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# SUBMANIFOLDS IN SEMI-RIEMANNIAN MANIFOLDS WITH GOLDEN STRUCTURE

Iqra Ansari, Shadab A. Khan, Sheeba Rizvi and M. Aamir Qayyoom

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**Abstract** A Riemannian manifold possesses a golden Riemannian structure when it is equipped with a tensor field $J$ of type $(1,1)$ that satisfies the golden section relation $J^2 = J + I$, together with a pure Riemannian metric $g$. we explore some fundamental characteristics of the induced structure on submanifolds that are submerged within golden Semi-Riemannian manifolds. Effective relations for induced structures on submanifolds of codimension 2 are derived. We additionally provide an illustration of a submanifold within a golden Semi-Riemannian manifold.

## 1 Introduction

The investigation of submanifolds is a captivating subject within the realm of differential geometry. The origins of this field can be traced back to the study of the geometry of plane curves and surfaces, which was initially established by Fermat. Subsequently, the field has undergone several developments in the realms of differential geometry and mechanics, particularly. The field of research is active and continues to play a significant part in the advancement of contemporary differential geometry. The submanifolds of an approximately $r$-para-contact Riemannian manifold were investigated by Ahmad M et. al. in their studies referenced as [1], [2], [3], [5], [6], [8] and [17]. These investigations focused on the properties of the submanifolds in relation to the manifold's semi-symmetric and quater symmetric connections. Hretcanu [23] conducted a study on the submanifolds present in almost product Riemannian manifolds. The investigation of CR-submanifolds within the context of LP-Sasakian manifolds was conducted by Ahmad, Ozgur, and Haseeb [27]. The construction of the golden structure on a differentiable Riemannian manifold $(\overline{M}, g)$ was carried out by Crasmareanu and Hretcanu [15]. This construction can be seen as a specific instance of a polynomial structure [21] that is based on the golden ratio. The integrability conditions of golden Riemannian structures were explored by Gezer et al. [19]. The Golden structure has been examined in previous studies, including in references [4], [10], [18], [20], [22] and [29]. Hretcanu and Crasmareanu [14] provided a definition of metallic structure that encompasses the concept of golden structure. The investigation of submanifolds within metallic manifolds was conducted by Blaga and Hretcanu [24] and [25]. Metallic structure also studied in [7] and [9] . Hretcanu and Crasmareanu [16] conducted a study on certain properties pertaining to invariant submanifolds within a Riemannian manifold that possesses a golden structure. The lightlike hypersurface of a Golden Riemannian manifold was investigated by Poyraz and Erol [28]. The slant submanifolds of golden Riemannian manifolds were investigated by Bahadur and Uddin [13]. In this research, we investigate the submanifold of a golden semi-Riemannian manifold, building upon previous studies in the field.The present paper is structured in the following manner: In the second section, the concept of golden Riemannian manifolds is introduced and defined. In Section 3, we present a discussion on the various features that arise from the induced structure $(P, g, \xi, u, a)$ on a submanifold that is submerged within a golden semi-Riemannian manifold. In the preceding section, we have presented an illustrative instance of a golden semi-Riemannian structure defined on Euclidean space and its associated submanifolds.

## 2 Golden Semi-Riemannian Manifolds

**Definition 2.1.** A polynomial structure on a manifold $M$ is called a golden structure if it is determined by an (1,1) tensor field $J$ which satisfies the equation

$$J^2 = J + I, \tag{2.1}$$

Let $I$ denote the identity operator acting on the Lie algebra $X(M)$, which consists of the vector fields defined on the manifold $M$. A Riemannian metric $g$ is said to be $J$-compatible if:

$$g(JX, Y) = g(X, JY) \tag{2.2}$$

For any elements $X$ and $Y$ in the set $X(M)$, if we replace $Y$ with $JY$, then we obtain the following result:

$$g(JX, JY) = g(X, JY) + g(X, Y) \tag{2.3}$$

A Riemannian manifold $(M, g)$ equipped with a golden structure $J$ such that the Riemannian metric $g$ is compatible with $J$ is referred to as a golden Riemannian manifold. The pair $(g, J)$ is denoted as the golden Riemannian structure on $M$.

**Definition 2.2.** A manifold is considered semi-Riemannian (or pseudo-Riemannian) if its infinitesimal distance, denoted as $(ds)^2$, is comparable to that of a pseudo-Euclidean space with (p,q) metric, where q is not equal to zero. In other words, a manifold is semi-Riemannian if its infinitesimal distance satisfies this condition.

$$(ds)^2 = \sum_{j=1}^{p}(dx^j)^2 - \sum_{j=p+1}^{n}(dx^j)^2$$

When the summand located on the rightmost side is not equal to zero.

**Definition 2.3.** A Riemannian manifold endowed with a semi-Riemannian metric g and a golden structure $J$ so that the Riemannian metric $g$ is $J$-compatible called a golden semi-Riemannian manifold.

# 3 Properties of Induced Strcuture on Submanifold in Golden Semi-Riemannian Manifolds.

Let us consider the scenario where $M$ is a submanifold of dimension $n$ and codimension $r$, which is isometrically immersed in a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, J)$ of dimension $(n + r)$, where $n$ and $r$ are natural numbers.The notation $T_xM$ is used to represent the tangent space of the manifold $M$ at a specific point $x$ belonging to $M$. Similarly, $T_x^{\perp}M$ denotes the normal space of $M$ at the point $x$.Consider the differential $i$ of the immersion $i : M \to \bar{M}$.The Riemannian metric $g$ induced on the manifold $M$ is defined as $g(X, Y) = g(iX, iY)$ for all $X, Y \in \chi(M)$.We shall consider a local orthonormal basis $N_1, N_2, ...., N_r$ for the normal space $T_x^{\perp}M$. It is assumed that the indices $\alpha, \beta, \gamma$ iterate through the range $1, 2, ..., r$.For any $X$ belonging to the tangent space $T_xM$, the vector fields $J(iX)$ and $J(N_\alpha)$ can be expressed as a decomposition of their tangential and normal components.

$$J(iX) = iP(X) + \sum_{\alpha=1}^{r} u_\alpha(X)N_\alpha, \tag{3.1}$$

$$J(N_\alpha) = (\xi_\alpha) + \sum_{\beta=1}^{n} a_{\alpha\beta}N_\beta, \tag{3.2}$$

Where $P$ is a $(1, 1)$ tensor field on $M$, $\xi \in \xi(M)$, $u_\alpha$ are $1 - forms$ on $M$ and $(a_{\alpha\beta})_r$ is an $rXr$ matrix of smooth real fuctions on $M$.

