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NUMERICAL ALGORITHM TO COMPUTE THE INVERSE OF TRIDIAGONAL QUASI-TOEPLITZ MATRIX

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Abstract The aim of this paper is to compute the inverse of the Tridiagonal Quasi-Toeplitz matrix by direct method. The proposed algorithm constructs a decomposition of the given matrix, thanks to the special structure of the considered matrix, into a sum of a band Toeplitz matrix and the rest of size $n \times n$. Then, the inverse of the matrix turns into the inverse of the obtained band matrix using the well known Schur complement and Sherman–Morrison–Woodbury formula. Illustrative example describing the different steps of the algorithm and numerical results are performed to show the effectiveness and accuracy of the algorithm.

1 Introduction

The structure of matrices is frequently brought about by the use of an approximation strategy or just by physics-related characteristics of the underlying problem. We can list the symmetrical matrices, bands, Toeplitz, circular, Hankel, Gauss and Vandermonde [1]. Quasi-Toeplitz matrix has a special structure and make up an intriguing class of Toeplitz matrices which can be encountered in many different applications. The objective of this paper is to propose a fast algorithm to reverse this very interesting type of Toeplitz matries. Then, we take advantage of these results to calculate the solutions of linear systems with Tridiagonal Quasi-Toeplitz matrices.

Different studies have researched Toeplitz matrices in great detail and they appear in a wide variety of problems, including image processing, control theory, integral equations, the orthogonal polynomials, partial differential equations (elliptical or parabolic), Padé's approximation, and other branches of numerical analysis.

In recent years, a number of quick inversion methods for Toeplitz matrices have been reported. Friedlander et al. use the generalized Levinson-Szego method [2], in addition to the multichannel Levinson algorithm [3], to create effective inversion recursions for the sums of Toeplitz and Hankel matrix products [4]. A system of linear equations with a strongly regular symmetric coefficient matrix, which is the sum of a real Toeplitz matrix and a real Hankel matrix, was solved using an approach provided by Gohberg et al. [5].The split Levinson and Schur algorithms are used to strongly regular Toeplitz-plus-Hankel matrices [6, 7, 8, 9].

The banded-Toeplitz matrix is an intriguing class of matrices that can be used to create quick algorithms [10, 11, 12, 13, 14, 15, 16, 17]. Over time, some extremely quick numerical techniques have been created for the resolution of a Toeplitz type system, which is crucial in many applications such that, numerical solutions of differential equations [18], digital solutions of multiple Markov chains from the modelling of girl waiting problems [19, 20], and the problem of image processing[21], where the Toeplitz matrix band is used as a preconditioned to accelerate the convergence of preconditioned conjugate gradient (PGC) techniques [22, 23, 24].

The main objective of this study is to calculate the inverse of the Tridiagonal Quasi-Toeplitz matrix and then take advantage of this results to solve the corresponding linear systems. It consists to take advantage of the special structure of Quasi-Toeplitz matrix and decompose it as a sum of two matrices where one of the two is a band matrix. Thus, the calculation of the inverse of the considered matrix amounts to the calculation of the inverse of this band matrix which will be carried out by an algorithm consisting in introducing the latter into a larger special triangular matrix and then use the Shur complement. Subsequently, the use of the well known Sherman–Morrison–Woodbury formula will make it possible to calculate the inverse of the special Toeplitz matrix considered.

The rest of this paper is organized as follows: In the second section we present some definitions of the Toeplitz matrix and the particular cases of them and the mathematical model of the considered problem. The third section is a general explanation and details to a very important algorithm, which is called Schur complement. In the fourth section, we present a Triangular method to calculate the inverse of the band-Toeplitz matrix. The fifth section is devoted to the proposed method of reversing the Quasi-Toeplitz matrix. Illustrative example describing the different steps of the algorithm, in addition to nmerical results for the resolution of a system with Quasi-Toeplitz symmetrical and asymmetrical, using the new approach, are presented and discussed in the sixth section.

