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ASYMPTOTIC BEHAVIOR OF POSITIVE GLOBAL SOLUTIONS OF AN INHOMOGENEOUS NONLINEAR EQUATION

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Abstract This paper concerns the asymptotic behavior near infinity of positive global solutions of the p-Laplacian equation

$$div\left(|\nabla v|^{p-2}\,\nabla v\right)+g(v)=0 \qquad \text{in } \mathbb{R}^N\backslash\{0\},$$

where $N>p>2,\ \delta>N(p-1)/(N-p)$ and $g(v)=v^\delta+h(|x|)$ such that h is a continuous and strictly positive function defined on $\mathbb{R}^N\backslash\{0\}$ satisfying $h(|x|)=o\left(|x|^{-p\delta/(\delta+1-p)}\right)$ as $|x|\to+\infty$. More precisely, we give an explicit behavior near infinity of radial solutions v that satisfy $\lim\limits_{|x|\to 0} v(x)=+\infty$.

1 Introduction

In this paper, we study the asymptotic behavior of positive global solutions of the following radial problem

$$\left(|v'|^{p-2}v'\right)'(r)+\frac{N-1}{r}|v'|^{p-2}v'(r)+v^\delta(r)+h(r)=0,\ r>0 \tag{1.1}$$

$$\lim_{r\to 0} v(r)=+\infty, \qquad \lim_{r\to 0} r^{(N-1)/(p-1)}v'(r)=0, \tag{1.2}$$

where $N>p>2,\ \delta>N(p-1)/(N-p),\ h$ is a continuous and strictly positive function on $(0,+\infty)$ satisfying $\lim\limits_{r\to+\infty} r^{p\,\delta/(\delta+1-p)}h(r)=0$.

This work constitutes a continuity of the work elaborated in [6] and [7] where the authors proved the existence of a positive global solution of problem (1.1)-(1.2). They noticed that the inhomogeneous term h has a crucial impact on the existence and asymptotic behavior of the solutions v to the problem (1.1)-(1.2). More precisely, if $N>p>2,\ \delta>N(p-1)/(N-p)$ and $h(r)=K\,r^{-p\delta/(\delta+1-p)}$, with

$$K=\frac{\delta+1-p}{p-1}\left(\frac{p-1}{\delta}\left(N-\frac{p\delta}{\delta+1-p}\right)\left(\frac{p}{\delta+1-p}\right)^{p-1}\right)^{\delta/(\delta+1-p)}.$$

Then equation (1.1) has an explicit solution

$$\widetilde{v}(r)=\left(\frac{p-1}{\delta}\left(N-\frac{p\delta}{\delta+1-p}\right)\left(\frac{p}{\delta+1-p}\right)^{p-1}\right)^{1/(\delta+1-p)} r^{-p/(\delta+1-p)}. \tag{1.3}$$

Moreover, they proved that if $h(r)\underset{+\infty}{\sim} Lr^{-p\,\delta/(\delta+1-p)}$ for some constant $L>0$, then under some conditions, the problem (1.1)-(1.2) has a solution v that behaves like $r^{-p/(\delta+1-p)}$ near infinity. In the case where h is negligible in front of $r^{-p\,\delta/(\delta+1-p)}$, it is difficult to find an equivalent of the solution v near infinity because v can be negligible in front of $r^{-p/(\delta+1-p)}$. This open question strongly depends on the behavior of h near infinity.

When $p = 2$, equation (1.1) becomes an inhomogeneous second-order elliptic equation that appears naturally in probability theory in the study of stochastic processes. In particular, equation (1.1) arises recently in a paper by Tzong-Yow Lee [16] establishing limit theorems for super-brownian motion. Moreover, Bernard [3] gave interesting results on the existence and nonexistence of this type of equations. Also, in [2], Bae studied the existence of global positive solutions, and in paper [1], he established the asymptotic behavior near the origin and infinity of positive radial solutions. We also invite the readers to see [15, 9, 18] for more details and the references therein. If the inhomogeneous term h is identically null, equation (1.1) becomes the classic Emden-Fowler equation. Existence results were obtained on $\mathbb{R}^N$ and $\mathbb{R}^N\backslash\{0\}$ in articles [10, 11, 12]. In the case $N > 2$, two critical values $N/(N-2)$ and $(N+2)/(N-2)$ appear. Gidas-Spruck [13] presented local and entire results in the non-radial case when $\delta < (N+2)/(N-2)$. Caffarelli, Gidas, and Spruck [8] have just extended them to the critical case $\delta = (N+2)/(N-2)$.

When $p > 2$ and the inhomogeneous term h is identically null, Ni and Serrin [17] studied the following equation

$$\left(|u'|^{p-2}u'\right)'(r) + \frac{N-1}{r}|u'|^{p-2}u'(r) + u^{\delta}(r) = 0, \quad r > 0. \tag{1.4}$$

They have proved the existence of two critical cases $N(p-1)/(N-p)$ and $(N(p-1)+p)/(N-p)$. Guedda and Véron [14] studied the existence of entire solutions and asymptotic behavior near the origin of radial solutions when $\delta < N(p-1)/(N-p)$. The non-radial case was proved by Bidaut-veron and Pohozaev [5].

This paper deals with the case where $p > 2$ and the inhomogeneous term h is not identically null. We present the asymptotic behavior near infinity of positive solutions v which tend to infinity at zero, while recalling that the asymptotic behavior near the origin has been studied in the paper Bouzelmate and Gmira [6] where they proved that v' must be negligible in front of $r^{(1-N)/(p-1)}$ near the origin when $N \geq p$.

The paper is organized as follows. The section 2 contains some preliminary results which are essential for the continuity of the work. In section 3, we study the asymptotic behavior of solutions of problem (1.1)-(1.2) and their derivatives. We present four main theorems that deal with the behavior of the solution v in the case where $\lim\limits_{r\to+\infty} r^{p\,\delta/(\delta+1-p)}h(r) = 0$. We prove under some assumptions that $\lim\limits_{r\to+\infty} r^{p/(\delta+1-p)}v(r) \geq 0$. If $\lim\limits_{r\to+\infty} r^{p/(\delta+1-p)}v(r) = 0$, we give equivalents of v and v' near infinity in the case where $\lim\limits_{r\to+\infty} r^{m}\,h(r) = l > 0$ for some constant $m > p\,\delta/(\delta+1-p)$. The study strongly depends on the position of m with respect to N. The section 4 gives a conclusion of the work presented.

2 Preliminaries

In this section, we present some useful computational tools to prove the main theorems. We also recall some preliminary results that appear in the papers [7] and [13].

Define the following function

$$F_{\lambda}(r) = \lambda v(r) + rv'(r), \quad r \geq 0. \tag{2.1}$$

This function plays an important role to study the asymptotic behavior of the function v, more exactly the monotonicity of $r^{\lambda}v$ because

$$\left(r^{\lambda}v(r)\right)' = r^{\lambda-1}F_{\lambda}(r). \tag{2.2}$$

To study the sign of F_{λ}, we give this following equation for any $r > 0$ such that $v'(r) \neq 0$,

$$(p-1)\,|v'|^{p-2}\,(r)F'_{\lambda}(r) = (p-1)\left(\lambda - \frac{N-p}{p-1}\right)|v'|^{p-2}\,v'(r) - rv^{\delta}(r) - r\,h(r). \tag{2.3}$$

Suppose that there exits $r_0 > 0$ such that $F_\lambda(r_0) = 0$, then equation (1.1) gives

$$(p-1)\, r_0^{p-1}\, |v'|^{p-2}\, (r_0) F_\lambda'(r_0) \;=\; (p-1)\left(\frac{N-p}{p-1} - \lambda\right) |\lambda|^{p-2} \lambda\, v^{p-1}(r_0) \\ -\, r_0^p v^\delta(r_0) - r_0^p\, h(r_0). \tag{2.4}$$

Now, we introduce the following change of variable which will be very useful. Let us define, for any real λ the function

$$\psi_\lambda(t) = r^\lambda v(r) \quad \text{where } \lambda \neq 0 \text{ and } t = \ln r. \tag{2.5}$$

Therefore ψ_λ verifies the following equation

$$y_\lambda'(t) + \Gamma_\lambda\, y_\lambda(t) + e^{(p-\lambda(\delta+1-p))t} \psi_\lambda^\delta(t) + j_\lambda(t) = 0, \tag{2.6}$$

where

$$j_\lambda(t) = e^{(p+\lambda(p-1))t} h(e^t), \tag{2.7}$$

$$y_\lambda(t) = |k_\lambda|^{p-2} k_\lambda(t), \tag{2.8}$$

$$k_\lambda(t) = \psi_\lambda'(t) - \lambda \psi_\lambda(t) \tag{2.9}$$

and

$$\Gamma_\lambda = N - p - \lambda(p-1). \tag{2.10}$$

It is easy to see that

$$k_\lambda(t) = r^{\lambda+1} v'(r) \tag{2.11}$$

and

$$\psi_\lambda'(t) = r^\lambda F_\lambda(r). \tag{2.12}$$

Now, we present some essential lemmas that initiated the study of the problem (1.1)-(1.2) and were already seen in [7] and [13].

Lemma 2.1 ([7]). *Let v be a solution of problem (1.1)-(1.2). Then*

$$v(r) > 0 \quad \textit{and} \quad v'(r) < 0, \quad \textit{for any } r > 0, \tag{2.13}$$

Moreover, there exists a constant $M > 0$ such that

$$0 < v(r) \leq M\, r^{-p/(\delta+1-p)}. \tag{2.14}$$

Lemma 2.2 ([7]). *Let v be a solution of problem (1.1)-(1.2). Then*

$$F_{(N-p)/(p-1)}(r) > 0 \quad \textit{for large } r.$$

Lemma 2.3 ([7]). *Let v be a solution of problem (1.1)-(1.2). Then the function $r^{p/(\delta+1-p)+1} v'(r)$ is bounded near infinity.*

Lemma 2.4 ([7]). *Let v be a solution of problem (1.1)-(1.2). Suppose that $r^{p/(\delta+1-p)} v(r)$ converges when $r \to +\infty$. Then $r^{p/(\delta+1-p)+1} v'(r)$ converges also when $r \to +\infty$ and*

$$\lim_{r\to+\infty} r^{p/(\delta+1-p)+1} v'(r) = \frac{-p}{\delta+1-p} \lim_{r\to+\infty} r^{p/(\delta+1-p)} v(r). \tag{2.15}$$

Lemma 2.5. *Let v be a solution of problem (1.1)-(1.2). If $\lim\limits_{r\to+\infty} r^{p/(\delta+1-p)} v(r) = b$. Then $b = 0$ or $b = \Lambda$, where*

$$\Lambda = \left(\left(\frac{p}{\delta+1-p}\right)^{p-1} \left(N - \frac{p\delta}{\delta+1-p}\right)\right)^{1/(\delta+1-p)}. \tag{2.16}$$