**Proposition 3.1.** *The structure denoted by $\sum$ is defined as $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$.The induced metric on the submanifold $M$, which is derived from the golden semi-Riemannian structure $(g, J)$ on $\bar{M}$, can be described by the following set of equations:*

$$P^2(X) = P(X) + X - \sum_{\alpha} u_\alpha(X)\xi_\alpha,$$
$$u_\alpha(PX) = u_\alpha(X) - \sum_{\beta} a_{\alpha\beta}u_\beta(X),$$
$$a_{\alpha\beta} = a_{\beta\alpha}$$
$$u_\beta(\xi_\alpha) = \delta_{\alpha\beta} + a_{\alpha\beta} - \sum_{\gamma} a_{\alpha\gamma}a_{\gamma\beta},$$
$$P(\xi_\alpha) = \xi_\alpha - \sum_{\beta} a_{\alpha\beta}\xi_\beta,$$
$$u_\alpha(X) = g(X, \xi_\alpha),$$
$$g(PX, Y) = g(X, PY),$$
$$g(PX, PY) = g(X, PY) + g(X, Y) + \sum_{\alpha} u_\alpha(X)u_\alpha(Y),$$

*for every $X, Y \in \chi(M)$, where $\delta_{\alpha\beta}$ is Kronecker delta.*

**Definition 3.2.** A submanifold $M$ within a manifold $\bar{M}$ equipped with a structural tensor field $J$ (i.e., $J$ is a tensor field defined on $barM$) is said to be invariant under $J$ if, for every $x \in M$, the image of the tangent space $T_x$ under the action of $J$ is contained within the tangent space $T_x(M)$.

**Remark 3.3.** The induced structure $\sum = (P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ On the submanifold $M$, we consider the golden Riemannian structure. The pair $(g, J)$ is said to be invariant if and only if the components $u_\alpha$ are equal to zero, which is equivalent to $\xi_\alpha$ being equal to zero for every $\alpha$ in the range $(1, ...., r)$.
The Gauss and Weingarten formulas are

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{\alpha=0}^{r} h_\alpha(X, Y)N_\alpha, \tag{3.3}$$

$$\bar{\nabla}_X N_\alpha = -A_\alpha X + \nabla_X^{\perp} N_\alpha \tag{3.4}$$

If $N_1, ..., N_r$ and $N'_1, ..., N'_r$ are two local orthogonal bases on a normal space $T_X^{\perp}$, then the decomposition of $N'_\alpha$ in the basis $N_1, ..., N_r$ is as follows.

$$N'_\alpha = \sum_{\gamma=1}^{r} K_\alpha^\gamma N_\gamma$$

for any $\alpha \in 1, ..., r$, where $(K_\alpha^\gamma)$ is an $rXr$ matrix and we have

$u_\alpha^{'} = \sum_\gamma K_\alpha^\gamma u_\gamma, \xi_\gamma^{'} = \sum_\gamma K_\alpha^\gamma \xi_\gamma$ and $a_{\alpha\beta}^{'} = \sum_\gamma K_\alpha^\gamma a_{\gamma\delta} K_\beta^\delta$ Thus, if $\xi_1, \xi_2, ..., \xi_r$ are linearly independent vector fields, then $\xi_1^{'}, \xi_2^{'}, ..., \xi_r^{'}$ are also linearly independent. We know that $a_{\alpha\beta}$ is symmetric in $\alpha$ and $\beta$, under a suitable transformation, we can find $a_{\alpha\beta}$ can be reduced to $a_{\alpha\beta}^{'} = \lambda_\alpha \delta_{\alpha\beta}$, where $\lambda_\alpha(\alpha \in 1, ..., r)$ are eigen values of the matrix $(a_{\alpha\beta})_r$ and in this case we have $u_\beta^{'}(\xi_\alpha) = \delta_{\alpha\beta}(1 + \lambda_\alpha - \lambda_\alpha\lambda_\beta)$ and from this we obtain $u_\alpha^{'}(\xi_\alpha) = (1 + \lambda_\alpha - \lambda_\alpha^2)$.

**Remark 3.4.** From Proposition 3.1, it follows that if $M$ is a non-invariant submanifold of codimension $r$ immersed in a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, J)$, where $M$ is $n$-dimensional and the tangent vector fields $xi_1, xi_2, ..., xi_r$ are linearly independent, then we obtain.

$$\| \xi_\alpha \|^2 = 1 + a_{\alpha\alpha} - \sum_\gamma a_{\alpha\gamma}^2$$

and, for $\alpha \neq \beta$, we have

$$\sum_\gamma a_{\alpha\gamma} a_{\gamma\beta} = a_{\alpha\beta}.$$

For the normal connection $\nabla_X^\perp N_\alpha$, we have the decomposition

$$\nabla_X^\perp N = \sum_{\beta=1}^r l_{\alpha\beta}(X) N_\beta \tag{3.5}$$

for every $X \in \chi(M)$. Thus, we obtain

$$l_{\alpha\beta} = -l_{\beta\alpha}$$

for any $\alpha, \beta \in 1, ..., r$.

**Theorem 3.5.** *Consider an $n$-dimensional submanifold $M$ of codimension $r$ in a golden semi-Riemannian manifold with structure $(\bar{M}, \bar{g}, J)$. If the structure $J$ is parallel with respect to the Levi-Civita connection $\bar{\nabla}$ defined on $\bar{g}$, then the induced structure $(P, g, u_\alpha, xi_\alpha, (a_\alpha\beta)_r)$ on $M$ induced by the structure $J$ possesses the following properties:*

$$(\nabla_X P)(Y) = \sum_\alpha [g(A_\alpha X, Y)\xi_\alpha + u_\alpha(Y) A_\alpha X], \tag{3.6}$$

$$(\bigtriangledown_X u_\alpha)(Y) = \sum_\beta [h_\beta(X, Y) a_{\beta\alpha} - u_\beta(Y) l_{\alpha\beta}(X)] - h_\alpha(X, PY), \tag{3.7}$$

$$\bigtriangledown_X \xi_\alpha = -P(A_\alpha X) + \sum_\beta a_{\alpha\beta} A_\beta X + \sum_\beta l_{\alpha\beta}(X)\xi_\beta, \tag{3.8}$$

$$X(a_{\alpha\beta}) = -u_\alpha(A_\beta X) - u_\beta(A_\alpha X) + \sum_\gamma [l_{\alpha\gamma}(X) a_{\gamma\beta} + l_{\beta\gamma}(X) a_{\alpha\gamma}], \tag{3.9}$$

*for any $X \in \chi(M)$.*

*Proof.* . Using (3.1) and $\bar{\nabla} J = 0$, we obtain

$$J(\bar{\nabla}_X Y) = \bar{\nabla}_X (PY) + \bar{\nabla}_X \sum_\alpha (u_\alpha(Y)) N_\alpha$$

$$J(\bar{\nabla}_X Y) = \bar{\nabla}_X PY + \sum_\alpha [u_\alpha(Y) \bar{\nabla}_X N_\alpha + N_\alpha \bar{\nabla}_X (u_\alpha(Y))]$$