2 Mathematical formulation

2.1 Definitions and notations

Definition 2.1. A matrix $T \in \mathbb{R}^{n\times n}$ is called Toeplitz, if $T = (t_{ij}) = (t_{i-j}),\ \forall i, j = 1, \ldots, n.$ i.e

$$T = \begin{bmatrix} t_0 & t_{-1} & \ldots & \ldots & t_{2-n} & t_{1-n} \\ t_1 & t_0 & t_{-1} & \ddots & \ddots & t_{2-n} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ t_{n-2} & \ddots & \ddots & t_1 & t_0 & t_{-1} \\ t_{n-1} & t_{n-2} & \ldots & \ldots & t_1 & t_0 \end{bmatrix}$$

If the element located at the intersection of row i and column j of T is noted $T_{i,j}$, we have:

$$T_{i,j} = T_{i+1,j+1} = T_{i-j},\ \forall i, j = 1, \ldots, n$$

Definition 2.2. A lower triangular matrix $T_{inf} \in \mathbb{R}^{n\times n}$ of type Toeplitz is defined as:

$$T_{inf} = \begin{bmatrix} t_0 & 0 & \ldots & \ldots & 0 & 0 \\ t_1 & t_0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ t_{n-2} & \ddots & \ddots & t_1 & t_0 & 0 \\ t_{n-1} & t_{n-2} & \ldots & \ldots & t_1 & t_0 \end{bmatrix}$$

$t_j \in \mathbb{K},\ \forall j = 0, \ldots, n-1 \quad and \quad t_0 \neq 0$

Definition 2.3. $T_{bande} \in \mathbb{R}^{n\times n}$, is a $2k+1$-banded-Toeplitz matrix with $t_{-k} \neq 0$, $t_k \neq 0$ and $k \in \mathbb{N}$, if it is in the form:

$$T_{bande} = \begin{bmatrix} t_0 & t_{-1} & \ldots & t_{-k} & 0 & \ldots & 0 & 0 \\ t_1 & t_0 & t_{-1} & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ t_k & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & t_{-k} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & t_1 & t_0 & t_{-1} \\ 0 & 0 & \ldots & 0 & t_k & \ldots & t_1 & t_0 \end{bmatrix}$$

2.2 Mathematical model

We are interested in computing the iverse of a very important class of Toeplitz matrix, it's the Tridiagonal Quasi-Toeplitz matrix. A matrix $QT \in \mathbb{R}^{n\times n}$ is called Quasi-Toeplitz, if $QT = t_{i,j} = t_{i-j}$, $\forall i,j = 1,\ldots,n$, where, $QT = t_{i,j} = t_{i-j} = 0$, $\forall i-j = -3,\ldots,2-n \; and \; i-j = 3,\ldots,n-2$
i.e :

$$QT = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & 0 & \dots & 0 & \gamma \\ t_1 & t_0 & t_{-1} & t_{-2} & \ddots & \ddots & 0 \\ t_2 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & t_{-2} \\ 0 & \ddots & \ddots & t_2 & t_1 & t_0 & t_{-1} \\ \gamma & 0 & \dots & 0 & t_2 & t_1 & t_0 \end{bmatrix}$$

where, $\gamma \neq 0$

Remark 2.4. Following the form of the Toeplitz matrix in general, we will notice that we can define the Quasi-Toeplitz matrix by its first row and its first column. Then we can replace QT with the matrix $Mat(l,c)$, where $l = [t_0, t_{-1}, t_{-2}, 0, \ldots, \ldots, 0, \gamma] \in \mathbb{R}^{1,n}$ is the first line of matrix QT, and $c = [t_0, t_1, t_2, 0, \ldots, \ldots, 0, \gamma]^T \in \mathbb{R}^{n,1}$ its first colon.

3 Schur complement

In linear algebra and the theory of matrices, the Schur complement of a block matrix is defined as follows:

Definition 3.1. Let M be a block matrix, and p, q are non-negative integers, and suppose A_1, A_2, A_3, A_4 are respectively $p\times p$, $p\times q$, $q\times p$ and $q\times q$ matrices of complex numbers.

$$M = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

So, M is a $(p+q)\times(p+q)$ matrix.
If A_1 is invertible, the Schur complement of the block A_1 of the matrix M is the $q\times q$ matrix defined by :

$$M/A_1 = A_4 - A_3A_1^{-1}A_2$$

If A_2 is invertible, the Schur complement of the block A_2 of the matrix M is the $q\times q$ matrix defined by :

$$M/A_2 = A_3 - A_4A_2^{-1}A_1$$

If A_3 is invertible, the Schur complement of the block A_3 of the matrix M is the $q\times q$ matrix defined by:

$$M/A_3 = A_2 - A_1A_3^{-1}A_4$$

If A_4 is invertible, then the Schur complement of the block A_4 of the matrix M is the $p\times p$ matrix defined by:

$$M/A_4 = A_1 - A_2A_4^{-1}A_3$$

3.1 Schur's complement role to reverse the block matrix.

Block matrices can be inverted with the use of Schur complements.