Proof. Taking $\lambda = p/(\delta+1-p)$ in logarithmic change (2.5), we obtain the following equation

$$y'_{p/(\delta+1-p)}(t) + \Gamma_{p/(\delta+1-p)}\, y_{p/(\delta+1-p)}(t) + \psi^{\delta}_{p/(\delta+1-p)}(t) + j_{p/(\delta+1-p)}(t) = 0. \tag{2.17}$$

We know by Lemma 2.4 that $\lim_{t\to+\infty} k_{p/(\delta+1-p)}(t) = -p/(\delta+1-p)\, b$. Then by (2.8), we have $\lim_{t\to+\infty} y_{p/(\delta+1-p)}(t) = -\left(p/(\delta+1-p)\right)^{p-1} b^{p-1}$. Since $\lim_{t\to+\infty} j_{p/(\delta+1-p)}(t) = 0$, then $y'_{p/(\delta+1-p)}(t)$ necessarily converges to 0. By tending $t \to +\infty$ in equation (2.17), we obtain

$$b^{\delta} - \Lambda^{\delta+1-p}\, b^{p-1} = 0, \tag{2.18}$$

where Λ is given by (2.16). Hence $b = 0$ or $b = \Lambda$. □

Lemma 2.6 ([13]). *Let G a positive differentiable function satisfying*

(i) $\displaystyle\int_{t_0}^{+\infty} G(t)\,dt < +\infty$ *for large t_0.*

(ii) *$G'(t)$ is bounded for large t.*

Then, $\lim_{t\to+\infty} G(t) = 0$.

3 Main Results

In this section, we study the asymptotic behavior of the solution v of problem (1.1)-(1.2) when $\lim_{r\to+\infty} r^{p\,\delta/(\delta+1-p)}\, h(r) = 0$. We give a complete study that allow us to obtain the equivalents of v and v' according to the behavior of the inhomogeneous term h near infinity and the position of δ with respect to the critical values $N(p-1)/(N-p)$ and $(N(p-1)+p)/(N-p)$.

We use ideas from [4], [6] and [7] and we introduce the following hypotheses:

(C_1) $\displaystyle\delta > \frac{N(p-1)+p}{N-p}$ and $\displaystyle\int_1^{+\infty} \left(r^{p\delta/(\delta+1-p)} h\right)_r^+ dr < \infty.$

(C_2) $\displaystyle\frac{N(p-1)}{N-p} < \delta < \frac{N(p-1)+p}{N-p}$ and $\displaystyle\int_1^{+\infty} \left(r^{p\delta/(\delta+1-p)} h\right)_r^- dr < \infty.$

(C_3) $r^{p\delta/(\delta+1-p)+1} h'(r)$ is bounded for large r and $\displaystyle\int_1^{+\infty} r^{p\delta/(\delta+1-p)-1} h(r)dr < \infty.$

(C_4) $\displaystyle\delta \ge \frac{N(p-1)+p}{N-p} + 1$ and $\displaystyle\int_1^{+\infty} \left(r^{p\delta/(\delta+1-p)} h\right)_r^+ dr < \infty.$

Then we have the following main theorem.

Theorem 3.1. *Let v be a solution of problem (1.1)-(1.2). If one of the following cases arises:*

(i) (C_1) *and* (C_3),

(ii) (C_2) *and* (C_3),

(iii) (C_4),

then

(i) $\lim_{r\to+\infty} r^{p/(\delta+1-p)} v(r) = 0$ *or* $\lim_{r\to+\infty} r^{p/(\delta+1-p)} v(r) = \Lambda.$

(ii) $\lim_{r\to+\infty} r^{p/(\delta+1-p)+1} v'(r) = 0$ *or* $\displaystyle\lim_{r\to+\infty} r^{p/(\delta+1-p)+1} v'(r) = \frac{-p}{\delta+1-p}\Lambda$, *where Λ is given by (2.16).*

Proof. Due to Lemmas 2.4 and 2.5, we concentrate to show that $\psi_{p/(\delta+1-p)(t)}$ converges when $t \to +\infty$.

• Suppose that the case (i) or the case (ii) occurs.
Let us define the following energy function associated with equation (2.17):

$$
\begin{aligned}
E_1(t) &= \frac{p-1}{p}\left|k_{p/(\delta+1-p)}(t)\right|^p + \Gamma\, y_{p/(\delta+1-p)}(t)\psi_{p/(\delta+1-p)}(t) \\
&+ \frac{\delta}{\delta+1} A\,\Gamma^{1/\delta}\left|y_{p/(\delta+1-p)}(t)\right|^{(\delta+1)/\delta} + \frac{1}{\delta+1}\psi^{\delta+1}_{p/(\delta+1-p)}(t),
\end{aligned}
\tag{3.1}
$$

where

$$
A = \frac{\delta(N-p) - (N(p-1)+p)}{\delta+1-p} \tag{3.2}
$$

and

$$
\Gamma = \Gamma_{p/(\delta+1-p)} = N - \frac{p\delta}{\delta+1-p}. \tag{3.3}
$$

Therefore

$$
\begin{aligned}
E_1'(t) &= -AY_1(t) - j_{p/(\delta+1-p)}(t)\psi'_{p/(\delta+1-p)}(t) \\
&\quad + Aj_{p/(\delta+1-p)}(t)\left(\Gamma^{1/\delta}\left|y_{p/(\delta+1-p)}(t)\right|^{1/\delta} - \psi_{p/(\delta+1-p)}(t)\right),
\end{aligned}
\tag{3.4}
$$

where

$$
Y_1(t) = \left(\psi_{p/(\delta+1-p)}(t) - \Gamma^{1/\delta}\left|y_{p/(\delta+1-p)}(t)\right|^{1/\delta}\right)\left(\psi^{\delta}_{p/(\delta+1-p)}(t) - \Gamma\left|y_{p/(\delta+1-p)}(t)\right|\right). \tag{3.5}
$$

We proceed in three steps.
Step 1. $E_1(t)$ converges when $t \to +\infty$.
Since $\psi_{p/(\delta+1-p)}(t)$, $k_{p/(\delta+1-p)}(t)$ and $y_{p/(\delta+1-p)}(t)$ are bounded for large t, then $E_1(t)$ is bounded for large t.
Integrating (3.4) on (T,t) for large T, we obtain

$$
\begin{aligned}
E_1(t) &= C_1(T) - AS_1(t) - j_{p/(\delta+1-p)}(t)\psi_{p/(\delta+1-p)}(t) + \int_T^t j'_{p/(\delta+1-p)}(s)\psi_{p/(\delta+1-p)}(s)\,ds \\
&+ A\int_T^t j_{p/(\delta+1-p)}(s)\left(\Gamma^{1/\delta}\left|y_{p/(\delta+1-p)}(s)\right|^{1/\delta} - \psi_{p/(\delta+1-p)}(s)\right)\,ds,
\end{aligned}
\tag{3.6}
$$

where

$$
C_1(T) = E_1(T) + j_{p/(\delta+1-p)}(T)\psi_{p/(\delta+1-p)}(T) \tag{3.7}
$$

and

$$
S_1(t) = \int_T^t Y_1(s)\,ds. \tag{3.8}
$$

Since $A \neq 0$, we have by (3.6),

$$
\begin{aligned}
S_1(t) &= -\frac{E_1(t)}{A} - \frac{1}{A}j_{p/(\delta+1-p)}(t)\psi_{p/(\delta+1-p)}(t) + \frac{1}{A}\int_T^t j'_{p/(\delta+1-p)}(s)\psi_{p/(\delta+1-p)}(s)\,ds \\
&+ \int_T^t j_{p/(\delta+1-p)}(s)\left(\Gamma^{1/\delta}\left|y_{p/(\delta+1-p)}(s)\right|^{1/\delta} - \psi_{p/(\delta+1-p)}(s)\right)\,ds + \frac{C_1(T)}{A}.
\end{aligned}
\tag{3.9}
$$

Since the function $s \to s^{\delta}$ is monotone, then $Y_1(t) \geq 0$. Therefore, the function S_1 is positive and increasing. On the other hand, by logarithmic change, hypothesis (C_1) gives $A > 0$ and $\int_T^{+\infty}\left(j'_{p/(\delta+1-p)}(s)\right)^+ ds < +\infty$, hypothesis (C_2) implies $A < 0$ and $\int_T^{+\infty}\left(j'_{p/(\delta+1-p)}(s)\right)^- ds < +\infty$, and hypothesis (C_3) gives $\int_T^{+\infty} j_{p/(\delta+1-p)}(s)\,ds < +\infty$. These assumptions, with the fact that $\psi_{p/(\delta+1-p)}(t)$, $y_{p/(\delta+1-p)}(t)$ and $E_1(t)$ are bounded for large t, $\lim_{t\to+\infty} j_{p/(\delta+1-p)}(t) = 0$ and $-\left(j'_{p/(\delta+1-p)}(s)\right)^- \leq j'_{p/(\delta+1-p)}(s) \leq \left(j'_{p/(\delta+1-p)}(s)\right)^+$, give that $S_1(t)$ is bounded for large t. Therefore, $S_1(t)$ converges when $t \to +\infty$. Hence by letting $t \to +\infty$ in (3.9), we obtain $\lim_{t\to+\infty} E_1(t)$ exists and is finite.

Step 2. $\lim_{t\to+\infty} y'_{p/(\delta+1-p)}(t) = 0$.

Recall that for any $1 < \varrho \le 2$, there is a c_ϱ such that

$$\left(|a|^{\varrho-2}a - |b|^{\varrho-2}b\right)(a-b) \ge c_\varrho (a-b)^2 \left(|a|+|b|\right)^{\varrho-2}, \tag{3.10}$$

for any $a, b \in \mathbb{R}$ such that $|a|+|b| > 0$.