Using (3.3) and (3.4), we obtain

$$\begin{aligned} J[\nabla_X Y + \sum_{\alpha=1}^r h_\alpha(X, Y) N_\alpha] \quad = \quad & \nabla_X (PY) + \sum_{\alpha=1}^r h_\alpha(X, PY) N_\alpha + \\ & \sum_\alpha [u_\alpha(Y)(-A_\alpha X + \nabla_X^\perp N_\alpha) + N_\alpha(\nabla_X (u_\alpha(Y) + \\ & \sum_{\beta=1}^r h_\beta(X, u(Y)) N_\beta]. \end{aligned}$$

Using (3.1), (3.2) and (3.5), we obtain

$$\begin{aligned} \sum_{\alpha=1}^r h_\alpha(X, Y)\xi_\alpha + \sum_{\alpha=1}^r h_\alpha(X, Y) \sum_{\beta=1}^r a_{\alpha\beta} N_\beta \quad = \quad & (\nabla_X P)(Y) + \sum_{\alpha=1}^r h_\alpha(X, PY) N_\alpha - \sum_\alpha u_\alpha(Y) A_\alpha X + \\ & \sum_\alpha u_\alpha(Y)[\sum_\beta l_{\alpha\beta}(X) N_\beta] + \sum_\alpha (\nabla_X u_\alpha)(Y) N_\alpha. \end{aligned}$$

By comparing the tangential and normal components, equations (3.6) and (3.7) are derived.
Using (3.2) and $\bar{\nabla} J = 0$, we obtain

$$J(\bar{\nabla}_X (N_\alpha)) = \bar{\nabla}_X \xi_\alpha + \bar{\nabla}_X \sum_\beta a_{\alpha\beta} N_\beta.$$

Using (3.3), (3.4) and (3.5), we obtain

$$-P(A_\alpha X) - \sum_\alpha u_\alpha(A_\alpha X)N_\alpha + \sum_\beta l_{\alpha\beta}(X)\xi_\beta + \sum_\beta l_{\alpha\beta}(X)\sum_\gamma a_{\beta\gamma}N_\gamma =$$

$$\nabla_X\xi_\alpha + \sum_\alpha h_\alpha(X,\xi_\alpha)N_\alpha + \sum_\beta X(a_{\alpha\beta})N_\beta - \sum_\beta a_{\alpha\beta}A_\beta X + \sum_\beta a_{\alpha\beta}\sum_\gamma l_{\beta\gamma}(X)N_\gamma.$$

By identifying the tangential and normal components in the aforementioned equation, we can derive equations (3.8) and (3.9) respectively.

**Definition 3.6.** If we have the equality $N_P(X,Y) - 2\sum_\alpha du_\alpha(X,Y)\xi_\alpha = 0$ for any $X, Y \in \chi(M)$, then the $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced structure on submanifold $M$ in a golden Riemannian manifold $(\bar{M}, \bar{g}, J)$ is said to be normal.

**Remark 3.7.** The condition of compatibility, $\bar{\nabla}J = 0$, where $\bar{\nabla}$ represents the Levi-Civita connection in relation to the metric $\bar{g}$, signifies the integrability of the structure $J$. This integrability is equivalent to the Nijenhuis torsion tensor field of $J$ vanishing.

$$N_J(X,Y) = [JX, JY] + J^2[X,Y] - J[JX,Y] - J[X,JY].$$

In order to proceed with this assumption, it is imperative to establish the following overarching lemma:

**Lemma 3.8.** *Assuming the presence of a golden structure $J$ on a manifold barM, alongside a linear connection $D$ featuring torsion $T$. If $N_J$ represents the Nijenhuis torsion tensor field of $J$, then we can derive.*

$$\begin{aligned} N_J(X,Y) &= (D_{JX}J)(Y) - (D_{JY}J)(X) - T[JX,JY] - JT(X,Y) - \\ &\quad T(X,Y) + J(D_YJ)(X) + J(T(JX,Y)) - J(D_XJ)(Y) + T(X,JY). \end{aligned}$$

*Proof.* . According to the definition of torsion $T$, It can be inferred that

$$[X,Y] = D_XY - D_YX - T(X,Y) \tag{3.10}$$

and from this we get

$$[JX,JY] = D_{JX}JY - D_{JY}JX - T(JX,JY), \tag{3.11}$$

$$[JX,Y] = D_{JX}Y - D_YJX - T(JX,Y), \tag{3.12}$$

$and$

$$[X,JY] = D_XJY - D_{JY}X - T(X,JY), \tag{3.13}$$

By utilising the relations $(D_XJ)(Y) = D_XJY - J(D_XY)$ and (2.1), and substituting the relations (3.10), (3.11), (3.12), and (3.13) into the formula for the Nijenhuis tensor field of $J$, we are able to derive the desired result.

$$\begin{aligned} N_J(X,Y) &= D_{JX}JY - D_{JY}JX - T(JX,JY) - (J+I)[X,Y] - \\ &\quad J[D_{JX}Y - D_YJX - T(JX,Y)] - J[D_XJY - D_{JY}X - T(X,JY)] \end{aligned}$$

$$\begin{aligned} N_J(X,Y) &= (D_{JX}J)(Y) + J(D_{JX}Y) - (D_{JY}J)(X) - J(D_{JY}X) - T[JX,JY] + \\ &\quad J(D_XY) - J(D_YX) - JT(X,Y) + D_XY - D_YX - T(X,Y) - J(D_{JX}Y) + \\ &\quad J((D_YJ)(X) + J(D_YX)) + J(D_{JY}X) + T[X,JY] \end{aligned}$$

$$\begin{aligned} N_J(X,Y) &= (D_{JX}J)(Y) - (D_{JY}J)(X) - T[JX,JY] - JT(X,Y) - T(X,Y) + \\ &\quad J(D_YJ)(X) + J(T(JX,Y)) - J(D_XJ)(Y) + T[X,JY]. \end{aligned}$$

**Proposition 3.9.** *Consider a submanifold $M$ of codimension $r$ in a golden semi-Riemannian manifold $(barM, barg, J)$. If the induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ on $M$ is normal and the normal connection $\nabla^\perp$ on $M$ is identically zero (i.e. $l_{\alpha\beta} = 0$), then we have the following equality.*

$$\sum_\alpha g(X,\xi_\alpha)(PA_\alpha - A_\alpha P)(Y) = \sum_\alpha g(Y,\xi_\alpha)(PA_\alpha - A_\alpha P)(X)$$