Proposition 3.2. *Let M be a block matrix*

$$M = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

If A_4 and its Schur complement M/A_4 are invertible, then M is invertible and

$$M^{-1} = \begin{bmatrix} (M/A_4)^{-1} & -(M/A_4)^{-1} A_2 A_4^{-1} \\ -A_4^{-1} A_3 (M/A_4)^{-1} & A_4^{-1} + A_4^{-1} A_3 (M/A_4)^{-1} A_2 A_4^{-1} \end{bmatrix}$$

Proof.

$$M = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} I & A_2 A_4^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 - A_2 A_4^{-1} A_3 & 0 \\ 0 & A_4 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_4^{-1} A_3 & I \end{bmatrix}$$

where I are identity matrices and 0 are matrices of zeros. As a result, the Schur complement $M/A_4 = A_1 - A_2 A_4^{-1} A_3$ appears in the upper-left $p \times p$ block.
Then, the inverse of M may be expressed involving A_4^{-1} and the inverse of Schur's complement, assuming it exists, as:

$$M^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}^{-1} = \left(\begin{bmatrix} I & -A_2 A_4^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 - A_2 A_4^{-1} A_3 & 0 \\ 0 & A_4 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_4^{-1} C & I \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} I & 0 \\ -A_4^{-1} A_3 & I \end{bmatrix} \begin{bmatrix} \left(A_1 - A_2 A_4^{-1} A_3\right)^{-1} & 0 \\ 0 & A_4^{-1} \end{bmatrix} \begin{bmatrix} I & -A_2 A_4^{-1} \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} \left(A_1 - A_2 A_4^{-1} A_3\right)^{-1} & -\left(A_1 - A_2 A_4^{-1} A_3\right)^{-1} A_2 A_4^{-1} \\ -A_4^{-1} A_3 \left(A_1 - A_2 A_4^{-1} A_3\right)^{-1} & A_4^{-1} + A_4^{-1} A_3 \left(A_1 - A_2 A_4^{-1} A_3\right)^{-1} A_2 A_4^{-1} \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} (M/A_4)^{-1} & -(M/A_4)^{-1} A_2 A_4^{-1} \\ -A_4^{-1} A_3 (M/A_4)^{-1} & A_4^{-1} + A_4^{-1} A_3 (M/A_4)^{-1} A_2 A_4^{-1} \end{bmatrix}$$

3.2 Factorization of a block matrix

A block matrix is frequently factorized into a combination of smaller block matrices using the Schur complements.

Proposition 3.3. *Let M be a block matrix*

$$M = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

If A_4 is invertible, then

$$M = \begin{bmatrix} I & A_2 A_4^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} M/A_4 & 0 \\ 0 & A_4 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_4^{-1} A_4 & iI \end{bmatrix}$$

where I are identity matrices and 0 are matrices of zeros.

Proof.

$$M/A_4 = A_1 - A_2A_4^{-1}A_3$$

Consequently, the three matrices product is

$$\begin{bmatrix} I & A_2A_4^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 - A_2A_4^{-1}A_3 & 0 \\ 0 & A_4 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_4^{-1}A_3 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & A_2A_4^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 - A_2A_4^{-1}A_3 & 0 \\ A_3 & A_4 \end{bmatrix}$$

$$= \begin{bmatrix} A_1 - A_2A_4^{-1}A_3 + A_2A_4^{-1}A_3 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

3.3 Factorization of the inverse of a block matrix

The inverse of a block matrix can be effectively factorized when the Schur complements are invertible.

Proposition 3.4. *Let M be a block matrix*

$$M = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

If A_4 is invertible, then

$$M^{-1} = \begin{bmatrix} I & 0 \\ -A_4^{-1}A_3 & I \end{bmatrix} \begin{bmatrix} (M/A_4)^{-1} & 0 \\ 0 & A_4^{-1} \end{bmatrix} \begin{bmatrix} I & -A_2A_4^{-1} \\ 0 & I \end{bmatrix}$$

where I are identity matrices and 0 are matrices of zeros.

