.1. *In a Bochner flat Lorentzian Kähler manifold satisfying Einstein field equation with cosmological constant, the Einstein soliton (g, ξ, a) for radiation fluid is:*
(i) steady: if $\lambda = 0$,
(ii) shrinking: if $\lambda < 0$,
(iii) or expanding: if $\lambda > 0$.

8 Conclusion remarks

The results of this paper emphasised the different conditions for the Einstein soliton to be steady, shrinking or expanding in terms of isotropic pressure, cosmological constant and energy density for perfect fluid in a Bochner flat Lorentzian Kähler manifold obeying Einstein field equation with cosmological constant and without cosmological constant. Also, results conclude the dependency of isotropic pressure on cosmological constant and energy density for Einstein soliton to be steady, shrinking or expanding in dark fluid, stiff matter, dust fluid and radiation fluid in a Bochner flat Lorentzian Kähler manifold.

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