*for any $X, Y \in \chi(M)$.*

*Proof.* . According to the definition provided in equation 3.6, we can deduce that

$$N_P(X,Y) - 2\sum_\alpha du_\alpha(X,Y)\xi_\alpha = 0$$

for any $X, Y \in \chi(M)$. Then we have

$$\sum_\alpha g(X,\xi_\alpha)(PA_\alpha - A_\alpha P)(Y) - \sum_\alpha g(Y,\xi_\alpha)(PA_\alpha - A_\alpha P)(X)$$

$$+ \sum_{\alpha\beta}[g(X,\xi_\beta)l_{\alpha\beta}(X) - g(Y,\xi_\beta)l_{\alpha\beta}(Y)]\xi_\alpha = 0$$

for any $X, Y \in \chi(M)$.
Moreover, if the normal connection $\nabla^\perp$ of $M$ is identically zero, i.e., $l_{\alpha\beta} = 0$, then we can deduce.

$$\sum_\alpha g(X,\xi_\alpha)(PA_\alpha - A_\alpha P)(Y) = \sum_\alpha g(Y,\xi_\alpha)(PA_\alpha - A_\alpha P)(X). \tag{3.14}$$

**Proposition 3.10.** *Given the previous result as a premise, it can be stated that Proposition 3.9 is independent of the basis chosen in the normal space $T_\perp x(M)$ for any x?M.*

*Proof.* . If $N_\alpha^{'}$ is another basis in $T_x^\perp(M)$, then we have

$$N_\alpha^{'} = \sum_\beta O_{\alpha\beta} N_\alpha, \tag{3.15}$$

where $(O_{\alpha\beta})_r$ is an orthogonal matrix. From the condition $\bar\nabla_X N_\alpha^{'} = 0$, we obtain

$$\bar\nabla_X N_\alpha^{'} = \sum_\beta O_{\alpha\beta} \bar\nabla_X N_\beta + \sum_\beta \bar\nabla_X O_{\alpha\beta} N_\beta \tag{3.16}$$

$$\sum_\beta X(O_{\alpha\beta} N_\alpha) = 0 \tag{3.17}$$

For any $X \in M$. $N_\beta$ is linearly independent set, then

$$O_{\alpha\beta} = constant.$$

On the other hand,

$$\bar\nabla_X N_\alpha^{'} = -A_\alpha^{'} X$$

$$\bar\nabla_X N_\alpha^{'} = \sum_\beta \bar\nabla_X (O_{\alpha\beta}) N_\beta - \sum_\beta O_{\alpha\beta} A_\beta X.$$

Thus, from the relations (3.14), $(3.15) and (3.16)$, we obtain

$$-A_\alpha^{'} X = \sum_\beta \bar\nabla_X (O_{\alpha\beta}) N_\beta - \sum_\beta O_{\alpha\beta} O_{\beta\alpha}, \tag{3.18}$$

$$A_\alpha^{'} X = \sum_\beta O_{\alpha\beta} A_\beta X.$$

Therefore, we have

$$J(N_\alpha^{'}) = i * \xi_\alpha + \sum_\beta a_{\alpha\beta}^{'} N_\beta^{'}.$$

Using (3.14), we obtain

$$J(N_\alpha^{'}) = i * \xi_\alpha^{'} + \sum_\beta \sum_\gamma a_{\alpha\beta}^{'} O_{\beta\gamma} N_\gamma. \tag{3.19}$$

Using (3.2) and (3.14), we get

$$J(N_\alpha^{'}) = \sum_\beta O_{\alpha\beta} \xi_\beta + \sum_\beta \sum_\gamma O_{\alpha\beta} a_{\beta\gamma} N_\gamma. \tag{3.20}$$

From (3.17) and (3.18), we obtain

$$\xi_\alpha^{'} = \sum_\beta O_{\alpha\beta} \xi_\beta \tag{3.21}$$

and

$$\sum_\beta \sum_\gamma a_{\alpha\beta}^{'} O_{\beta\gamma} = \sum_\beta \sum_\gamma O_{\alpha\beta} a_{\beta\gamma}.$$

On the basis $N_1^{'}, ...., N_r^{'}$, the condition of proposition 3.10 becomes

$$\sum_\alpha g(X, \xi_\alpha^{'})(PA_\alpha^{'} - A_\alpha^{'} P)(Y) = \sum_\alpha g(Y, \xi_\alpha^{'})(PA_\alpha^{'} - A_\alpha^{'} P)(X).$$

From (3.16) and (3.19), we get

$$\sum_\alpha g(X, O_{\alpha\beta}\xi_\beta)(PO_{\alpha\gamma}A_\gamma - O_{\alpha\gamma}A_\gamma P)(Y) - \sum_\alpha g(Y, O_{\alpha\beta}\xi_\beta)(PO_{\alpha\gamma}A_\gamma - O_{\alpha\gamma}A_\gamma P)(X) \quad =$$

$$\sum_\alpha O_{\alpha\beta} O_{\alpha\gamma} [g(X, \xi_\beta)(PA_\gamma - A_\gamma P)(Y) - g(Y, \xi_\beta)(PA_\gamma - A_\gamma P)] = 0.$$

From the orthogonality of the matrix $(O_{\alpha\beta})$, it follows that

$$\sum_\alpha [g(X, \xi_\alpha)(PA_\alpha - A_\alpha P)(Y) - g(Y, \xi_\alpha)(PA_\alpha - A_\alpha P)(X)] = 0.$$

Hence, it can be concluded that Proposition 3.9 is independent of the basis chosen in the normal space $T_x^\perp(M)$ for any $x \in M$.

**Lemma 3.11.** *Consider a submanifold $M$ embedded in a golden semi-Riemannian manifold $(\bar M, \bar g, J)$. Consider the structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced on $M$. Then*

$$g((PA_\alpha - A_\alpha P)(X), Y)$$

*on $M$ is skew-symmetric for any $X, Y \in \chi(M)$.*

*Proof.* .

$$g(PA_\alpha X - A_\alpha PX, Y) = g(PA_\alpha X, Y) - g(A_\alpha PX, Y)$$
$$g(PA_\alpha X - A_\alpha PX, Y) = g(X, A_\alpha PY) - g(PA_\alpha Y, X)$$
$$g(PA_\alpha X - A_\alpha PX, Y) = -g((PA_\alpha - A_\alpha P)(Y), X)).$$

So, $g((PA_\alpha - A_\alpha P)(X), Y)$ is skew-symmetric.