Proof. If we multiply the factorization of M into three matrices derived above by the factorization of M^{-1} proposed here, we obtain the identity matrix because

$$\begin{bmatrix} I & -A_2A_4^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & A_2A_4^{-1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} (M/A_4)^{-1} & 0 \\ 0 & A_4^{-1} \end{bmatrix} \begin{bmatrix} (M/A_4) & 0 \\ 0 & A_4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ -A_4^{-1}A_3 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_4^{-1}A_3 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

3.4 Determinants of block matrices

Proposition 3.5. *[25] Let's*

$$M = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

A_1 is non-singular matrix. So we have :

$$detM = detA_1.det(A_4 - A_4A_1^{-1}A_2) \tag{3.1}$$

and

$$rank(A_4 - A_3A_1^{-1}A_2) = rank(M) - rank(A_1) \tag{3.2}$$

First, we develop a formula for the determinant:

Theorem 3.6. *Let M, N, Q be matrix, where, $M = N + Q$.*
If $N = \begin{bmatrix} N_{1,1} & N_{1,2} \\ N_{2,1} & N_{2,2} \end{bmatrix}$ and $Q = \begin{bmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{bmatrix}$
Are conformal partition and $N_{1,1}, Q_{2,2}$ are maximal, then

$$det(M) = \frac{det\begin{bmatrix} N_{1,1} & N_{1,2} \\ N_{2,1} & N_{2,2} \end{bmatrix} det\begin{bmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{bmatrix}}{detN_{1,1}detQ_{2,2}} \tag{3.3}$$

Proof. By (3.2) and maximality of $N_{1,1}$ and $Q_{2,2}$

$$N_{1,1} = N_{2,1}N_{1,1}^{-1}N_{1,2} \qquad Q_{1,1} = Q_{1,2}Q_{2,2}^{-1}Q_{2,1}$$

Thus, I_1 and I_2 denoting identity matrices of appropriate sizes.

$$N = \begin{bmatrix} N_{1,1} & 0 \\ N_{2,1} & I_2 \end{bmatrix} \begin{bmatrix} I_1 & N_{1,1}^{-1}N_{1,2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 & 0 \\ -N_{2,2}^{-1}N_{2,1} & N_{2,2}^{-1} \end{bmatrix} \begin{bmatrix} I_1 & 0 \\ N_{2,1} & N_{2,2} \end{bmatrix}$$

$$Q = \begin{bmatrix} N_{1,1} & 0 \\ N_{2,1} & I_2 \end{bmatrix} \begin{bmatrix} N_{1,1}^{-1} & 0 \\ -N_{2,1}N_{1,1}^{-1} & I_2 \end{bmatrix} \begin{bmatrix} 0 & Q_{1,2Q_{2,2}^{-1}} \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} I_1 & 0 \\ Q_{2,1} & Q_{2,2} \end{bmatrix}$$

So, that

$$M = \begin{bmatrix} N_{1,1} & 0 \\ N_{2,1} & I_2 \end{bmatrix} \begin{bmatrix} I_1 - N_{1,1}^{-1}N_{1,2}N_{2,2}^{-1}N_{2,1} & N_{1,1}^{-1}(N_{1,2} + Q_{1,2})Q_{2,2}^{-1} \\ 0 & I_2 - N_{2,1}N_{1,1}^{-1}Q_{1,2}Q_{2,2}^{-1} \end{bmatrix} \begin{bmatrix} I_1 & 0 \\ Q_{2,1} & Q_{2,2} \end{bmatrix} \tag{3.4}$$

Since,

$$det(I_1 - N_{1,1}^{-1}N_{1,2}Q_{2,2}^{-1}Q_{2,1}) = \frac{det(N_{1,1} - N_{2,1}Q_{2,2}^{-1}Q_{2,1})}{detN_{1,1}} = \frac{det\begin{bmatrix} N_{1,1} & N_{1,2} \\ Q_{2,1} & Q_{2,2} \end{bmatrix}}{detN_{1,1}detQ_{2,2}}$$

By (3.1) and

$$det(I_2 - N_{2,1}N_{1,1}^{-1}Q_{1,2}Q_{2,2}^{-1}) = \frac{det\begin{bmatrix} N_{1,1} & Q_{1,2} \\ N_{2,1} & Q_{2,2} \end{bmatrix}}{detN_{1,1}detQ_{2,2}}$$

(3.3) followes from (3.4).