Therefore, we have

$$\begin{aligned}&\left(\psi_{p/(\delta+1-p)}(t) - \Gamma^{1/\delta}\left|y_{p/(\delta+1-p)}(t)\right|^{1/\delta}\right)\left(\psi^\delta_{p/(\delta+1-p)}(t) - \Gamma\left|y_{p/(\delta+1-p)}(t)\right|\right) \ge \\ &c_\delta\left(\psi^\delta_{p/(\delta+1-p)}(t) - \Gamma\left|y_{p/(\delta+1-p)}(t)\right|\right)^2\left(\psi^\delta_{p/(\delta+1-p)}(t) + \Gamma\left|y_{p/(\delta+1-p)}(t)\right|\right)^{-(1-1/\delta)}.\end{aligned} \tag{3.11}$$

As $y_{p/(\delta+1-p)}(t) < 0$ for large t, then according to (2.17) and (3.5), we have for large t

$$Y_1(t) \ge c_\delta\left(y'_{p/(\delta+1-p)}(t) + j_{p/(\delta+1-p)}(t)\right)^2\left(\psi^\delta_{p/(\delta+1-p)}(t) + \Gamma\left|y_{p/(\delta+1-p)}(t)\right|\right)^{-(1-1/\delta)}. \tag{3.12}$$

Using the fact that $\psi_{p/(\delta+1-p)}(t)$ and $y_{p/(\delta+1-p)}(t)$ are bounded for large t and $1-1/\delta > 0$. Then there exists a constant $C > 0$ such that for large t,

$$\left(y'_{p/(\delta+1-p)}(t) + j_{p/(\delta+1-p)}(t)\right)^2 \le CY_1(t).$$

Which yields that

$$\int_T^t \left(y'_{p/(\delta+1-p)}(s) + j_{p/(\delta+1-p)}(s)\right)^2 ds \le C\, S_1(t).$$

Consequently

$$\begin{aligned}\int_T^t y'^2_{p/(\delta+1-p)}(s)\,ds &\le CS_1(t) - 2\int_T^t y'_{p/(\delta+1-p)}(s) j_{p/(\delta+1-p)}(s)\,ds - \int_T^t j^2_{p/(\delta+1-p)}(s)\,ds \\ &\le CS_1(t) - 2\int_T^t y'_{p/(\delta+1-p)}(s) j_{p/(\delta+1-p)}(s)\,ds.\end{aligned}$$

Since $S_1(t)$ and $y'_{p/(\delta+1-p)}(t)$ are bounded for large t and $\int_T^t j_{p/(\delta+1-p)}(s)\,ds < +\infty$ from (C_3), then $\int_T^t y'^2_{p/(\delta+1-p)}(s)\,ds$ is bounded. Moreover, since $\int_T^t y'^2_{p/(\delta+1-p)}(s)\,ds$ is increasing, then $\int_T^{+\infty} y'^2_{p/(\delta+1-p)}(s)\,ds < +\infty$.

On the other hand, deriving equation (2.17), we get

$$y''_{p/(\delta+1-p)}(t) + \Gamma y'_{p/(\delta+1-p)}(t) + \delta\psi^{\delta-1}_{p/(\delta+1-p)}(t)\psi'_{p/(\delta+1-p)}(t) + j'_{p/(\delta+1-p)}(t) = 0. \tag{3.13}$$

Since $j'_{p/(\delta+1-p)}(t)$ is bounded from (C_3) and $y'_{p/(\delta+1-p)}(t)$, $\psi_{p/(\delta+1-p)}(t)$, $\psi'_{p/(\delta+1-p)}(t)$ are bounded for large t, then $y''_{p/(\delta+1-p)}(t)$ is bounded for large t. Hence, using Lemma 2.6 we have $\lim_{t\to+\infty} y'_{p/(\delta+1-p)}(t) = 0$.

Step 3. $\psi_{p/(\delta+1-p)}(t)$ converges when $t \to +\infty$.

Since $\lim_{t\to+\infty} j_{p/(\delta+1-p)}(t) = 0$, then by tending $t \to +\infty$ in equation (2.17), we obtain

$$\lim_{t\to+\infty} \Gamma y_{p/(\delta+1-p)}(t) + \psi^\delta_{p/(\delta+1-p)}(t) = 0. \tag{3.14}$$

We argue by contradiction, and we suppose that $\psi_{p/(\delta+1-p)}(t)$ oscillates for large t. Then there exist two sequences $\{\eta_j\}$ and $\{\xi_j\}$ that go to $+\infty$ as $j \to +\infty$ such that $\{\eta_j\}$ and $\{\xi_j\}$ are local minimum and local maximum of $\psi_{p/(\delta+1-p)}$, respectively, satisfying $\eta_j < \xi_j < \eta_{j+1}$ and

$$\begin{aligned}&0 \le \liminf_{t\to+\infty} \psi_{p/(\delta+1-p)}(t) = \lim_{j\to+\infty} \psi_{p/(\delta+1-p)}(\eta_j) = \alpha < \\ &\limsup_{t\to+\infty} \psi_{p/(\delta+1-p)}(t) = \lim_{j\to+\infty} \psi_{p/(\delta+1-p)}(\xi_j) = \beta < +\infty.\end{aligned} \tag{3.15}$$

Now, since $\psi'_{p/(\delta+1-p)}(\eta_j) = \psi'_{p/(\delta+1-p)}(\xi_j) = 0$, the relation (3.14) implies that $\chi(\alpha) = \chi(\beta) = 0$, where

$$\chi(s) = \Lambda^{\delta+1-p}\, s^{p-1} - s^{\delta} = 0, \quad s \geq 0, \tag{3.16}$$

and Λ is given by (2.16). As $\alpha < \beta$, then necessarily $\alpha = 0$ and $\beta = \Lambda$. On the other hand, by (3.1), we have $\lim_{j\to+\infty} E_1(\eta_j) = 0$ and $\lim_{j\to+\infty} E_1(\xi_j) = -1/(\delta+1)\ (p/(\delta+1-p))^{p-1}\Lambda^p < 0$, which cannot take place because $E_1(t)$ converges when $t \to +\infty$. Hence, $\psi_{p/(\delta+1-p)}(t)$ converges when $t \to +\infty$.

- Suppose that the case (iii) occurs.

Similarly to the cases (i) and (ii), it suffices to show that $\psi_{p/(\delta+1-p)}(t)$ converges when $t \to +\infty$. Define the following energy function,

$$\begin{aligned} E_2(t) &= \frac{p-1}{p}\left|k_{p/(\delta+1-p)}(t)\right|^p + \frac{p}{\delta+1-p}\, y_{p/(\delta+1-p)}(t)\psi_{p/(\delta+1-p)}(t) \\ &-\frac{A}{p}\left(\frac{p}{\delta+1-p}\right)^{p-1}\psi^p_{p/(\delta+1-p)}(t) + \frac{1}{\delta+1}\psi^{\delta+1}_{p/(\delta+1-p)}(t), \end{aligned} \tag{3.17}$$

where A is given by (3.2).
A simple calculation gives

$$E_2'(t) = -AY_2(t) - j_{p/(\delta+1-p)}(t)\psi'_{p/(\delta+1-p)}(t), \tag{3.18}$$

where

$$\begin{aligned} Y_2(t) = &\left[|k_{p/(\delta+1-p)}(t)|^{p-1} - \left(\frac{p}{\delta+1-p}\right)^{p-1}\psi^{p-1}_{p/(\delta+1-p)}(t)\right]\times \\ &\left[|k_{p/(\delta+1-p)}(t)| - \frac{p}{\delta+1-p}\psi_{p/(\delta+1-p)}(t)\right]. \end{aligned} \tag{3.19}$$

Integrating relation (3.18) on (T,t) for large T, we obtain

$$E_2(t) = C_2(T) - AS_2(t) - j(t)\psi_{p/(\delta+1-p)}(t) + \int_T^t j'(s)\psi_{p/(\delta+1-p)}(s)\,ds, \tag{3.20}$$

where

$$C_2(T) = E_2(T) + j_{p/(\delta+1-p)}(T)\psi_{p/(\delta+1-p)}(T) \tag{3.21}$$

and

$$S_2(t) = \int_T^t Y_2(s)\,ds. \tag{3.22}$$

Since the function $s \to s^{p-1}$ is monotone, then $Y_2(t) \geq 0$. Therefore, S_2 is positive and increasing. In the same way as the cases (i) and (ii), we prove that $S_2(t)$ is bounded for large t by using (C_4) which gives $A > 0$ and $\int_T^{+\infty}\left(j'_{p/(\delta+1-p)}(s)\right)^+ ds < +\infty$. Therefore $S_2(t)$ converges when $t \to +\infty$ and thereby $E_2(t)$ converges to a real number noted d when $t \to +\infty$.
Assume by contradiction that $\psi_{p/(\delta+1-p)}(t)$ oscillates for large t. Then there exist two sequences $\{\eta_j\}$ and $\{\xi_j\}$ that go to $+\infty$ as $j \to +\infty$ such that $\{\eta_j\}$ and $\{\xi_j\}$ are local minimum and local maximum of $\psi_{p/(\delta+1-p)}$, respectively, satisfying $\eta_j < \xi_j < \eta_{j+1}$ and (3.15). Since $\psi'_{p/(\delta+1-p)}(\eta_j) = \psi'_{p/(\delta+1-p)}(\xi_j) = 0$, then by expression (3.17) of E_2, we obtain

$$\lim_{j\to+\infty} E_2(\eta_j) = \zeta(\alpha) \;\text{ and }\; \lim_{j\to+\infty} E_2(\xi_j) = \zeta(\beta), \tag{3.23}$$

where

$$\zeta(s) = \frac{s^{\delta+1}}{\delta+1} - \frac{\Lambda^{\delta+1-p}}{p}s^p, \quad s \geq 0. \tag{3.24}$$

Since $\lim_{t\to+\infty} E_2(t) = d$, then

$$\zeta(\alpha) = \zeta(\beta) = d, \tag{3.25}$$

Therefore, there exists $\gamma \in (\alpha, \beta)$ and $t_j \in (\eta_j, \xi_j)$ such that $\psi_{p/(\delta+1-p)}(t_j) = \gamma$, $\zeta'(\gamma) = 0$ and $\zeta(\gamma) \neq d$. It is easy to see that $\zeta'(0) = \zeta'(\Lambda) = 0$, hence $\psi_{p/(\delta+1-p)}(t_j) = \gamma = \Lambda$.
Now, we distinguish two cases.

• If $\liminf_{t\to+\infty} \psi_{p/(\delta+1-p)}(t) = \alpha = 0$, then using (3.25), we have $\lim_{t\to+\infty} E_2(t) = 0$. On the other hand, using expression (3.17) of E_2 and the fact that $k_{p/(\delta+1-p)}(t) < 0$, we obtain

$$E_2(t_j) < \left|k_{p/(\delta+1-p)}(t_j)\right|^{p-1} \left(\left|k_{p/(\delta+1-p)}(t_j)\right| - \frac{p}{\delta+1-p}\, \psi_{p/(\delta+1-p)}(t_j) \right) - \frac{A}{p}\Big(\frac{p}{\delta+1-p}\Big)^{p-1} \psi^p_{p/(\delta+1-p)}(t_j) + \frac{1}{\delta+1} \psi^{\delta+1}_{p/(\delta+1-p)}(t_j).$$

That is, thanks to (2.9),

$$E_2(t_j) < -\psi'_{p/(\delta+1-p)}(t_j) \left|k_{p/(\delta+1-p)}(t_j)\right|^{p-1} - \frac{A}{p}\Big(\frac{p}{\delta+1-p}\Big)^{p-1} \psi^p_{p/(\delta+1-p)}(t_j) + \frac{1}{\delta+1} \psi^{\delta+1}_{p/(\delta+1-p)}(t_j).$$

But $\psi_{p/(\delta+1-p)}(t_j) = \Lambda$ and $\psi'_{p/(\delta+1-p)}(t_j) \geq 0$, hence $E_2(t_j) < \rho(\Lambda)$, where

$$\rho(s) = \frac{s^{\delta+1}}{\delta+1} - \frac{A}{p}\Big(\frac{p}{\delta+1-p}\Big)^{p-1} s^p, \quad s \geq 0. \tag{3.26}$$

Since $\delta \geq (N(p-1)+p)/(N-p)+1$, then $\rho(\Lambda) < 0$. Therefore, $\lim_{i\to+\infty} E_2(t_j) \leq \rho(\Lambda) < 0$. This is impossible because $\lim_{t\to+\infty} E_2(t) = 0$.