**Proposition 3.12.** *Let $M$ be a submanifold of codimension $r \geq 2$ in a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, J)$, where the structure $J$ is parallel to the Levi-Civita connection $nablaperp$ and vanishes identically on the normal bundle $T \perp (M)$ (i.e., $l_\alpha \beta = 0$). In this case, the tangent vector fields $xi_1, xi_2, ..., xi_r$ are linearly independent if and only if the determinant of the matrix $(I_r + A)$*

*Proof.* . Let $K_1, ..., K_r$ denote a set of real numbers that possess certain properties.

$$K_1\xi_1 + K_2\xi_2 + .... + K_r\xi_r = 0 \tag{3.22}$$

in any point $x \in M$. From the equality (3.6), we obtain

$$g(\xi_\alpha, \xi_\beta) = u_\beta(\xi_\alpha) = \delta_{\alpha\beta} + a_{\alpha\beta} - \sum_\gamma a_{\alpha\gamma}a_{\gamma\beta}. \tag{3.23}$$

Multiplying the equality (3.20) by $\xi_\alpha$ for any $\alpha \in (1, 2, ..., r)$ and using the equality (3.21), we obtain

$$\begin{cases} k_1(1 + a_{11} - \sum_\gamma a_{1\gamma}a_{\gamma 1}) + k_2(a_{12} - \sum_\gamma a_{1\gamma}a_{\gamma 2}) + ...... + k_r(a_{1r} - \sum_\gamma a_{1\gamma}a_{\gamma r}) = 0 \\ k_1(a_{21} - \sum_\gamma a_{2\gamma}a_{\gamma 2}) + k_2(1 + a_{22} - \sum_\gamma a_{2\gamma}a_{\gamma 2}) + ...... + k_r(a_{2r} - \sum_\gamma a_{r\gamma}a_{\gamma r}) = 0 \\ ------------------------------------ \\ k_1(a_{r1} - \sum_\gamma a_{r\gamma}a_{\gamma 1}) + k_2(a_{r2} - \sum_\gamma a_{r\gamma}a_{\gamma 2}) + ...... + k_r(1 + a_{rr} - \sum_\gamma a_{r\gamma}a_{\gamma r}) = 0. \end{cases}$$

The linear system of equations possesses a solitary solution. The condition for $k_1 = k_2 = \ldots = k_r = 0$ is that the determinant of the matrix does not equal zero. In addition, it should be noted that the determinant of the linear system of equations can be obtained by calculating the determinant of the matrix provided below.

$$I_r + \begin{pmatrix} a_{11} & a_{12} & a_{13}.....a_{1r} \\ a_{21} & a_{22} & a_{23}.....a_{2r} \\ . & . & ...... \\ . & . & ..... \\ a_{r1} & a_{r2} & a_{r3}.....a_{rr} \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12}.....a_{1r} \\ a_{21} & a_{22}.....a_{2r} \\ . & . \\ . & . \\ a_{r1} & a_{r2}.....a_{rr} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12}.....a_{1r} \\ a_{21} & a_{22}.....a_{2r} \\ . & . \\ . & . \\ a_{r1} & a_{r2}.....a_{rr} \end{pmatrix},$$

that is determinant of the matrix

$$I_r + A - A^2.$$

**Lemma 3.13.** *Consider a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, J)$, where $\bar{M}$ is the manifold, $\bar{g}$ is the metric, and $J$ is the structure. Let $M$ be a $n$-dimensional submanifold of co-dimension 2 in $\bar{M}$. The submanifold $M$ possesses a normal induced structure denoted by $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta}))$. Furthermore, the structure $J$ is parallel to the Levi-Civita connection $\bar{\nabla}$. If the normal connection $\nabla^\perp$ is identically zero (i.e., $l_{\alpha\beta} = 0$), it is not necessary for the following equation to hold true.*

$$g(Y, \xi_1)(PA_1 - A_1P)(X) + g(Y, \xi_2)(PA_2 - A_2X)(X) + g((PA_1 - AP_1)(X), Y)\xi_1$$
$$+g((PA_2 - A_2P)X, Y)\xi_2 \quad = \quad 0. \tag{3.24}$$

*for any $X, Y \in \chi(M)$.*

*Proof.* . By applying Lemma 3.11, we can derive.

$$g(Y, \xi_1)(PA_1 - A_1P)(Y) + g(X, \xi_2)(PA_2 - A_2P)(Y)$$

$$= g(Y, \xi_1)(PA_1 - A_1P)(X) + g(Y, \xi_2)(PA_2 - A_2P)(X)$$

for any $X, Y \in \chi(M)$. Multiplying by $Z \in \chi(M)$ we have

$$g(X, \xi_1)g((PA_1 - A_1P)(Y), Z) + g(X, \xi_2)g((PA_2 - A_2P)(Y), Z)$$

$$= g(Y, \xi_1)g((PA_1 - A_1P)(X), Z) + g(Y, \xi_2)g((PA_2 - A_2P)(X), Z) \tag{3.25}$$

for any $X, Y, Z \in \chi(M)$.

By inverting Y with respect to Z in the final equation, we derive

$$g(X, \xi_1)g((PA_1 - A_1P)Z, Y) + g(X, \xi_2)g((PA_2 - A_2P)Z, Y)$$

$$= g(Z, \xi_1)g((PA_1 - A_1P)X, Y) + g(Z, \xi_2)g((PA_2 - A_2P)(X), Y). \tag{3.26}$$

The inclusion of equalities By referring to equations (3.23) and (3.24), we derive

$$g(X, \xi_1)g((PA_1 - A_1P)(Y), Z) + g(X, \xi_2)g((PA_2 - A_2P)(Y), Z)$$

$$+g(X, \xi_1)g((PA_1 - A_1P)Z, Y) + g(X, \xi_2)g((PA_2 - A_2P)Z, Y)$$

$$= g(Y, \xi_1)g((PA_1 - A_1P)(X), Z) + g(Y, \xi_2)g((PA_2 - A_2P)(X), Z)$$

$$+g(Z, \xi_1)g((PA_1 - A_1P)X, Y) + g(Z, \xi_2)g((PA_2 - A_2P)(X), Y).$$

By virtue of the property of skew-symmetry, we are able to derive.

$$g([g(Y,\xi_1)((PA_1 - A_1P)(X), Z) + (g(Y,\xi_2)(PA_2 - A_2P)(X), Z)$$

$$+g((PA_1 - PA_1)X, Y)\xi_1 + g((PA_2 - A_2P)X, Y)\xi_2], Z) = 0$$

However, if $g$ is a semi-Riemannian metric, it follows that we have.

$$g([(Y,\xi_1)((PA_1 - A_1P)(X), Z) + (g(Y,\xi_2)(PA_2 - A_2P)(X), Z)$$

$$+g((PA_1 - A_1P)(X), Y)\xi_1) + g((PA_2 - A_2P)(X), Y)\xi_2], Z) = 0$$

Or

$$g([(Y,\xi_1)((PA_1 - A_1P)(X), Z) + (g(Y,\xi_2)(PA_2 - A_2P)(X), Z)$$

$$+g((PA_1 - A_1P)(X), Y)\xi_1) + g((PA_2 - A_2P)(X), Y)\xi_2], Z) \neq 0.$$

for any $Z \in \chi(M)$.
Therefore, it is not imperative that the equality (3.22) is valid.