Remark 3.7. From (3.4), we can also obtain a formula for M^{-1}.

Remark 3.8. In view of (3.1) can be written in other forms :

$$detM = det(N_{1,1} - Q_{1,2}Q_{2,2}^{-1}N_{2,1})det(Q_{2,2} - Q_{2,1}N_{1,1}^{-1}Q_{1,2}) \tag{3.5}$$

$$detM = det(N_{1,1} - N_{1,2}Q_{2,2}^{-1}Q_{2,1})det(Q_{2,2} - N_{2,1}N_{1,1}^{-1}Q_{1,1}) \tag{3.6}$$

Thus, if

$$M = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

We can choose :

$$N = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix}$$

We obtain (3.3) Schur's formula (3.1)

4 Inverse of a banded Toeplitz matrix

In the following, we present an algorithm which computes the inverse of 5-banded Toeplitz matrix $T_5 \in \mathbb{R}^{n\times n}$, given as follow:

$$T_5 = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & 0 & \dots & 0 & 0 \\ t_1 & t_0 & t_{-1} & t_{-2} & \ddots & \ddots & 0 \\ t_2 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & t_{-2} \\ 0 & \ddots & \ddots & t_2 & t_1 & t_0 & t_{-1} \\ 0 & 0 & \dots & 0 & t_2 & t_1 & t_0 \end{bmatrix}$$

The approach consist in embedding the banded Toeplitz matrix in a larger triangular Toeplitz matrix [26]. Specifically, we embed T_5 in a lower triangular Toeplitz matrix M of size $(n+2)\times(n+2)$ where the first column of M is given by:

$$r = [t_{-2}, t_{-1}, t_0, t_1, t_2, 0, \dots, \dots, 0, 0]^T \in \mathbb{R}^{n+2,1}$$

More precisely, if we note S a matrix of size 2×2 given by:

$$S = \begin{bmatrix} t_{-2} & 0 \\ t_{-1} & t_{-2} \end{bmatrix}$$

We can write M as follows:

$$M = \begin{bmatrix} t_{-2} & 0 & \dots & \dots & \dots & \dots & 0 & 0 \\ t_{-1} & t_{-2} & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ t_0 & t_{-1} & t_{-2} & \ddots & \ddots & \ddots & \ddots & \vdots \\ t_1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ t_2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & t_2 & t_1 & t_0 & t_{-1} & t_{-2} & 0 \\ 0 & \dots & 0 & t_2 & t_1 & t_0 & t_{-1} & t_{-2} \end{bmatrix} = \begin{bmatrix} S & 0 & \dots & 0 \\ & & & \vdots \\ & T_5 & & 0 \\ & & & S \end{bmatrix} \tag{4.1}$$

To compute the inverse of the matrix T_5, we present the following result:

Proposition 4.1. *[26]*
Let M be the matrix defined in (4.1); So, the matrix M^{-1} is a lower triangular matrix, defined by its first column: $[a_1, a_2, a_3, \dots, \dots, a_{n-1}, a_n, a_{n+1}, a_{n+2}]^T$. In addition, M^{-1} can be partitioned as follows:

$$M^{-1} = \begin{bmatrix} a_1 & 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 \\ a_2 & a_1 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ a_3 & a_2 & a_1 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a_{n-1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a_n & a_{n-1} & \ddots & \ddots & \ddots & \ddots & a_1 & 0 & 0 \\ a_{n+1} & a_n & a_{n-1} & \ddots & \ddots & a_3 & a_2 & a_1 & 0 \\ a_{n+2} & a_{n+1} & a_n & a_{n-1} & \dots & \dots & a_3 & a_2 & a_1 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

where,

$$A_1 = \begin{bmatrix} a_1 & 0 \\ a_2 & a_1 \\ \vdots & \ddots \\ \vdots & \ddots \\ \vdots & \ddots \\ a_{n-1} & \ddots \\ a_n & a_{n-1} \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 & \dots & \dots & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ a_1 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a_{n-3} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2} & a_{n-3} & \dots & \dots & a_1 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} a_{n+1} & a_n \\ a_{n+2} & a_{n+1} \end{bmatrix} \qquad A_4 = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & \dots & \dots & a_1 & 0 \\ a_n & a_{n-1} & a_{n-2} & \dots & \dots & a_2 & a_1 \end{bmatrix}$$

are matrices of size $n \times 2$, $n \times n$, 2×2, $2 \times n$, respectively.