• If $\liminf_{t\to+\infty} \psi_{p/(\delta+1-p)}(t) > 0$, then there exists $\varepsilon > 0$ such that $\psi_{p/(\delta+1-p)}(t) \geq \varepsilon$ for large t. Combining this with equation (2.17) and the fact that $j_{p/(\delta+1-p)}(t)$ is positive, we get for large t,

$$y'_{p/(\delta+1-p)}(t) + \left(N - \frac{p\delta}{\delta+1-p}\right) y_{p/(\delta+1-p)}(t) \leq -\varepsilon^q.$$

Integrating this last inequality on (T, t) for large T and taking into account $y_{p/(\delta+1-p)}(t) < 0$ and $N > p\,\delta/(\delta+1-p)$, we obtain

$$\left|y_{p/(\delta+1-p)}(t)\right| \geq \frac{\varepsilon^\delta}{N - \dfrac{p\delta}{\delta+1-p}} + M(T)\, e^{-\left(N-p\delta/(\delta+1-p)\right)t} \quad \text{for } t > T,$$

where

$$M(T) = \left[\left|y_{p/(\delta+1-p)}(T)\right| - \frac{\varepsilon^\delta}{N - \dfrac{p\delta}{\delta+1-p}} \right] e^{\left(N-p\delta/(\delta+1-p)\right)T}.$$

Therefore, by (2.8), we have that $\left|k_{p/(\delta+1-p)}(t)\right|^{2-p}$ is bounded for large t. Now, we prove that $\lim_{t\to+\infty} \psi'_{p/(\delta+1-p)}(t) = 0$, which amounts to show that $\lim_{t\to+\infty} Y_2(t) = 0$. For this we apply Lemma 2.6. Since $S_2(t)$ converges when $t \to +\infty$, then $\int_T^{+\infty} Y_2(s)\, ds < +\infty$. We show that $Y_2'(t)$ is bounded for large t. Using expression (3.19) of Y_2, we have

$$\begin{aligned} Y_2(t) &= |y_{p/(\delta+1-p)}|^{p/(p-1)}(t) - \frac{p}{\delta+1-p}\, |y_{p/(\delta+1-p)}|(t)\psi_{p/(\delta+1-p)}(t) \\ &\quad + \left(\frac{p}{\delta+1-p}\right)^{p-1} \psi^{p-1}_{p/(\delta+1-p)}(t)\psi'_{p/(\delta+1-p)}(t). \end{aligned} \tag{3.27}$$

Deriving relation (3.27), we obtain

$$\begin{aligned} Y_2'(t) &= \frac{p}{p-1}\, k_{p/(\delta+1-p)}(t) y'_{p/(\delta+1-p)}(t) + \frac{p}{\delta+1-p}\, y_{p/(\delta+1-p)}(t)\psi'_{p/(\delta+1-p)}(t) \\ &+ \frac{p}{\delta+1-p}\, \psi_{p/(\delta+1-p)}(t) y'_{p/(\delta+1-p)}(t) \\ &+ (p-1)\left(\frac{p}{\delta+1-p}\right)^{p-1} \psi^{p-2}_{p/(\delta+1-p)}(t)\psi'^{2}_{p/(\delta+1-p)}(t) \\ &+ \left(\frac{p}{\delta+1-p}\right)^{p-1} \psi^{p-1}_{p/(\delta+1-p)}(t)\psi''_{p/(\delta+1-p)}(t). \end{aligned} \tag{3.28}$$

Since $\psi_{p/(\delta+1-p)}(t)$, $k_{p/(\delta+1-p)}(t)$ and $j_{p/(\delta+1-p)}(t)$ are bounded for large t, according to (2.9) and (2.17), $\psi'_{p/(\delta+1-p)}(t)$ and $y'_{p/(\delta+1-p)}(t)$ are bounded for large t. Moreover, $\psi''_{p/(\delta+1-p)}(t)$ is bounded also for large t by using the fact that

$$\psi''_{p/(\delta+1-p)}(t) = \frac{1}{p-1}|k_{p/(\delta+1-p)}(t)|^{2-p} y'_{p/(\delta+1-p)}(t) + \frac{p}{\delta+1-p}\psi'_{p/(\delta+1-p)}(t) \tag{3.29}$$

and $|k_{p/(\delta+1-p)}(t)|^{2-p}$ is bounded for large t. Therefore, by equation (3.28), $Y_2'(t)$ is bounded for large t. Hence, by Lemma 2.6, we obtain $\lim\limits_{t\to+\infty} Y_2(t) = 0$ and thereby $\lim\limits_{t\to+\infty} \psi'_{p/(\delta+1-p)}(t) = 0$. This gives $\lim\limits_{j\to+\infty} E_2(t_j) = \zeta(\Lambda) = \zeta(\gamma)$. But this contradicts the fact that $\zeta(\Lambda) \neq d = \lim\limits_{t\to+\infty} E_2(t)$. Consequently, $\psi_{p/(\delta+1-p)}(t)$ converges when $t \to +\infty$. The proof is complete. □

Now, a main question arises: Could we find equivalents of v and v' near infinity in the case where $\lim\limits_{r\to+\infty} r^{p/(\delta+1-p)}\, v(r) = 0$? The answer to this question strongly depends on the behavior of the inhomogeneous term h. For this, we assume that there exists $m > p\,\delta/(\delta+1-p)$ satisfying the following hypotheses.

(H_b) $r^m\, h(r)$ is bounded for large r.

(H_c) $\lim\limits_{r\to+\infty} r^m\, h(r) = l > 0.$

The study depends on the position of m with respect to N. We start with the case where $p\,\delta/(\delta+1-p) < m < N$.

Theorem 3.2. *Assume that* $p\,\delta/(\delta+1-p) < m < N$ *and* (H_c) *holds. Let* v *be a solution of problem (1.1)-(1.2) satisfying* $\lim\limits_{r\to+\infty} r^{p/(\delta+1-p)} v(r) = 0$. *Then*

(i) $v(r) \underset{+\infty}{\sim} \dfrac{p-1}{m-p}\left(\dfrac{l}{N-m}\right)^{1/(p-1)} r^{-(m-p)/(p-1)}.$

(ii) $v'(r) \underset{+\infty}{\sim} -\left(\dfrac{l}{N-m}\right)^{1/(p-1)} r^{-(m-1)/(p-1)}.$

The proof requires the following results.

Lemma 3.3. *Let* v *be a solution of problem (1.1)-(1.2) satisfying* $\lim\limits_{r\to+\infty} r^{p/(\delta+1-p)} v(r) = 0$. *Suppose that* $r^{\theta(p-1)+p}\, h(r)$ *is bounded for large* r *and* $\lim\limits_{r\to+\infty} r^{\theta} v(r) = +\infty$ *for some* $p/(\delta+1-p) < \theta \le (N-p)/(p-1)$. *Then* $F_{p/(\delta+1-p)}(r) < 0$ *and* $F_\theta(r) > 0$, *for large* r.

Proof. The proof will be done in two steps.
Step 1. $F_{p/(\delta+1-p)}(r) < 0$ for large r.
Using relation (2.2) and the fact that $\lim\limits_{r\to+\infty} r^{p/(\delta+1-p)} v(r) = 0$, it suffices to show that $F_{p/(\delta+1-p)}(r) \neq 0$. Suppose by contradiction that there exists a large r such that $F_{p/(\delta+1-p)}(r) = 0$. Using the

relation (2.4) with $\lambda = p/(\delta+1-p)$ and multiplying by $r^{\theta(p-1)}$, we get

$$(p-1)\, r^{(\theta+1)(p-1)}\, |v'(r)|^{p-2}\, F'_{p/(\delta+1-p)}(r) = r^{\theta(p-1)} v^{p-1}(r) \Big[\Lambda^{\delta+1-p} - r^p v^{\delta+1-p}(r) - r^{p+\theta(p-1)} h(r) \left(r^\theta v(r)\right)^{1-p} \Big]. \tag{3.30}$$

Since $\lim_{r\to+\infty} r^{p/(\delta+1-p)} v(r) = 0$, $r^{\theta(p-1)+p} h(r)$ is bounded for large r and $\lim_{r\to+\infty} r^\theta v(r) = +\infty$, then $F'_{p/(\delta+1-p)}(r) > 0$. Hence $F_{p/(\delta+1-p)}(r) \neq 0$ for large r.
Step 2. $F_\theta(r) > 0$ for large r.
We start with the first case $p/(\delta+1-p) < \theta < (N-p)/(p-1)$. In the same way as the first step, using (2.2), it suffices to show that $F_\theta(r) \neq 0$ for large r since $\lim_{r\to+\infty} r^\theta v(r) = +\infty$. Suppose that there exists a large r such that $F_\theta(r) = 0$. We have by (2.4)

$$(p-1)\, r^{(\theta+1)(p-1)}\, |v'(r)|^{p-2}\, F'_\theta(r) = r^{\theta(p-1)} v^{p-1}(r) \Big[\Gamma_\theta \theta^{p-1} - r^p v^{\delta+1-p}(r) - r^{p+\theta(p-1)} h(r) \left(r^\theta v(r)\right)^{1-p} \Big], \tag{3.31}$$

where Γ_θ is given by (2.10). Using our hypothesis and the fact that $\Gamma_\theta > 0$ (because $p/(\delta+1-p) < \theta < (N-p)/(p-1)$), we obtain $F'_\theta(r) > 0$. Therefore, $F_\theta(r) \neq 0$ for large r.
The case $\theta = (N-p)/(p-1)$ is given by Lemma 2.2. The proof is over. □

Proposition 3.4. *Assume that (H_b) holds. Let v be a solution of problem (1.1)-(1.2) satisfying $\lim_{r\to+\infty} r^{p/(\delta+1-p)} v(r) = 0$. Then $r^{(m-p)/(p-1)} v(r)$ is bounded for large r. Moreover, we have*

$$\liminf_{r\to+\infty} r^m h(r) \le (N-m) \left(\frac{m-p}{p-1}\right)^{p-1} \limsup_{r\to+\infty} r^{m-p} v^{p-1}(r) \tag{3.32}$$

and

$$\limsup_{r\to+\infty} r^m h(r) \ge (N-m) \left(\frac{m-p}{p-1}\right)^{p-1} \liminf_{r\to+\infty} r^{m-p} v^{p-1}(r). \tag{3.33}$$

Proof. Taking $\theta = (m-p)/(p-1)$ and using the change (2.5) (for $\lambda = \theta$), we show that $\psi_\theta(t)$ is bounded for large t. We argue by contradiction and we distinguish two cases.
- If $\lim_{t\to+\infty} \psi_\theta(t) = +\infty$.