**Lemma 3.14.** *Let $M$ be a $n$-dimensional submanifold of codimension 2 in a golden semi-Riemannian manifold $(\bar{M}, \bar{g}, J)$. The submanifold has a normal induced structure denoted as $(P, g, u_\alpha, xi_alpha, (a_\alpha\beta)_r)$. The structure $J$ is parallel to the Levi-Civita connection $\nabla\ \perp$ and vanishes identically, i.e., $l_\alpha\beta = 0$. However, it is important to note that the following equations*

$$(PA_1 - A_1P)\xi_1 = 0,$$
$$(PA_2 - A_2P)\xi_2 = 0,$$
$$(PA_1 - A_1P)\xi_2 = 0,$$
$$(PA_2 - A_2P)\xi_1 = 0,$$

*Proof.* . With $X = Y = \xi_1$ in equality (3.22)

$$g(\xi_1,\xi_1)(PA_1 - A_1P)(\xi_1) + g(\xi_1,\xi_1)(PA_2 - A_2P)(\xi_1)$$

$$+g((PA_1 - A_1P)(\xi_1), \xi_1) + g((PA_2 - A_2P)(\xi_1), \xi_1)\xi_2 = 0.$$

Utilizing, $g(\xi_1,\xi_1) = a + \sigma \neq 0, g(\xi_1,\xi_1) = 0$

$$g((PA_1 - A_1P)(\xi_1), \xi_1) = -g((PA_1 - A_1P)(\xi_1), \xi_2)$$

Since, $g$ is semi-Riemannian metric, we have

$$g((PA_1 - A_1P)(\xi_1), \xi_2) = 0$$

or

$$g((PA_1 - A_1P)(\xi_1), \xi_2) \neq 0$$

Hence,

$$(PA_1 - A_1P)\xi_1 = 0$$

or

$$(PA_1 - A_1P)\xi_1 \neq 0$$

With $X = Y = \xi_2$, in equality (3.22), we obtain

$$g(\xi_2,\xi_2)(PA_1 - A_1P)(\xi_2) + g(\xi_1,\xi_2)(PA_2 - A_2P)\xi_1$$

$$+g((PA_1 - A_1P)(\xi_1), \xi_2)\xi_1 + g((PA_2 - A_2P)(\xi_1), \xi_1)\xi_2 = 0.$$

Using that $g(\xi_2,\xi_2) = b + \sigma \neq 0, g(\xi_1,\xi_2) = 0$.

$$g((PA_2 - A_2P)\xi_2, \xi_2) = -g((PA_2 - A_2P)\xi_2, \xi_1)$$

Since, $g$ is semi-Riemannian metric, we have

$$g((PA_2 - A_2P)\xi_2, \xi_1) = 0$$

or

$$g((PA_2 - A_2P)\xi_2, \xi_1) \neq 0$$

Hence,

$$(PA_2 - A_2P)\xi_2 = 0$$

or

$$(PA_2 - A_2P)\xi_2 \neq 0$$

If we put $X = \xi_1$ and $Y = \xi_2$ in equality (3.22), we obtain

$$g(\xi_2,\xi_1)(PA_1 - A_1P)\xi_1 + g(\xi_1,\xi_2)(PA_2 - A_2P)\xi_1$$

$$+g((PA_1 - A_1P)(\xi_1), \xi_2)\xi_2 + g((PA_2 - A_2P)(\xi_1), \xi_2)\xi_2 = 0.$$

Using that $g(\xi_2, \xi_2) = b + \sigma \neq 0, g(\xi_1, \xi_2) = 0$,
Since, $g$ is semi-Riemannian metric, we have

$$g((PA_2 - A_2P)\xi_1, \xi_2) = 0$$

or

$$g((PA_2 - A_2P)\xi_1, \xi_2) \neq 0$$

Hence,

$$(PA_2 - A_2P)\xi_1 = 0.$$

or

$$(PA_2 - A_2P)\xi_1 \neq 0.$$

Again $X = \xi_2$ and $Y = \xi_1$, we obtain

$$g(\xi_1, \xi_1)(PA_1 - A_1P)\xi_2 + g(\xi_2, \xi_2)(PA_2 - A_2P)\xi_1$$

$$+g((PA_1 - A_1P)(\xi_2), \xi_1)\xi_1 + g((PA_2 - A_2P)(\xi_2), \xi_1)\xi_2 = 0.$$

Using that $g(\xi_1, \xi_1) = a + \sigma \neq 0, g(\xi_1, \xi_2) = 0$, we obtain
Since, $g$ is semi-Riemannian metric, we have

$$g((PA_1 - A_1P)\xi_2, \xi_1) = 0$$

or

$$g((PA_1 - A_1P)\xi_2, \xi_1) \neq 0$$

Hence,

$$(PA_1 - A_1P)\xi_2 = 0.$$

or

$$(PA_1 - A_1P)\xi_2 \neq 0.$$

Therefore, it is not imperative that the aforementioned equations (Lemma 3.14) hold true.

**Proposition 3.15.** *Consider a submanifold M of dimension n and codimension 2 in a golden semi-Riemannian manifold ($\bar{M}$,$\bar{g}$,J). The submanifold M possesses a normal induced structure denoted by (P,g,$u_\alpha$,$xi_\alpha$,$(a_\alpha\beta)_r$). In the given context, it can be observed that structure J is in parallel with the Levi-Civita connection $\nabla^\perp$, resulting in the complete vanishing of $l_{\alpha\beta}$ and the non-zero value of $\sigma$, while also satisfying the condition that the trace of A is equal to zero. If P commutes with the Weingarten operator $A_\alpha(\alpha \in 1, 2)$, it is not necessarily the case that the following relations occur.*

$$(PA_1 - A_1P)(X) = 0, \tag{3.27}$$

$$(PA_2 - A_2P)(X) = 0 \tag{3.28}$$

$\forall X \in \chi(M)$

*Proof.* .

$$g((PA_\alpha - A_\alpha P), \xi_\beta) = g(PA_\alpha X, \xi_\beta) - g((A_\alpha P)X, \xi_\beta)$$
$$g((PA_\alpha - A_\alpha P), \xi_\beta) = -[g(PA_\alpha\xi_\beta, X) - g((A_\alpha P)\xi_\beta, X)]$$
$$g((PA_\alpha - A_\alpha P)X, \xi_\beta) = -g((PA_\alpha - A_\alpha P)\xi_\beta, X),$$

where $\alpha\beta \in 1, 2$, from the last Lemma

$$(PA_\alpha - A_\alpha P)\xi_\beta = 0$$

for any $\alpha\beta \in 1, 2$.