Then, we investigate the structure of the matrix M^{-1} to compute the inverse of the banded Toeplitz matrix T_5. This result is presented in Theorem (4.2)

Theorem 4.2. *[26] Let T_5 be a non-singular banded Toeplitz matrix and M its associated lower triangular matrix. Suppose that M^{-1} is partitioned as follows:*

$$M^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

*where A_1, A_2, A_3, and A_4 are matrices of size $n \times 2$, $n \times n$, 2×2, $2 \times n$, respectively. If T_5 is non-singular, then A_3^{-1} is also non-singular and the **Schur complement** of the block A_3 for the matrix M^{-1} is defined by:*

$$T_5^{-1} = A_2 - A_1 A_3^{-1} A_4 \tag{4.2}$$

Proof. See [26] and section 3.

5 Main algorithm

In this section, we present our approach to calculate the inverse of matrix QT. This technique is based on a new decomposition of this Quasi-Toeplitz matrix and the application of the Sherman-Morrison-Woodbury formula. Indeed, the considered decomposition consists to express the matrix QT in the form of a 5-banded Toepliz matrix and the rest which will be decomposed in a well-determined form. Then, the application of the Sherman-Morrison-Woodbury formula reduced the computation of the inverse of QT to the inverse of T_5 as a banded Toeplitz matrix given on the decomposition (5.1) defined as follows:

$$QT = T_5 + R \tag{5.1}$$

where,

$$T_5 = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & 0 & \dots & 0 & 0 \\ t_1 & t_0 & t_{-1} & t_{-2} & \ddots & \ddots & 0 \\ t_2 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & t_{-2} \\ 0 & \ddots & \ddots & t_2 & t_1 & t_0 & t_{-1} \\ 0 & 0 & \dots & 0 & t_2 & t_1 & t_0 \end{bmatrix} \quad R = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & \gamma \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 & 0 & 0 & 0 \\ \gamma & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}$$

$T_5 \in \mathbb{R}^{n\times n}$ is a 5-band Toeplitz matrix and $R \in \mathbb{R}^{n\times n}$.
To apply the Sherman-Morrison-Woodbury formula on equation (5.1).
We consider $U, C \in \mathbb{R}^{n\times n}$ such that $R = UC^T$; with,

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \gamma \end{bmatrix} \quad and \quad C^T = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & \gamma \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 & 0 & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, we apply the formula of Sherman-Morrison-Woodbury to get:

$$QT^{-1} = (T_5 + UC^T)^{-1} = T_5^{-1} - T_5^{-1}U(I_n + C^T T_5^{-1} U)^{-1} T_5^{-1} \tag{5.2}$$

where, I_n is an identity matrix of size $n \times n$.

The different steps of the proposed algorithm to calculate the inverse of a Quasi-Toeplitz matrix are given as follows:
Step 1. Input: $n,\ t_0, t_{-1}, t_{-2}, t_1, t_2,$ and γ
Step 2. Recover T_5 and R from QT.
Step 3. Recover U and C from R.
Step 4. Give the matrix M as defined in (4.1).
Step 5. Calculate the inverse M^{-1}.
Step 6. Recover A_1, A_2, A_3 *et* A_4 from M^{-1}.
Step 7. Calculate T_5^{-1}, applying the expression (4.2)

$$T_5^{-1} = A_2 - A_1 A_3^{-1} A^4$$

.
Step 8. Output: Calculate QT^{-1}, applying the expression (5.2)

$$QT^{-1} = (T_5 + UC^T)^{-1} = T_5^{-1} - T_5^{-1}U(I_n + C^T T_5^{-1}U)^{-1}T_5^{-1}$$

6 Numerical results

In this section, three examples are considered to show the efficiency of the proposed algorithm. The first example is a typical test of a matrix of size $(8, 8)$ illustrating the different steps of the proposed algorithm. However, examples 2 and 3 are numerical tests carried out on Matlab on symmetric and asymmetric Quasi-Toeplitz matrices, of different sizes, implemented on an Intel(R) Core(TM) $i3 - 3110M$ CPU @ with a 2.40 GHz processor and 4 GO of RAM, to show the efficiency of the algorithm in terms of CPU-time and the solution of the considered system.