As $p/(\delta+1-p) < \theta < (N-p)/(p-1)$ and $r^{\theta(p-1)+p} h(r)$ is bounded for large r by hypothesis (H_b), combining this with Lemma 3.3, we have $F_{p/(\delta+1-p)}(r) < 0$ and $F_\theta(r) > 0$, for large r. Consequently, since $v'(r) < 0$ on $(0, +\infty)$, then for large r,

$$\frac{p}{\delta+1-p} < \frac{r|v'|}{v} < \theta. \tag{3.34}$$

Using the change (2.5), we have for large t,

$$\left(\frac{p}{\delta+1-p}\right)^{p-1} < |y_\theta(t)| \psi_\theta^{1-p}(t) < \theta^{p-1}. \tag{3.35}$$

Now, taking $\lambda = \theta$ in equation (2.6) and multiplying by $\psi_\theta^{1-p}(t)$, we get

$$\left(y_\theta(t)\psi_\theta^{1-p}(t)\right)' + (p-1)\, |k_\theta(t)|^p\, \psi_\theta^{-p}(t) + (N-p)\, y_\theta(t)\psi_\theta^{1-p}(t) + J_\theta(t) = 0, \tag{3.36}$$

where

$$J_\theta(t) = e^{(p-\theta(\delta+1-p))t} \psi_\theta^{\delta+1-p}(t) + j_\theta(t)\psi_\theta^{1-p}(t). \tag{3.37}$$

Since $\lim_{t\to+\infty} e^{(p-\theta(\delta+1-p))t} \psi_\theta^{\delta+1-p}(t) = 0$ (because $\lim_{r\to+\infty} r^{p/(\delta+1-p)} v(r) = 0$), $j_\theta(t)$ is bounded for large t and $\lim_{t\to+\infty} \psi_\theta(t) = +\infty$, then $\lim_{t\to+\infty} J_\theta(t) = 0$.
For simplicity, we set

$$\varphi_\theta(t) = |y_\theta(t)| \psi_\theta^{1-p}(t). \tag{3.38}$$

Then by (3.35), we have for large t

$$\left(\frac{p}{\delta+1-p}\right)^{p-1} < \varphi_\theta(t) < \theta^{p-1} \tag{3.39}$$

and by (3.36), we have

$$\varphi_\theta'(t) = (p-1)\,|\varphi_\theta(t)|^{p/(p-1)} - (N-p)\,\varphi_\theta(t) + J_\theta(t). \tag{3.40}$$

Since $\varphi_\theta(t) > 0$ for large t, we obtain for large t

$$\varphi_\theta'(t) = (p-1)\tau\left(\varphi_\theta(t)\right) + J_\theta(t) \tag{3.41}$$

where

$$\tau(s) = s^{p/(p-1)} - \frac{N-p}{p-1}s, \qquad s \geq 0. \tag{3.42}$$

A simple study of the function τ implies that there exists $c > 0$ such that $\tau(s) < -c$ for $(p/(\delta+1-p))^{p-1} < s < \theta^{p-1} < ((N-p)/(p-1))^{p-1}$. Using (3.39), (3.41) and the fact that $\lim_{t\to+\infty} J_\theta(t) = 0$, we see that there exists a constant $c_1 > 0$ such that for large t, $\varphi_\theta'(t) < -c_1$. Integrating this last inequality on (T,t) for large T, we get $\lim_{t\to+\infty} \varphi_\theta(t) = -\infty$, which gives a contradiction with the fact that $\varphi_\theta(t)$ is bounded for large t by (3.39).

- If $\limsup_{t\to+\infty} \psi_\theta(t) = +\infty$.

Then there exists a sequence $\{r_i\}$ going to $+\infty$ as $i \to +\infty$ such that $\{r_i\}$ is a local maximum of ψ_θ satisfying $\lim_{t\to+\infty} \psi_\theta(r_i) = +\infty$.
Taking $t = r_i$ in equation (2.6) with $\lambda = \theta$, we get

$$y_\theta'(r_i) = -\Gamma_\theta\, y_\theta(r_i) - e^{(p-\theta(\delta+1-p))r_i}\psi_\theta^\delta(r_i) - j_\theta(r_i). \tag{3.43}$$

Since $\psi_\theta'(r_i) = 0$, then by (2.9) and (2.8), we have

$$\frac{\psi_\theta^{p-1}(r_i)}{y_\theta(r_i)} = -\theta^{1-p}.$$

As a consequence, equation (3.43) can be written as

$$y_\theta'(r_i) = y_\theta(r_i)\left[-\Gamma_\theta + \theta^{1-p}e^{(p-\theta(\delta+1-p))r_i}\psi_\theta^{\delta+1-p}(r_i) - \frac{j_\theta(r_i)}{y_\theta(r_i)}\right]. \tag{3.44}$$

Using our hypotheses, we have $\lim_{i\to+\infty} \dfrac{y_\theta'(r_i)}{y_\theta(r_i)} = -\Gamma_\theta$, then $y_\theta'(r_i) > 0$ for large i. On the other hand, we have $k_\theta'(r_i) = v_\theta''(r_i) \leq 0$, which yields that $y_\theta'(r_i) \leq 0$. This is a contradiction.
We deduce that $\psi_\theta(t)$ is bounded for large t.

Now, we show the estimate (3.32). Assume by contradiction that

$$\liminf_{r\to+\infty} r^m h(r) > (N-m)\left(\frac{m-p}{p-1}\right)^{p-1} \limsup_{r\to+\infty} r^{m-p}v^{p-1}(r).$$

Taking $\lambda = \theta = (m-p)/(p-1)$ in (2.5), there exists $\varepsilon_0 > 0$ such that for large t,

$$j_\theta(t) = e^{mt}\,h(e^t) \geq (N-m)\,\theta^{p-1}\psi_\theta^{p-1}(t) + \varepsilon_0. \tag{3.45}$$

First, we show that $\psi_\theta(t)$ is strictly monotone for large t, which amounts to prove that $F_\theta(r) \neq 0$ for large r by (2.2). Suppose by contradiction that there exists a large r such that $F_\theta(r) = 0$. Then combining this with relation (2.4), we obtain

$$(p-1)r^{m-1}|v'|^{p-2}F_\theta'(r) = (N-m)\,\theta^{p-1}\psi_\theta^{p-1}(t) - e^{(p\delta-m(\delta+1-p))/(p-1))t}\psi_\theta^\delta(t) - j_\theta(t). \tag{3.46}$$

Using inequality (3.45), we obtain for large r,

$$(p-1)r^{m-1}|v'|^{p-2}F_\theta'(r) < (N-m)\,\theta^{p-1}\psi_\theta^{p-1}(t) - j_\theta(t) \le -\varepsilon_0 < 0. \tag{3.47}$$

Therefore, $F_\theta(r) \neq 0$ for large r, that is $\psi_\theta(t)$ is strictly monotone for large t. Moreover, since $\psi_\theta(t)$ is bounded for large t, then $\lim_{t\to+\infty} \psi_\theta(t) = b_1 \ge 0$ and $\lim_{t\to+\infty} \psi_\theta'(t) = 0$. Therefore, by (2.9), that $\lim_{t\to+\infty} k_\theta(t) = -\theta b_1$ and thereby $\lim_{t\to+\infty} y_\theta(t) = -\theta^{p-1}b_1^{p-1}$.
On the other hand, according to (2.17), we have

$$y_\theta'(t) = -(N-m)y_\theta(t) - e^{(p\delta-m(\delta+1-p))/(p-1))t}\psi_\theta^\delta(t) - j_\theta(t). \tag{3.48}$$

Hence, combining with (3.45), we get for large t,

$$y_\theta'(t) \le \phi(t) - \varepsilon_0, \tag{3.49}$$

where

$$\phi(t) = -(N-m)y_\theta(t) - e^{(p\delta-m(\delta+1-p))/(p-1))t}\psi_\theta^\delta(t) - (N-m)\,\theta^{p-1}\psi_\theta^{p-1}(t). \tag{3.50}$$

Since $m > p\delta/(\delta+1-p)$, $\lim_{t\to+\infty} \psi_\theta(t) = b_1$ and $\lim_{t\to+\infty} y_\theta(t) = -\theta^{p-1}b_1^{p-1}$, then $\lim_{t\to+\infty} \phi(t) = 0$. This implies that there exists a constant $c_2 > 0$ such that $y_\theta'(t) \le -c_2$ for large t. Integrating the last inequality on (T,t) for large T, we obtain $\lim_{t\to+\infty} y_\theta(t) = -\infty$. This is impossible and the estimate (3.32) holds.

Finally, to prove estimate (3.33), we assume by contradiction that

$$\limsup_{r\to+\infty} r^m h(r) < (N-m)\left(\frac{m-p}{p-1}\right)^{p-1} \liminf_{r\to+\infty} r^{m-p}v^{p-1}(r).$$

Then with $\theta = (m-p)/(p-1)$, there exists $\varepsilon_2 > 0$ such that for large t

$$j_\theta(t) \le (N-m)\,\theta^{p-1}\psi_\theta^{p-1}(t) - \varepsilon_2. \tag{3.51}$$

In a similar manner, we prove that the last inequality gives $\psi_\theta(t)$ is strictly monotone for large t. Indeed, suppose by contradiction that there exists a large r such that $F_\theta(r) = 0$. Then, according to (3.46) and (3.51) , we get for large r,

$$(p-1)r^{m-1}|v'|^{p-2}F_\theta'(r) \ge \varepsilon_2 - e^{(p\delta-m(\delta+1-p))/(p-1))t}\psi_\theta^\delta(t). \tag{3.52}$$

Since $\lim_{t\to+\infty} e^{(p\delta-m(\delta+1-p))/(p-1))t}\psi_\theta^\delta(t) = 0$ (because $m > p\delta/(\delta+1-p)$ and $\psi_\theta(t)$ is bounded for large t), then for large r

$$(p-1)r^{m-1}|v'|^{p-2}F_\theta'(r) > \frac{\varepsilon_2}{2} > 0.$$

Therefore $F_\theta(r) \neq 0$ for small r and thereby $\psi_\theta(t)$ is strictly monotone for large t. Hence $\lim_{t\to+\infty} \psi_\theta(t) = b_1 \ge 0$ and $\lim_{t\to+\infty} \psi_\theta'(t) = 0$.
We use the same reasoning as in the previous case, equation (3.48) and estimate (3.51) to get the contradiction. Hence, the estimate (3.33) is verified. This completes the proof. □

Now we can prove Theorem 3.2.

