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Renormalized solutions for some non-coercive parabolic equation in the anisotropic Sobolev spaces

Moussa Chrif, Hassane Hjiaj and Mohamed Sasy

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Abstract This paper is devoted to the study of the following nonlinear and non-coercive parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div}\,(a(x,t,u,\nabla u)) + |u|^{p_0-2}u = f - \text{div}\,(\phi(x,t,u)) & \text{in } \Omega \times (0,T) = Q_T, \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T); \end{cases}$$

in the anisotropic Sobolev space, where $f \in L^1(Q_T)$, and $\phi = (\phi_1, \phi_2 \ldots \phi_N)$ is a Carathéodory function acted from $Q_T \times \mathbb{R}$ into $\mathbb{R}^N$, that verifies some growth condition. We prove the existence of renormalized solutions for our parabolic equation, and we conclude some regularity results.

1 Introduction

Let Ω be a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$, we set $Q_T = \Omega \times (0,T)$ a cylinder of $\mathbb{R}^{N+1}$, with $T > 0$. Boccardo, Gallouët and Vasquez have studied in [8] the existence and regularity of renormalized solutions for the nonlinear parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p-2}\nabla u) + \alpha_0|u|^{p_0-2}u = f & \text{in } Q_T, \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T); \end{cases}$$

where the data $f \in L^1(Q_T)$ and the exponents $p > 1 + \frac{N}{N+1}$, $p_0 > \frac{p(N+1)-N}{N}$, with $\alpha_0 > 0$. Blanchard, Murat and Redwane have proved in [10] the existence and uniqueness of renormalized solution for the nonlinear parabolic

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div}\,(a(x,t,u,\nabla u)) + \text{div}(\phi(u)) = f - \text{div}\, g & \text{in } Q_T, \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

in the isotropic Sobolev space, with $-\,\text{div}\,(a(x,t,u,\nabla u))$ is a Leray-Lions operator, where $\phi(\cdot)$, f, g and u_0 belong respectively to $C^0(\mathbb{R}, \mathbb{R}^N)$, $L^1(Q_T)$, $(L^{p'}(Q_T))^N$ and $L^1(\Omega)$. Other problems have been considered in [5], [8] and [9].
It should be noted that the concept of the renormalized solution was originally introduced by DiPerna and Lions [13] in their study of the Boltzmann equation, and was later adapted by Boccardo, Giachetti, Diaz, and Murat [6] to address elliptic problems with L^1 data. Recently, anisotropic Sobolev spaces have garnered significant attention due to their diverse applications in fields such as electro-rheological fluids and image processing (for a more detailed discussion, we refer the reader to [17], [18] and [19]).
The existence of entropic and renormalized solutions for certain nonlinear parabolic problems in Sobolev spaces has been demonstrated by the authors, in [2],[7], and [15].

In [12] Chrif, Hjiaj and El Manouni, have studied the existence of entropy and renormalized solutions for the following nonlinear Dirichlet parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div}\, a(x,t,u,\nabla u) + g(x,t,u) = f & \text{in } Q_T, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T); \end{cases}$$

in the anisotropic parabolic Sobolev spaces $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, with $f \in L^1(Q_T)$, $u_0 \in L^1(\Omega)$ and $g(x,t,s)$ is a Carathéodory function, that verifies same growth condition. We refer the reader also to [11].
Our objective in this paper is to prove the existence of renormalized solutions for the following nonlinear parabolic Drechlet problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div}\,(a(x,t,u,\nabla u)) + |u|^{p_0-2}u = f - \text{div}\,(\phi(x,t,u)) & \text{in } Q_T, \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T); \end{cases}$$

in the anisotropic Sobolev space $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, where $-\text{div}\, a(x,t,u,\nabla u)$ is an operator of Leray-Lions type acting from $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ into its dual $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$, while $(a_i(x,t,s,\xi))_{i=1,\ldots,N}$ are Carathéodory functions that verify the degenerate coercivity. where $f \in L^1(Q_T)$ and $\phi = (\phi_1, \phi_2 \ldots \phi_N)$ is Carathéodory function acted from $Q_T \times \mathbb{R}$ into $\mathbb{R}^N$, that verifies only some growth condition.