6.1 Example 1:

In this first example, we consider a symmetrical quasi-Toeplitz matrix of size 8×8 to illustrate the different steps of the proposed algorithm.
Step 1. The consideed matrix is given by:

$$l = [t_0 = 1, t_1 = 2, t_2 = 1, 0, \dots, \dots, 0, \gamma = 7]^T$$

i.e

$$QT = Mat(l) = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 0 & 7 \\ 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 7 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

Step 2. In this step, we recover T_5 and R from the matrix QT. We obtain:

$$T_5 = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 3. Factorisation of R in the form UC^T

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}, \quad C^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 4. The triangular matrix M of size $8+2=10$ obtained as defined in (4.1) is given by:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 1 \end{bmatrix}$$

Step 5. In this step, we compute the inverse of M, where we obtain:

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & 3 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & -6 & 3 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -22 & 12 & -6 & 3 & -2 & 1 & 0 & 0 & 0 & 0 \\ 41 & -22 & 12 & -6 & 3 & -2 & 1 & 0 & 0 & 0 \\ -78 & 41 & -22 & 12 & -6 & 3 & -2 & 1 & 0 & 0 \\ 147 & -78 & 41 & -22 & 12 & -6 & 3 & -2 & 1 & 0 \\ -276 & 147 & -78 & 41 & -22 & 12 & -6 & 3 & -2 & 1 \end{bmatrix}$$

Step 6. We recover the matrices A_1, A_2, A_3 and A_4 from the matrix M^{-1} as defineded in Theorem 4.1.

$$A_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 3 & -2 \\ -6 & 3 \\ 12 & -6 \\ -22 & 12 \\ 41 & -22 \\ -78 & 41 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ -6 & 3 & -2 & 1 & 0 & 0 & 0 & 0 \\ 12 & -6 & 3 & -2 & 1 & 0 & 0 & 0 \\ -22 & 12 & -6 & 3 & -2 & 1 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 147 & -78 \\ -276 & 147 \end{bmatrix} ; \; A_4 = \begin{bmatrix} 41 & -22 & 12 & -6 & 3 & -2 & 1 & 0 \\ -78 & 41 & -22 & 12 & -6 & 3 & -2 & 1 \end{bmatrix}$$

Step 7. We compute the inverse of T_5 from the considered matries in step 6, using the formula: $T_5^{-1} = A_2 - A_1 A_3^{-1} A_4$, to obtain the following result:

$$T_5^{-1} = \begin{bmatrix} 0.7037 & 0.4444 & -0.5926 & -0.6667 & 0.3333 & 0.7407 & 0.1111 & -0.9630 \\ 0.4444 & -0.3333 & 0.2222 & 0.0000 & -0.0000 & -0.1111 & -0.0000 & 0.1111 \\ -0.5926 & 0.2222 & 0.1481 & 0.6667 & -0.3333 & -0.5185 & -0.1111 & 0.7407 \\ -0.6667 & -0.0000 & 0.6667 & 0.0000 & -0.0000 & -0.3333 & -0.0000 & 0.3333 \\ 0.3333 & 0.0000 & -0.3333 & -0.0000 & 0.0000 & 0.6667 & 0.0000 & -0.6667 \\ 0.7407 & -0.1111 & -0.5185 & -0.3333 & 0.6667 & 0.1481 & 0.2222 & -0.5926 \\ 0.1111 & -0.0000 & -0.1111 & -0.0000 & 0.0000 & 0.2222 & -0.3333 & 0.4444 \\ -0.9630 & 0.1111 & 0.7407 & 0.3333 & -0.6667 & -0.5926 & 0.4444 & 0.7037 \end{bmatrix}$$