Proof. (i) Taking $\theta = (m-p)/(p-1)$. By Proposition 3.4 we have that $\psi_\theta(t)$ is bounded for large t. Suppose by contradiction that $\psi_\theta(t)$ oscillates for large t. Then there exist two sequences $\{\eta_i\}$ and $\{\xi_i\}$ that tend to $+\infty$ as $i \to +\infty$ such that $\{\eta_i\}$ and $\{\xi_i\}$ are local minimum and local maximum of ψ_θ, respectively, satisfying $\eta_i < \xi_i < \eta_{i+1}$ and

$$0 \le \liminf_{t\to+\infty} \psi_\theta(t) = \lim_{i\to+\infty} \psi_\theta(\eta_i) = \alpha < \limsup_{t\to+\infty} \psi_\theta(t) = \lim_{i\to+\infty} \psi_\theta(\xi_i) = \beta < +\infty. \tag{3.53}$$

Since $\psi'_\theta(\eta_i) = \psi'_\theta(\xi_i) = 0$, $\psi''_\theta(\eta_i) \geq 0$ and $\psi''_\theta(\xi_i) \leq 0$, then using (2.8) and (2.9), we have

$$\lim_{i\to+\infty} y_\theta(\eta_i) = -\theta^{p-1}\alpha^{p-1},$$

$$\lim_{i\to+\infty} y_\theta(\xi_i) = -\theta^{p-1}\beta^{p-1},$$

$$y'_\theta(\eta_i) \geq 0 \;\text{ and }\; y'_\theta(\xi_i) \leq 0.$$

Hence according to (3.48), we have

$$-(N-m)y_\theta(\eta_i) - e^{(p\delta-m(\delta+1-p))/(p-1))\eta_i}\psi_\theta^\delta(\eta_i) - j_\theta(\eta_i) \geq 0 \tag{3.54}$$

and

$$-(N-m)y_\theta(\xi_i) - e^{(p\delta-m(\delta+1-p))/(p-1))\xi_i}\psi_\theta^\delta(\xi_i) - j_\theta(\xi_i) \leq 0. \tag{3.55}$$

Letting $i \to +\infty$ in (3.54) and (3.55) and since $\lim_{t\to+\infty} j_\theta(t) = l$, we get

$$\beta^{p-1} \leq \frac{l}{(N-m)\theta^{p-1}} \leq \alpha^{p-1}.$$

Which contradicts (3.53). Consequently, $\psi_\theta(t)$ converges when $t \to +\infty$.
Using again Proposition 3.4 we have

$$\liminf_{t\to+\infty} \psi_\theta^{p-1}(t) \leq \frac{l}{(N-m)\theta^{p-1}} \leq \limsup_{t\to+\infty} \psi_\theta^{p-1}(t).$$

Hence, $\displaystyle\lim_{t\to+\infty} \psi_\theta(t) = \frac{1}{\theta}\left(\frac{l}{N-m}\right)^{1/(p-1)} = \frac{p-1}{m-p}\left(\frac{l}{N-m}\right)^{1/(p-1)}$.

(ii) We show that $\displaystyle\lim_{t\to+\infty} k_\theta(t) = -\left(\frac{l}{N-m}\right)^{1/(p-1)}$.
Since $\psi_\theta(t)$ is bounded for large t and $F_{(N-p)/(p-1)}(r) > 0$ for large r from Lemma 2.2, then $k_\theta(t)$ is bounded for large t. Assume by contradiction that $k_\theta(t)$ oscillates for large t. Then there exist two sequences $\{s_i\}$ and $\{\rho_i\}$ that go to $+\infty$ as $i \to +\infty$ such that $\{s_i\}$ and $\{\rho_i\}$ are local minimum and local maximum of k_θ, respectively, satisfying $s_i < \rho_i < s_{i+1}$ and

$$\liminf_{t\to+\infty} k_\theta(t) = \lim_{i\to+\infty} k_\theta(s_i) = l_1 < \limsup_{t\to+\infty} k_\theta(t) = \lim_{i\to+\infty} k_\theta(\rho_i) = L_1. \tag{3.56}$$

Hence, $y'_\theta(s_i) = y'_\theta(\rho_i) = 0$ (because $k'_\theta(s_i) = k'_\theta(\rho_i) = 0$), $\lim_{i\to+\infty} y_\theta(s_i) = |l_1|^{p-2} l_1$ and $\lim_{i\to+\infty} y_\theta(\rho_i) = |L_1|^{p-2} L_1$. Since $m > p\delta/(\delta+1-p)$, ψ_θ converges and $\lim_{t\to+\infty} e^{mt}f(e^t) = l$, we claim, by taking respectively $t = s_i$ and $t = \rho_i$ in equation (2.6) and letting $i \to +\infty$, that

$$(N-m)\,|l_1|^{p-2}\,l_1 = -l = (N-m)\,|L_1|^{p-2}\,L_1.$$

As $m < N$ and $l > 0$, then

$$|l_1|^{p-2}\,l_1 = |L_1|^{p-2}\,L_1 < 0.$$

That is equivalent to $l_1 = L_1$. This gives a contradiction to (3.56). Consequently, $k_\theta(t)$ converges when $t \to +\infty$, therefore by, (2.9), we necessarily have $\lim_{t\to+\infty} \psi'_\theta(t) = 0$ (because ψ_θ converges).

As a consequence, $\displaystyle\lim_{t\to+\infty} k_\theta(t) = -\left(\frac{l}{N-m}\right)^{1/(p-1)}$, which is equivalent by (2.11) to

$$\lim_{r\to 0} r^{(m-1)/(p-1)} v'(r) = -\left(\frac{l}{N-m}\right)^{1/(p-1)}.$$

The proof is over. □

Now we consider the case $m = N$ and look for equivalents to v and v' near infinity.

Theorem 3.5. *Assume that $m = N$ and (H_c) holds. Let v be a solution of problem (1.1)-(1.2) such that $\lim_{r\to+\infty} r^{p/(\delta+1-p)}v(r) = 0$. Then*

(i) $\dfrac{r^{(N-p)/(p-1)}v(r)}{(\ln r)^{1/(p-1)}} \underset{+\infty}{\sim} \dfrac{p-1}{N-p}\, l^{1/(p-1)}$.

(ii) $\dfrac{r^{(N-1)/(p-1)}v'(r)}{(\ln r)^{1/(p-1)}} \underset{+\infty}{\sim} -l^{1/(p-1)}$.

The following proposition will be useful for the proof of this theorem.

Proposition 3.6. *Assume that $m = N$ and (H_b) holds. Let v be a solution of problem (1.1)-(1.2) such that $\lim_{r\to+\infty} r^{p/(\delta+1-p)}v(r) = 0$. Then $\dfrac{r^{(N-p)/(p-1)}v(r)}{(\ln r)^{1/(p-1)}}$ is bounded for large r. Moreover, we have*

$$\liminf_{r\to+\infty} r^N h(r) \le \left(\frac{N-p}{p-1}\right)^{p-1} \limsup_{r\to+\infty} \frac{r^{N-p}v^{p-1}(r)}{\ln r} \tag{3.57}$$

and

$$\limsup_{r\to+\infty} r^N h(r) \ge \left(\frac{N-p}{p-1}\right)^{p-1} \liminf_{r\to+\infty} \frac{r^{N-p}v^{p-1}(r)}{\ln r}. \tag{3.58}$$

Proof. Taking the change (2.5) for $\lambda = (N-p)/(p-1)$, we see that $\psi_{(N-p)/(p-1)}$ satisfies the following equation

$$y'_{(N-p)/(p-1)}(t) + e^{\left(N-\delta(N-p)/(p-1)\right)t}\psi^{\delta}_{(N-p)/(p-1)}(t) + e^{Nt}h(e^t) = 0. \tag{3.59}$$

Therefore, $y'_{(N-p)/(p-1)}(t) < 0$. Moreover, since $y_{(N-p)/(p-1)}(t) < 0$ (because $v'(r) < 0$), then $\lim_{t\to+\infty} y_{(N-p)/(p-1)}(t) \in [-\infty, 0[$. We distinguish two cases.

Case 1. $-\infty < \lim_{t\to+\infty} y_{(N-p)/(p-1)}(t) < 0$.

Then, $\lim_{r\to+\infty} r^{(N-1)/(p-1)}v'(r)$ is finite, so using Hôspital's rule (because $\lim_{r\to+\infty} v(r) = 0$ and $N > p$), we have $\lim_{r\to+\infty} r^{(N-p)/(p-1)}v(r)$ is finite. This implies that $\lim_{r\to+\infty} \dfrac{r^{(N-p)/(p-1)}v(r)}{(\ln r)^{1/(p-1)}} = 0$.

Case 2. $\lim_{t\to+\infty} y_{(N-p)/(p-1)}(t) = -\infty$.

Then $\lim_{r\to+\infty} r^{(N-1)/(p-1)}v'(r) = -\infty$ and by Hôspital's rule, $\lim_{r\to+\infty} r^{(N-p)/(p-1)}v(r) = +\infty$. Therefore, according to Lemma 3.3, we have $F_{p/(\delta+1-p)}(r) < 0$ for large r. Consequently, for large t,

$$\frac{p}{\delta+1-p}\psi_{(N-p)/(p-1)}(t) < \left|k_{(N-p)/(p-1)}(t)\right|.$$

Hence for large t,

$$\frac{\psi^{p-1}_{(N-p)/(p-1)}(t)}{t} < \left(\frac{p}{\delta+1-p}\right)^{1-p} \frac{\left|y_{(N-p)/(p-1)}(t)\right|}{t}. \tag{3.60}$$

Therefore, it is clear that, to show that $\psi^{p-1}_{(N-p)/(p-1)}(t)/t$ is bounded for large t, it suffices to prove that $\left|y_{(N-p)/(p-1)}(t)\right|/t$ is bounded for large t.