This paper is organized as follows: In section 2 we presents some definitions and results related to the anisotropic parabolic spaces. In section 3 we presents the essential assumptions and technical lemmas required to establish the main result. The section 4 focuses on demonstrating the existence of renormalized solutions for the parabolic problem in anisotropic spaces, and we conclude some regularity results.

2 Preliminaries

Let Ω be an open bounded domain in $I\!R^N$ $(N \geq 2)$ with boundary $\partial\Omega$.
Let $p_1, p_2, \ldots, p_N$ be N real exponents, such that $1 < p_i < \infty$ for $i = 1, \ldots, N$.
We set $\vec{p} = (p_1, \ldots, p_N)$, and

$$\underline{p} = \min\{p_1, p_2, \ldots, p_N\} \qquad \text{and} \qquad p_0 = \max\{p_1, p_2, \ldots, p_N\}.$$

Moreover, we denote

$$D^0 u = u \qquad \text{and} \qquad D^i u = \frac{\partial u}{\partial x_i} \qquad \text{for} \qquad i = 1, \ldots, N.$$

The anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ is defined as

$$W^{1,\vec{p}}(\Omega) = \left\{ u \in L^{p_0}(\Omega) \quad \text{such that} \quad D^i u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \ldots, N \right\},$$

this space is equipped with a norm

$$\|u\|_{1,\vec{p}} = \sum_{i=0}^{N} \|D^i u\|_{p_i}. \tag{2.1}$$

We set $W_0^{1,\vec{p}}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ for the norm (2.1).
The Sobolev spaces $W^{1,\vec{p}}(\Omega)$ and $W_0^{1,\vec{p}}(\Omega)$ are separable and reflexive Banach space.

Lemma 2.1. *Let Ω be a bounded open set in $\mathbb{R}^N$. Then the following embedding are compact.*

- *if* $\overline{p} < N$ *then* $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ *for any* $r \in [1, \underline{p}^*[$, *where* $\frac{1}{\underline{p}^*} = \frac{1}{\underline{p}} - \frac{1}{N}$.
- *if* $\overline{p} = N$ *then* $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ *for any* $r \in [1, +\infty[$,
- *if* $\overline{p} > N$ *then the embedding* $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$.

The proof is based on the continuous embedding of $W_0^{1,\vec{p}}(\Omega)$ into $W_0^{1,\underline{p}}(\Omega)$, and the compact embedding theorem for Sobolev spaces.

Definition 2.2. The dual of the anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$ is denoted by $W^{-1,\vec{p'}}(\Omega)$, where $\vec{p'} = (p'_1, p'_2, \ldots, p'_N)$ giving by :

$$W^{-1,\vec{p'}}(\Omega) = \Big\{ F = F_0 - \sum_{i=1}^{N} D^i F_i \ \ / \ \ F_0 \in L^{p'_0}(\Omega) \text{ and } F_i \in L^{p'_i}(\Omega) \text{ for } i = 1, 2, \ldots, N \Big\}.$$

Moreover, for all $u \in W_0^{1,\vec{p}}(\Omega)$ we have

$$\langle F, u \rangle = \sum_{i=0}^{N} \int_\Omega F_i D^i u \, dx.$$

The norm on the dual space is defined by

$$\|F\|_{-1,\vec{p'}} = \inf \left\{ \sum_{i=0}^{N} \|F_i\|_{p'_i} \ \text{ with } \ F = F_0 - \sum_{i=1}^{N} D^i F_i, \ where \ F_0 \in (L^{p'_0}(\Omega) \ \text{ and } \ F_i \in L^{p'_i}(\Omega) \right\}.$$

Now, we introduce the anisotropic parabolic space $L^{\vec{p}}(0, T; W^{1,\vec{p}}(\Omega))$ by

$$L^{\vec{p}}(0, T; W^{1,\vec{p}}(\Omega)) = \left\{ u \text{ measurable function } / \sum_{i=0}^{N} \int_0^T \left\| D^i u \right\|_{L^{p_i}(\Omega)}^{p_i} dt < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^{\vec{p}}(0,T;W^{1,\vec{p}}(\Omega))} = \sum_{i=0}^{N} \left\| D^i u \right\|_{L^{p_i}(Q_T)}.$$

The functional space $L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega))$ is defined by

$$L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega)) = \left\{ u \in L^{\vec{p}}(0, T; W^{1,\vec{p}}(\Omega)) / u = 0 \text{ on } \partial\Omega \times [0, T] \right\}.$$

Note that $L^{\vec{p}}(0, T; W^{1,\vec{p}}(\Omega))$ and $L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega))$ are separable and reflexive Banach spaces.