$$g((PA_1 - A_1P)X, \xi_\beta) = -g((PA_1 - A_1P)\xi_\beta, X),$$

Since, $g$ is semi-Riemannian metric, therefore we have,

$$g((PA_1 - A_1P)X, \xi_\beta) = 0$$

or

$$g((PA_1 - A_1P)X, \xi_\beta) \neq 0.$$

Hence,

$$(PA_1 - A_1P)X = 0.$$

or

$$(PA_1 - A_1P)X \neq 0.$$

Similarly

$$(PA_2 - A_2P)X = 0.$$

Therefore, it is not necessary for the relations (3.25) and (3.26) to occur.
For any $\alpha\beta \in 1, 2$. In this paper, we make the assumption that $M$ is a submanifold of codimension 2 in the golden semi-Riemannian manifold $(\bar{M}, \bar{g}, J)$, where $M$ has $n$ dimensions. We also consider the induced structure $(P, g, u_\alpha, xi_\alpha, (a_\alpha\beta)_2)$ on $M$, where $alpha$ and $beta$ are elements of the set $1, 2$ It is assumed that the normal connection disappears completely, resulting in $(l_{\alpha\beta} = 0)$. Under these circumstances, the relationships described in Proposition 3.1 can be expressed in the following formats:

$$P^2X = P(X) + X - m_1(X)\xi_1 - m_2(X)\xi_2, \tag{3.29}$$
$$m_1(PX) = m_1(X) - a_{11}m_1(X) - a_{12}m_2(X),$$
$$m_2(PX) = m_2(X) - a_{21}m_1(X) - a_{22}m_2(X),$$
$$m_1(\xi_1) = 1 + a_{11} + a_{11}^2 + a_{12}^2,$$

$$m_2(\xi_2) = 1 + a_{22} + a_{12}^2 + a_{22}^2,$$
$$m_1(\xi_1) = u_2(\xi_1) = a_{21} - a_{21}(a_{11} + a_{22}),$$
$$P(\xi_1) = \xi_1 - a_{11}\xi_1 - a_{12}\xi_2,$$
$$P(\xi_2) = \xi_2 - a_{21}\xi_1 - a_{22}\xi_2,$$
$$g(PX, PY) = g(X, PY) + g(X, Y) + m_1(X)m_1(Y) + m_2(X)m_2(Y)$$

for any $X, Y \in \chi(M)$. We denote by $A = \begin{pmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{pmatrix}$.

Furthermore, by applying Theorem 3.5 under the assumption that the normal connection $\nabla^\perp$ vanishes completely (i.e., $l_{\alpha\beta}$), we can derive.

$$(\nabla_X P)(Y) = g(A_1X, Y)\xi_1 + g(A_2X, Y)\xi_2 + g(Y, \xi_1)A_1X + g(Y, \xi_2)A_2X,$$
$$(\nabla_X m_1)(Y) = -g(A_1X, PY) + a_{11}g(A_1X, Y) + a_{21}g(A_2X, Y),$$
$$(\nabla_X m_2)(Y) = -g(A_2X, PY) + a_{12}g(A_1X, Y) + a_{22}g(A_2X, Y),$$
$$\nabla_X \xi_1 = -P(A_1X) + a_{11}A_1X + a_{12}A_2X,$$
$$\nabla_X \xi_2 = -P(A_2X) + a_{21}A_1X + a_{22}A_2X,$$
$$X(a_{12}) = -2m_1(A_1X),$$
$$X(a_{22}) = -2m_2(A_2X).$$

**Remark 3.16.** One possible simplifying assumption for these relations is that the sum of the elements $a_{11}$ and $a_{22}$ is equal to zero. Hence, it follows that the trace of matrix A is equal to zero. Given this assumption, let us represent $a_{11}$ as $-a_{22}$, where $k$ is a constant.It can be observed from the given relations that $a_{12} = a_{21} = l$ and $1 - k^2 - l^2 = \sigma$.

$$m_1(\xi_1) = m_2(\xi_2) = k + \sigma \Leftrightarrow g(\xi_1, \xi_1) = g(\xi_2, \xi_2) = l + \sigma,$$
$$m_1(\xi_2) = m_2(\xi_1) = l,$$
$$m_1(PX) = (1 - k)m_1(X) - lm_2(X),$$
$$m_2(PX) = (1 - k)m_2(X) - lm_1(X),$$

and

$$P(\xi_1) = (1 - k)\xi_1 - l\xi_2, \tag{3.30}$$
$$P(\xi_2) = (1 - k)\xi_2 - l\xi_1. \tag{3.31}$$

**Proposition 3.17.** *Consider a submanifold $M$ of codimension 2 in a golden Semi-Riemannian manifold $(\bar{M}, \bar{g}, J)$. The structure $J$ is parallel to the Levi-Civita connection $\bar{\nabla}$ defined on $\bar{M}$. The submanifold $M$ has a normal induced structure $(P, g, u_\alpha, xi_\alpha, (a_\alpha\beta)_2)$. If the normal connection $\nabla^\perp$ is identically zero ($l_{\alpha\beta} = 0$), $\sigma$ is not equal to zero, and the trace of $A$ is zero, then it is not necessary for the following relation to hold.*

$$(a + \sigma)A_1\xi_1 + bA_1\xi_2 = h_1(\xi_1, \xi_1)\xi_1 + h_1(\xi_1, \xi_2)\xi_2)\xi_2, \tag{3.32}$$
$$(a + \sigma)A_1\xi_2 + bA_1\xi_1 = h_1(\xi_1, \xi_2)\xi_1 + h_1(\xi_2, \xi_2)\xi_2)\xi_2, \tag{3.33}$$
$$(a + \sigma)A_2\xi_1 + bA_2\xi_2 = h_2(\xi_1, \xi_1)\xi_1 + h_2(\xi_1, \xi_2)\xi_2)\xi_2, \tag{3.34}$$
$$(a + \sigma)A_2\xi_2 + bA_2\xi_1 = h_2(\xi_1, \xi_2)\xi_1 + h_2(\xi_2, \xi_2)\xi_2). \tag{3.35}$$

*Proof.* . By utilising the value of 3.25 and employing the operator P, it can be deduced that.

$$P^2A_1X = PA_1PX$$

for any $X \in \chi(M)$.
By utilising the equality (3.27) and substituting $X = \xi_1$ and $X = \xi_2$, we are able to derive the following expressions.

$$P(A_1\xi_1) + A_1\xi_1 - u_1(A_1\xi_1)\xi_1 - u_2(A_1\xi_1)\xi_2 = PA_1P\xi_1.$$

By utilising equation (3.28), we obtain.

$$(2 - P)A_1\xi_1 + (P - 1)aA_1\xi_1 + (P - 1)bA_1\xi_2 = h_1(\xi_1, \xi_1)\xi_1 + h_1(\xi_1, \xi_2)\xi_2. \tag{3.36}$$

Now,

$$P(A_1\xi_2) + A_1\xi_2 - u_1(A_1\xi_2)\xi_2 - u_2(A_1\xi_2)\xi_2 = PA_1P\xi_2.$$