According to Proposition 3.4, if there exists $p/(\delta+1-p) < \varrho < (N-p)/(p-1)$ such that $r^{p+\varrho(p-1)}h(r)$ is bounded for large r, then $r^{\varrho}v(r)$ is bounded for large r. In particular for $\varrho = N/\delta$, we have $r^{p+N(p-1)/\delta}h(r)$ is bounded for large r (because $p + N(p-1)/\delta < N$ and $r^N h(r)$ is bounded for large r) and therefore $r^{N/\delta}v(r)$ is bounded for large r. This is equivalent to $e^{\left(N-\delta(N-p)/(p-1)\right)t}\psi^{\delta}_{(N-p)/(p-1)}(t)$ is bounded for large t. Hence, by equation (3.59), there exists a constant $C > 0$ such that for large t, we have

$$-C < y'_{(N-p)/(p-1)}(t) < 0.$$

Integrating this last inequality on (T,t) for large T and using the fact that $y_{(N-p)/(p-1)}(t) < 0$, we get $\left|y_{(N-p)/(p-1)}(t)\right| /t$ is bounded for large t. This implies by (3.60) that $\psi^{p-1}_{(N-p)/(p-1)}(t)/t$ is bounded for large t. That is $r^{(N-p)/(p-1)}v(r)/\left(\ln r\right)^{1/(p-1)}$ is bounded for large r.

Now, we show the estimate (3.57). Assume by contradiction that

$$\liminf_{r\to+\infty} r^N h(r) > \left(\frac{N-p}{p-1}\right)^{p-1} \limsup_{r\to+\infty} \frac{r^{N-p}v^{p-1}(r)}{\ln r}. \tag{3.61}$$

We make the following change

$$V(t) = \frac{r^{(N-p)/(p-1)}v(r)}{\left(\ln r\right)^{1/(p-1)}}, \quad t = \ln r. \tag{3.62}$$

Then V satisfies the following equation

$$W'(t) + \frac{W(t)}{t} + t^{(\delta+1-p)/(p-1)} e^{\left(N-\delta(N-p)/(p-1)\right)t} V^{\delta}(t) + \frac{e^{Nt}h(e^t)}{t} = 0, \tag{3.63}$$

where

$$W(t) = \left|H(t)\right|^{p-2} H(t) \tag{3.64}$$

and

$$H(t) = V'(t) - \frac{N-p}{p-1}V(t) + \frac{1}{p-1}\frac{V(t)}{t}. \tag{3.65}$$

Note that

$$H(t) = \left(\ln r\right)^{-1/(p-1)} r^{(N-1)/(p-1)} v'(r). \tag{3.66}$$

Using the change (3.62), inequality (3.61) implies that there exists $\varepsilon_3 > 0$ such that for large t,

$$e^{Nt}h(e^t) \geq \left(\frac{N-p}{p-1}\right)^{p-1} V^{p-1}(t) + \varepsilon_3. \tag{3.67}$$

Therefore, according to equation (3.63), we have

$$W'(t) \leq -\frac{W(t)}{t} - \left(\frac{N-p}{p-1}\right)^{p-1}\frac{V^{p-1}(t)}{t} - \frac{\varepsilon_3}{t}. \tag{3.68}$$

On the other hand, we know by Lemma 2.2, that $F_{(N-p)/(p-1)}(r) > 0$ for large r. Therefore, using (3.62), (3.66), (3.64) and the fact that $v'(r) < 0$, we have

$$\left|W(t)\right| = -W(t) < \left(\frac{N-p}{p-1}\right)^{p-1} V^{p-1}(t). \tag{3.69}$$

Since $V(t)$ is bounded for large t by (i), then $W(t)$ is bounded for large t. But according to (3.68) and (3.69), we have for large t,

$$W'(t) < -\frac{\varepsilon_3}{t}.$$

Integrating this last inequality on (T,t) for large T, we obtain $\lim\limits_{t\to+\infty} W(t) = -\infty$. This contradicts the fact that $W(t)$ is bounded for large t by (3.69). Consequently, the estimate (3.57) is satisfied.

Finally, we show the estimate (3.58). Suppose by contradiction that

$$\limsup_{r\to+\infty} r^N h(r) < \left(\frac{N-p}{p-1}\right)^{p-1} \liminf_{r\to+\infty} \frac{r^{N-p}v^{p-1}(r)}{\ln r}. \tag{3.70}$$

Then there exists $\varepsilon_4 > 0$ such that for large t,

$$e^{Nt}h(e^t) \leq \left(\frac{N-p}{p-1}\right)^{p-1} V^{p-1}(t) - \varepsilon_4. \tag{3.71}$$

This inequality implies since h is positive, that

$$V(t) \geq \frac{p-1}{N-p}\varepsilon_4^{1/(p-1)} > 0 \tag{3.72}$$

and by equation (3.63),

$$tW'(t) \geq \chi_1(t) + \varepsilon_4 \tag{3.73}$$

where

$$\chi_1(t) = -W(t) - t^{\delta/(p-1)}e^{\left(N-\delta(N-p)/(p-1)\right)t}V^\delta(t) - \left(\frac{N-p}{p-1}\right)^{p-1} V^{p-1}(t). \tag{3.74}$$

We propose to show that $\lim_{t\to+\infty} \chi_1(t) = 0$. For this, since

$$\lim_{t\to+\infty} t^{\delta/(p-1)}e^{\left(N-\delta(N-p)/(p-1)\right)t}V^\delta(t) = 0 \tag{3.75}$$

because $V(t)$ is bounded for large t and $\delta > (N(p-1)/(p-1)$, it suffices to prove that

$$\lim_{t\to+\infty} W(t) + \left(\frac{N-p}{p-1}\right)^{p-1} V^{p-1}(t) = 0. \tag{3.76}$$

This will be shown in four steps.
Step 1. $F_{p/(\delta+1-p)}(r) < 0$ for large r.
By equation (1.1) we have

$$\left(r^{N-1}v'|v'|^{p-2}\right)' = -r^{N-1}v^\delta(r) - r^{N-1}h(r). \tag{3.77}$$

Then the function $r^{N-1}|v'|^{p-2}v'(r)$ is decreasing and negative. Therefore, $\lim_{r\to+\infty} r^{N-1}|v'|^{p-2}v'(r) \in [-\infty, 0[$, which is equivalent to $\lim_{r\to+\infty} r^{(N-1)/(p-1)}v'(r) \in [-\infty, 0[$. This gives by Hôspital's rule that $\lim_{r\to+\infty} r^{(N-p)/(p-1)}v(r) \in]0, +\infty]$. If $\lim_{r\to+\infty} r^{(N-p)/(p-1)}v(r)$ is finite, then using the change (3.62), we have $\lim_{t\to+\infty} V(t) = 0$. But this contradicts (3.72). Therefore, necessarily $\lim_{r\to+\infty} r^{(N-p)/(p-1)}v(r) = +\infty$ and by Lemma 3.3, $F_{p/(\delta+1-p)}(r) < 0$ for large r.
Step 2. $\lim_{t\to+\infty} H'(t) = 0$.
Since $V(t)$ is bounded for large t, then by (3.69), $W(t)$ is bounded for large t. Using in addition the fact that $e^{Nt}h(e^t)$ is bounded for large t and (3.75), we get by (3.63), $\lim_{t\to+\infty} W'(t) = 0$. On the other hand, using the change (3.62), the first step and the fact that $v' < 0$, we obtain

$$|H(t)| > \frac{p}{\delta+1-p}V(t). \tag{3.78}$$

This implies, using (3.72),

$$|H(t)| > C = \frac{p(p-1)}{(N-p)(\delta+1-p)}\varepsilon_4^{1/(p-1)} > 0. \tag{3.79}$$

Therefore, since by (3.64) $H'(t) = \frac{1}{p-1}|H(t)|^{2-p}W'(t)$ (H' exists because $v' < 0$), we have

$$|H'(t)| < \frac{C^{2-p}}{p-1}|W'(t)|.$$

Hence, $\lim_{t\to+\infty} H'(t) = 0$.

Step 3. $\lim_{t\to+\infty} V'(t) = 0$.

Since $W(t)$ is bounded for large t (by (3.69)), then $H(t)$ is bounded for large t and therefore by (3.65), $V'(t)$ is bounded for large t. Suppose that $V'(t)$ oscillates for large t. Then there exist two sequences $\{s_i\}$ and $\{k_i\}$ that go to $+\infty$ as $i \to +\infty$ such that $\{s_i\}$ and $\{k_i\}$ are local minimum and local maximum of V', respectively, satisfying $s_i < k_i < s_{i+1}$ and

$$\liminf_{t\to+\infty} V'(t) = \lim_{i\to+\infty} V'(s_i) < \limsup_{t\to+\infty} V'(t) = \lim_{i\to+\infty} V'(k_i). \tag{3.80}$$

By deriving the equation (3.65), we obtain

$$H'(t) = V''(t) - \frac{N-p}{p-1}V'(t) + \frac{1}{p-1}\frac{V'(t)}{t} - \frac{1}{p-1}\frac{V(t)}{t^2}. \tag{3.81}$$

Since $V''(s_i) = V''(k_i) = 0$, then

$$H'(s_i) = -\frac{N-p}{p-1}V'(s_i) + \frac{1}{p-1}\frac{V'(s_i)}{s_i} - \frac{1}{p-1}\frac{V(s_i)}{s_i^2}$$

and

$$H'(k_i) = -\frac{N-p}{p-1}V'(k_i) + \frac{1}{p-1}\frac{V'(k_i)}{k_i} - \frac{1}{p-1}\frac{V(k_i)}{k_i^2}.$$

It follows, since $V(t)$ and $V'(t)$ are bounded for large t and $\lim_{t\to+\infty} H'(t) = 0$, that

$$\lim_{i\to+\infty} V'(s_i) = \lim_{i\to+\infty} V'(k_i) = 0.$$

This contradicts (3.80). We deduce that $V'(t)$ converges when $t \to +\infty$. Since $V(t)$ is bounded for large t, then necessarily $\lim_{t\to+\infty} V'(t) = 0$.

Step 4. $\lim_{t\to+\infty} W(t) + \left(\frac{N-p}{p-1}\right)^{p-1} V^{p-1}(t) = 0$.