Step 8. Compute QT^{-1} using the Sherman-Morrison-Woodbury formula:

$$QT^{-1} = (T_5 + UC^T)^{-1} = T_5^{-1} - T_5^{-1}U(I_n + C^T T_5^{-1}U)^{-1}T_5^{-1}$$

So,

$$QT^{-1} = \begin{bmatrix} 0.0810 & -0.3565 & -0.0284 & 0.2514 & 0.1577 & -0.1534 & -0.3253 & 0.2372 \\ -0.3565 & 0.9560 & 0.7216 & -0.6861 & -0.9048 & 0.0966 & 1.3622 & -0.3253 \\ -0.0284 & 0.7216 & -0.3409 & 0.0170 & -0.1080 & 0.1591 & 0.0966 & -0.1534 \\ 0.2514 & -0.6861 & 0.0170 & 0.1491 & 0.8054 & -0.1080 & -0.9048 & 0.1577 \\ 0.1577 & -0.9048 & -0.1080 & 0.8054 & 0.1491 & 0.0170 & -0.6861 & 0.2514 \\ -0.1534 & 0.0966 & 0.1591 & -0.1080 & 0.0170 & -0.3409 & 0.7216 & -0.0284 \\ -0.3253 & 1.3622 & 0.0966 & -0.9048 & -0.6861 & 0.7216 & 0.9560 & -0.3565 \\ 0.2372 & -0.3253 & -0.1534 & 0.1577 & 0.2514 & -0.0284 & -0.3565 & 0.0810 \end{bmatrix}$$

6.2 Numerical resolution of linear systems with Quasi-Toeplitz matrices

The purpose of a matrix inversion calculation is to use it to solve systems of linear equations. In the following examples we will use our new approach to solve the symmetric and asymmetric quasi-Toeplitz linear systems to show that our method is one of the ideal ways to solve this type of problem.
We use our algorithm to compute x solution of the system $Ax = b$. The **Error** $= \frac{\|x-x^*\|_2}{\|x^*\|_2}$ is considered, where $\| \, . \, \|_2$ is the Euclidean vector norm to evaluate the quality of the approximation. It can be verified that $x^* = [1, 1, 1, \dots, 1, 1, 1]^T$ is the exact solution for the two considered examples.

Example 2:

For this example, we consider the $n \times n$ symmetric Quasi-Toeplitz linear system $QTx = b$ with: $t_0 = 1, t_{-1} = t_1 = 1, t_{-2} = t_2 = 2, \gamma = -1$ and $b = [3, 5, 7, \dots, 7, 5, 3]^T$.

Example 3:

In this example, we consider a non symmetric Quasi-Toeplitz linear system $QTx = b$, with: $t_0 = -1, t_{-1} = -1, t_{-2} = 2, t_1 = 1, t_2 = -1, \gamma = 1$ and $b = [1, 1, 0, \dots, 0, -2, 0]^T$.

Table 1: Numerical result for example 2

n	**Error**	CPU-time(s)	$Cond(QT)$
10	$7.195068e^{-16}$	$1.375000e^{-02}$	7.4953
10^2	$5.370129e^{-15}$	$4.687500e^{-02}$	610.9946
10^3	$1.215850e^{-14}$	$1.093750e^{-01}$	918.5586
10^4	$5.594362e^{-14}$	$3.046875e^{+00}$	$1.0761e^{+04}$

Table 2: Numerical result for example 3

n	**Error**	CPU-time(s)	$Cond(QT)$
10	$1.110223e^{-16}$	$4.687500e^{-02}$	13.3223
10^2	$2.362976e^{-16}$	$4.687500e^{-02}$	110.2692
10^3	$2.294821e^{-15}$	$7.812500e^{-02}$	$1.0679e^{+03}$
10^4	$4.438856e^{-15}$	$2.796875e^{+00}$	$1.0643e^{+04}$

The numerical results of example 2 are presented in Table 1. From this result we can see that for all the values of n, the errors remain very low, and that the calculated solutions are a good approximation to the exact solution, which proves the effectiveness of our approach. Table 2 shows the results of example 3, thanks to this example, we show that our algorithm is still valid for asymmetric system.

7 Conclusion

In this paper, we have presented a new algorithm to compute the inverse of a class of Quasi-Toeplitz matrices. We have broken down the matrix into a sum of two matrices where the first one is a 5-band Toeplitz matrix and the rest which is a matrix of size $n \times n$. Then, we applied Sherman Morrison formula to find the inverse of the decomposed matrix which reduce the computation of the inverse of the initial Toeplitz matrix to the inverse of a banded Toeplitz matrix. Numerical examples are presented showing the efficiency and the accuracy of the proposed method.

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