Using (3.29), we obtain

$$A_1\xi_2 - A_1b\xi_1 - A_1a\xi_2 + A_1\xi_2 - PA_1(\xi_2 - b\xi_1 - a\xi_2) = h_1(\xi_1, \xi_2)\xi_1 + h_1(\xi_2, \xi_2)\xi_2. \tag{3.37}$$

The substitution $X \to PX$ is made in equation (3.25), resulting in

$$PA_1PX = A_1P^2X.$$

Using equality (3.27) and if we put $X = \xi_1$ and $X = \xi_2$ respectively, we get

$$PA_1P\xi_1 = A_1P\xi_1 + A_1\xi_1 - u_1(\xi_1)A_1\xi_1 - u_2(\xi_1)A_1\xi_2.$$

Using (3.28), we obtain

$$PA_1(\xi_1 - a\xi_1 - b\xi_2) = A_1(\xi_1 - a\xi_1 - b\xi_2) + A_1\xi_1 - u_1(\xi_1)A_1\xi_1 - u_2(\xi_1)A_1\xi_2,$$
$$(P - 2 + \sigma)A_1\xi_1 + (2 - P)aA_1\xi_1 + (2 - P)bA_1\xi_2 = 0 \tag{3.38}$$

and

$$PA_1PA\xi_2 = A_1P\xi_2 + A_1\xi_2 - u_1(\xi_2)A_1\xi_1 - u_2(\xi_2)A_1\xi_2.$$

Using (3.29), we obtain

$$PA_1((1-a)\xi_2 - b\xi_1) = A_1((1-a)\xi_2 - b\xi_1) + A_1\xi_2 - bA_1\xi_1 - (a+\sigma)A_1\xi_2,$$

$$(P-2+\sigma)A_1\xi_2 + (2-P)A_1a\xi_2 + (2-P)bA_1\xi_1 = 0. \tag{3.39}$$

Adding the relations (3.34) and (3.36), we obtain (3.30).
Adding (3.35) and (3.37), we obtain (3.31). Applying $P$ in the equality (3.26), it follows that

$$P^2A_2X = PA_2PX$$

For any element $X$ belonging to the set $\chi(M)$, by utilising equation (3.27) and considering the cases where $X$ is equal to $\xi_1$ and $\xi_2$ respectively, we can derive the following results.

$$(2-P)A_2\xi_1 + (P-1)aA_2\xi_1 + (P-1)bA_2\xi_2 = h_2(\xi_1,\xi_1)\xi_1 + h_2(\xi_1,\xi_2)\xi_2 \tag{3.40}$$

and

$$(2-P)A_2\xi_2 + (P-1)bA_2\xi_1 + (P-1)aA_2\xi_2 = h_2(\xi_1,\xi_2) + h_2(\xi_2,\xi_2)\xi_2.$$

The substitution $X \to PX$ is made in equation (3.26), resulting in

$$PA_2PX = A_2P^2X \tag{3.41}$$

By utilising equation (3.27) and substituting $X = \xi_1$ and $X = \xi_2$, we can derive the following result.

$$(P-2+\sigma)A_2\xi_1 + (2-P)aA_2\xi_1 + (2-P)bA_2\xi_2 = 0 \tag{3.42}$$

and

$$(P-2+\sigma)A_2\xi_2 + (2-P)aA_2\xi_2 + (2-P)bA_2\xi_1 = 0. \tag{3.43}$$

By summing the values of (3.38) and (3.40), we arrive at the result of (3.32). By combining equations (3.39) and (3.41), equation (3.33) can be derived.

**Theorem 3.18.** *Consider a submanifold $M$ of a golden semi-Riemannian manifold $\bar{M}$. Let $J$ be a structure on $M$ that is parallel to the Levi-Civita connection $\bar{\nabla}$ defined on $M$, meaning that $\bar{\nabla}J = 0$. If the vectors $\xi_\alpha$ ($\alpha = 1, 2, 3, ..., r$) are linearly independent, the function $T_r(P)$ is constant, and the manifold $M$ is totally umbilical, it is not necessarily the case that $M$ is totally geodesic.*

*Proof.* . Since

$$\nabla_X(a_{\alpha\beta}) = -m_\alpha(A_\beta X) - m_\beta(A_\alpha X) + \sum_\gamma [l_{\alpha\gamma}(X)a_{\gamma\beta}] + l_{\beta\gamma}(X)a_{\alpha\gamma}].$$

Putting $\alpha = \beta$, we have

$$\nabla_X(a_{\alpha\alpha}) = -2m_\alpha(A_\alpha X) + \sum_\gamma [l_{\alpha\gamma}(X)a_{\gamma\alpha}] + l_{\alpha\gamma}(X)a_{\alpha\gamma}].$$

Since, $a_{\alpha\beta}$ is symmetric and $l_{\alpha\beta}$ is skew-symmetric in $\alpha, \beta$, then $\sum_{\alpha\gamma} a_{\alpha\gamma}l_{\alpha\gamma}(X) = 0$. Since, $T_r(P) = constant$, we have $\sum_\alpha a_{\alpha\alpha} = constant$. Hence,

$$\sum_\alpha m_\alpha(A_\alpha X) = 0$$

$$\sum_\alpha g(X, A_\alpha\xi_\alpha) = 0$$

Given that $g$ is a semi-Riemannian metric, we can conclude that

$$\sum_\alpha g(X, A_\alpha\xi_\alpha) = 0$$

or

$$\sum_\alpha g(X, A_\alpha\xi_\alpha) \neq 0$$

Therefore, $\sum_\alpha A_\alpha\xi_\alpha = 0$ or $\sum_\alpha A_\alpha\xi_\alpha \neq 0$. Given that $\xi_\alpha$ is linearly independent, it follows that $A_\alpha$ can either be equal to zero or not equal to zero. Therefore, it is not necessary for $M$ to be totally geodesic.

# 4 Conclusion remarks

This paper aims is to obtain new transformations for Horn's double hypergeometric functions $G_1$, $G_2$ and $G_3$. Also, some new integral representations of Euler-type involving these functions in terms of hypergeometric functions of two variables have been discussed. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

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**Author information**

Iqra Ansari, Department of Mathematics and Statistics, Faculty of Science, Integral University, Lucknow, India.
E-mail: ansariiqra880@gmail.com

Shadab A. Khan, Department of Mathematics and Statistics, Faculty of Science, Integral University, Lucknow, India.
E-mail: sakhan.lko@gmail.com

Sheeba Rizvi, School of Liberal Art and Sciences, Era University, Lucknow, India.
E-mail: sheeba.rizvi7@gmail.com

M. Aamir Qayyoom, Department of Mathematics and Statistics, Faculty of Science, Integral University, Lucknow, India.
E-mail: aamir.qayyoom@gmail.com

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