Definition 2.3. The dual space of $L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega))$ is defined as follows

$$L^{\vec{p'}}(0, T; W^{-1,\vec{p'}}(\Omega)) = \left\{ F = f_0 - \sum_{i=1}^{N} D^i f_i, \text{ with } f_i \in L^{p'_i}(Q_T) \right\}.$$

We define a norm on the dual space by

$$\|F\|_{L^{\vec{p'}}(0,T;W^{-1,\vec{p'}}(\Omega))} = \inf \left\{ \sum_{i=0}^{N} \|f_i\|_{L^{p'_i}(Q_T)} / \quad F = f_0 - \sum_{i=1}^{N} D^i f_i \text{ with } f_i \in L^{p'_i}(Q_T) \right\}.$$

The duality of the spaces $L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega))$ and $L^{\vec{p'}}(0, T; W^{-1,\vec{p'}}(\Omega))$ is given by relation

$$\int_0^T \langle F, v \rangle \, dt = \sum_{i=0}^{N} \int_{Q_T} f_i D^i v dx \quad \text{ for all } \quad v \in L^{\vec{p}}(0, T; W_0^{1,\vec{p}}(\Omega)),$$

and we introduce the norm for $L^{\vec{p'}}(0, T; W^{-1,\vec{p'}}(\Omega))$ by

$$\|F\|_{L^{\vec{p'}}(0,T;W^{-1,\vec{p'}}(\Omega))} = \inf \Big\{ \sum_{i=0}^{N} \|f_i\|_{L^{p'_i}(Q_T)}, \ \text{ with } \ F = f_0 - \sum_{i=1}^{N} D^i f_i \ \text{ for } \ f_i \in L^{p'_i}(Q_T) \Big\}.$$

Lemma 2.4. *([20]) Let B_0, B and B_1 be a Banach spaces with $B_0 \subset B \subset B_1$. Let us set*

$$Y = \{u \ : \ u \in L^{p_0}(0,T;B_0) \quad and \quad u' \in L^{p_1}(0,T;B_1)\},$$

where $p_0 > 1$ and $p_1 > 1$ are reals numbers.
Assuming that the embedding $B_0 \hookrightarrow\hookrightarrow B$ is compact, then

$$Y \hookrightarrow\hookrightarrow L^{p_0}(0,T;B),$$

and this imbedding is compact.

Remark 2.5. Let $\underline{p} > \dfrac{2N}{N+2}$, we set

$$B_0 = W_0^{1,\vec{p}}(\Omega), \qquad B = L^2(\Omega) \qquad \text{and} \qquad B_1 = W^{-1,\vec{p}'}(\Omega),$$

with $p_0 = \underline{p}$, and $p_1 = p_0'$. In view of the Lemma 2.4, we obtain

$$\{u \ : \ u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \quad \text{and} \quad u' \in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))\} \subseteq Y \hookrightarrow\hookrightarrow L^1(Q_T). \tag{2.2}$$

Moreover, in view of [3], we have

$$\{u \ : \ u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \quad \text{and} \quad u' \in L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))\} \subseteq C([0,T];L^1(\Omega)). \tag{2.3}$$

Definition 2.6. Let $k > 0$, the truncation function $T_k(\cdot) : \mathbb{R} \longmapsto \mathbb{R}$ is defined as

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \le k, \\ k\frac{s}{|s|} & \text{if} \quad |s| > k. \end{cases}$$

Lemma 2.7 (see [14], Theorem 13.47). *Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that*

(i) $u_n \to u$ *a.e. in* Ω,

(ii) $u_n \ge 0$ *and* $u \ge 0$ *a.e. in* Ω,

(iii) $\displaystyle\int_\Omega u_n \, dx \to \int_\Omega u \, dx$,

then $u_n \longrightarrow u$ in $L^1(\Omega)$.