Recall that for any $\varrho > 1$, there exists a constant $C_\varrho > 0$ such that

$$\left||a|^{\varrho-2}a - |b|^{\varrho-2}b\right| \le C_\varrho \left(|a| + |b|\right)^{\varrho-2} |a-b| \tag{3.82}$$

for any a, $b \in \mathbb{R}$ such that $|a| + |b| > 0$. Hence, taking $\varrho = p > 2$, $a = (N-p)/(p-1)\, V(t)$ and $b = -H(t) = |H(t)|$, we obtain

$$\left|\left(\frac{N-p}{p-1}\right)^{p-1} V^{p-1}(t) + W(t)\right| \le c_p \left(\frac{N-p}{p-1}V(t) + |H(t)|\right)^{p-2} \left|\frac{N-p}{p-1}V(t) + H(t)\right|. \tag{3.83}$$

Since $V(t)$ and $H(t)$ are bounded for large t and $p > 2$, then there exists a constant $C > 0$ such that for large t,

$$\left|\left(\frac{N-p}{p-1}\right)^{p-1} V^{p-1}(t) + W(t)\right| \le C \left|\frac{N-p}{p-1} V(t) + H(t)\right|. \tag{3.84}$$

Using again the fact that $V(t)$ is bounded for large t and $\lim_{t\to+\infty} V'(t) = 0$, we deduce easily from (3.65) that

$$\lim_{t\to+\infty} H(t) + \frac{N-p}{p-1} V(t) = 0.$$

Which implies (3.76) and therefore by (3.75), $\lim_{t\to+\infty} \chi_1(t) = 0$.

Consequently, according to (3.73), there exists a constant $C > 0$ such that for large t,

$$tW'(t) \ge C.$$

Integrating this last inequality on (T, t) for large T, we obtain $\lim_{t\to+\infty} W(t) = +\infty$. Which contradicts the fact that $W(t)$ is bounded for large t. It follows that estimate (3.58) is satisfied. This completes the proof. □

Now, we return to the proof of Theorem 3.5.

Proof. (i) Using the change (3.62), we have that $V(t)$ is bounded for large t by Proposition 3.6. Assume that $V(t)$ oscillates for large t. Then there exist two sequences $\{\mu_i\}$ and $\{\nu_i\}$ that go to $+\infty$ as $i \to +\infty$ such that $\{\mu_i\}$ and $\{\nu_i\}$ are local minimum and local maximum of V, respectively, satisfying $\mu_i < \nu_i < \mu_{i+1}$ and

$$0 \le \liminf_{t\to+\infty} V(t) = \lim_{i\to+\infty} V(\mu_i) = \alpha_1 < \limsup_{t\to+\infty} V(t) = \lim_{i\to+\infty} V(\nu_i) = \beta_1 < +\infty. \tag{3.85}$$

Since $V'(\mu_i) = V'(\nu_i) = 0$, $V''(\mu_i) \ge 0$ and $V''(\nu_i) \le 0$, then using (3.64), (3.65) and (3.81), we have

$$\lim_{i\to+\infty} W(\mu_i) = -\left(\frac{N-p}{p-1}\right)^{p-1} \alpha_1^{p-1},$$

$$\lim_{i\to+\infty} W(\nu_i) = -\left(\frac{N-p}{p-1}\right)^{p-1} \beta_1^{p-1},$$

$$W'(\mu_i) = (p-1)\,|H(\mu_i)|^{p-2}\, H'(\mu_i) \ge -\,|H(\mu_i)|^{p-2}\, \frac{V(\mu_i)}{\mu_i^2}$$

and

$$W'(\nu_i) = (p-1)\,|H(\nu_i)|^{p-2}\, H'(\nu_i) \le -\,|H(\nu_i)|^{p-2}\, \frac{V(\nu_i)}{\nu_i^2}.$$

Therefore according to equation (3.63), we have

$$-\,|H(\mu_i)|^{p-2}\,\frac{V(\mu_i)}{\mu_i} \le \mu_i W'(\mu_i) = -W(\mu_i) - \mu_i^{\delta/(p-1)} e^{-\left(N-\delta(N-p)/(p-1)\right)\mu_i} V^{\delta}(\mu_i) - e^{N\mu_i} h(e^{\mu_i}) \tag{3.86}$$

and

$$-\,|H(\nu_i)|^{p-2}\,\frac{V(\mu_i)}{\mu_i} \ge \nu_i W'(\nu_i) = -W(\nu_i) - \nu_i^{\delta/(p-1)} e^{-\left(N-\delta(N-p)/(p-1)\right)\nu_i} V^{\delta}(\nu_i) - e^{N\nu_i} h(e^{\nu_i}). \tag{3.87}$$

Letting $i \to +\infty$ in the two previous inequalities and using the fact that $\lim_{t\to+\infty} e^{Nt} h(e^t) = l$, we obtain

$$\beta_1^{p-1} \le \left(\frac{p-1}{N-p}\right)^{p-1} l \le \alpha_1^{p-1}.$$

But this contradicts (3.85). Therefore, $V(t)$ converges when $t \to +\infty$.
On the other hand, we have, by Proposition 3.6,

$$\liminf_{t\to+\infty} V^{p-1}(t) \le \left(\frac{p-1}{N-p}\right)^{p-1} l \le \limsup_{t\to+\infty} V^{p-1}(t).$$

Hence $\lim_{t\to+\infty} V(t) = (p-1)/(N-p)\, l^{1/(p-1)}$.
(ii) Using the change (3.62) and by (i), we have $\lim_{t\to+\infty} V(t) = (p-1)/(N-p)\, l^{1/(p-1)}$. Now, we show that $\lim_{t\to+\infty} H(t) = -l^{1/(p-1)}$.
Since $F_{(N-p)/(p-1)}(r) > 0$ for large r Lemma 2.2, then $H(t)$ is bounded for large t. Suppose by contradiction that $H(t)$ oscillates for large t. Then there exist two sequences $\{m_i\}$ and $\{n_i\}$ that tend to $+\infty$ as $i \to +\infty$ such that $\{m_i\}$ and $\{n_i\}$ are local minimum and local maximum of H, respectively, satisfying $m_i < n_i < m_{i+1}$ and

$$\liminf_{t\to+\infty} H(t) = \lim_{i\to+\infty} H(m_i) = \alpha_2 < \limsup_{t\to+\infty} H(t) = \lim_{i\to+\infty} H(n_i) = \beta_2. \tag{3.88}$$

Therefore $W'(m_i) = W'(n_i) = 0$ (because $H'(m_i) = H'(n_i) = 0$), $\lim_{i\to+\infty} W(m_i) = |\alpha_2|^{p-2}\,\alpha_2$ and $\lim_{i\to+\infty} W(n_i) = |\beta_2|^{p-2}\,\beta_2$. Since $\delta > N(p-1)/(N-p)$, V converges and $\lim_{t\to+\infty} e^{Nt} h(e^t) =$

l, we deduce, by multiplying equation (3.63) by t, taking respectively $t = m_i$ and $t = n_i$ in equation (3.63) and letting $i \to +\infty$, that

$$|\alpha_2|^{p-2}\alpha_2 = -l = |\beta_2|^{p-2}\beta_2.$$

Therefore, $\alpha_2 = \beta_2$. That contradicts (3.88). Therefore, $H(t)$ converges when $t \to +\infty$. Hence, by (3.65), we have $\lim\limits_{t\to+\infty} V'(t) = 0$ (because V converges). Consequently $\lim\limits_{t\to+\infty} H(t) = -l^{1/(p-1)}$. This completes the proof. □

The last main result in this work concerns the search for the equivalents of v and v' near infinity in the case $m > N$.

Theorem 3.7. *Assume that $m > N$ and (H_c) holds. Let v be a solution of problem (1.1)-(1.2) such that $\lim\limits_{r\to+\infty} r^{p/(\delta+1-p)}v(r) = 0$. Then*

(i) $\lim\limits_{r\to+\infty} r^{(N-p)/(p-1)}v(r)$ is finite and strictly positive.

(ii) $\lim\limits_{r\to+\infty} r^{(N-1)/(p-1)}v'(r)$ is finite and strictly negative.

Proof. Since $h > 0$ and $v > 0$ (by Lemma 2.1), then by (3.77), the function $r^{N-1}|v'|^{p-2}v'(r)$ is decreasing and negative. Therefore, $\lim\limits_{r\to+\infty} r^{N-1}|v'|^{p-2}v'(r) \in [-\infty, 0[$. Assume by contradiction that
$\lim\limits_{r\to+\infty} r^{N-1}|v'|^{p-2}v'(r) = -\infty$. Then $\lim\limits_{r\to+\infty} \varphi(r) = +\infty$ where

$$\varphi(r) = r^{N-1}|v'(r)|^{p-1}. \tag{3.89}$$

Let $0 < \lambda_1 < \min\left(\delta(N-p)/(p-1) - N,\ m - N\right)$ (this is possible because $\delta > N(p-1)/(N-p)$ and $m > N$). We show that $\lim\limits_{r\to+\infty} r^{\lambda_1+1}\varphi'(r) = 0$.
We have by (3.77),

$$r^{\lambda_1+1}\varphi'(r) = r^{\lambda_1+N}v^{\delta}(r) + r^{\lambda_1+N}h(r). \tag{3.90}$$

Since $r^m\, h(r)$ is bounded for large r, then $r^N\, h(r)$ is also bounded for large r (because $m > N$). Therefore, according to Proposition 3.6 and the fact that $(\lambda_1+N)/\delta < (N-p)/(p-1)$, we have $\lim\limits_{r\to+\infty} r^{(\lambda_1+N)/\delta}v(r) = 0$. On the other hand, since $\lambda_1 + N < m$, then $\lim\limits_{r\to+\infty} r^{\lambda_1+N}h(r) = 0$. Therefore, we have by (3.90), $\lim\limits_{r\to+\infty} r^{\lambda_1+1}\varphi'(r) = 0$. Therefore, since φ' is strictly positive, there exists a constant $C > 0$ such that for large r,

$$0 < \varphi'(r) < Cr^{-\lambda_1-1}$$

Integrating this last inequality on (R, r) for large R and using the fact that $\lambda_1 > 0$, we obtain

$$\varphi(r) - \varphi(R) < \frac{-C}{\lambda_1}r^{-\lambda_1} + \frac{C}{\lambda_1}R^{-\lambda_1}.$$

By letting $r \to +\infty$, we obtain a contradiction with the fact that $\lim\limits_{r\to+\infty} \varphi(r) = +\infty$. Therefore, $\lim\limits_{r\to+\infty} r^{N-1}|v'|^{p-2}v'(r)$ is finite and strictly negative, that is, $\lim\limits_{r\to+\infty} r^{(N-1)/(p-1)}v'(r)$ is finite and strictly negative. Consequently, using Hôspital's rule (because $\lim\limits_{r\to+\infty} v(r) = 0$ and $N > p$), we have $\lim\limits_{r\to+\infty} r^{(N-p)/(p-1)}v(r)$ is finite and strictly positive. The proof is complete. □

4 Conclusion

In this paper, we presented a detailed study of the asymptotic behavior of global positive solutions of the problem (1.1)-(1.2) in the case where the inhomogeneous term h is strictly positive and negligible in front of $r^{-p\delta/(\delta+1-p)}$. The main results strongly depend on the sign and asymptotic behavior of the inhomogeneous term h. The case where the inhomogeneous term changes sign remains an open question to be treated in another paper.

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