Lemma 2.8. *Let $u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, then $T_k(u) \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ for any $k > 0$. Moreover, we have*

$$T_k(u) \longrightarrow u \qquad in \quad L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \quad as \quad k \to \infty.$$

Proposition 2.9. *We introduce a time mollification of a function $u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ for all $\mu \ge 0$ by*

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} u(x,s) \exp(\mu(s-t)) \chi_{(0,T)}(s) \, ds.$$

Then, the following assertions hold.

(i) If $u \in L^{p_0}(Q_T)$, then u_μ is measurable in Q_T, with $\dfrac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$, and

$$\int_{Q_T} |u_\mu|^{p_0} dxdt \le \int_{Q_T} |u|^{p_0} dxdt$$

(ii) If $u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, then $u_\mu \to u$ in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ as $\mu \to +\infty$.

(iii) If $u_n \to u$ in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, then $(u_n)_\mu \to u_\mu$ in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$.

(iv) $|(T_k(u))_\mu| \le k$ for all $u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$.

Proofs of Lemma 2.8 and Proposition 2.9 are is similar to those in the classical space $L^p(0,T;W_0^{1,p}(\Omega))$.

3 Essential Assumptions

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N (N \geq 2)$, and $T > 0$.
We consider a Leray-Lions operator $A : L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega)) \longmapsto L^{\vec{p}'}(0,T; W^{-1,\vec{p}'}(\Omega))$ given by

$$Au = -\sum_{i=1}^{N} D^i a_i(x,t,u,\nabla u) + |u|^{p_0-2}u, \tag{3.1}$$

where $a_i(x,t,s,\xi) : Q_T \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}$ is a Carathéodory function, which satisfies the following conditions:

$$|a_i(x,t,s,\xi)| \leq \beta(K_i(x,t) + |s|^{p_i-1} + |\xi_i|^{p_i-1}), \tag{3.2}$$

$$(a_i(x,t,s,\xi) - a_i(x,t,s,\xi^*))(\xi_i - \xi_i^*) > 0 \quad \text{for all } \xi_i \neq \xi_i^*, \tag{3.3}$$

$$a_i(x,t,s,\xi)\xi_i \geq \alpha(|s|)|\xi_i|^{p_i}; \tag{3.4}$$

where $K_i(x,t)$ is a nonnegative function lying in $L^{p_i'}(Q_T)$, and $\alpha(|\cdot|)$ is a positive decreasing function, such that $\alpha(|s|) \geq \frac{\alpha_0}{(1+|s|)^\delta}$ for all $s \in \mathbb{R}$. with $0 < \alpha_0$ and $0 < \delta < \underline{p} - 1$.
As a consequence of (3.4), and the continuity of the function $a_i(x,t,s,\cdot)$ with respect to ξ, we have

$$a_i(x,t,s,0) = 0 \qquad \text{for all} \quad i = 1,2,\ldots,N.$$

The Carathéodory function $\phi_i(x,t,s) : \Omega \times (0,T) \times \mathbb{R} \longmapsto \mathbb{R}$, satisfies the following condition

$$|\phi_i(x,t,s)| \leq c_i(x,t)(1+|s|)^\lambda, \tag{3.5}$$

where $c_i(x,t)$ is nonnegative function lying in $L^{r_i}(\Omega)$ for $r_i \geq \max\left(\frac{(p_0-1)p_i'}{p_0-(\lambda+\delta)p_i'+\delta}, p_i'\right)$, and $0 < \lambda < \frac{p_0-\delta(p_i'-1)}{p_i'}$, for all $i = 1,2,\ldots,N$.

We consider the quasilinear parabolic Dirichlet problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \sum_{i=1}^{N} D^i a_i(x,t,u,\nabla u) + |u|^{p_0-2}u = f - \sum_{i=1}^{N} D^i \phi(x,t,u) & \text{in } Q_T, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega; \end{cases} \tag{3.6}$$

where $f \in L^1(Q_T)$, and $u_0 \in L^1(\Omega)$.
Now, we present an essential lemma to establish the existence of solutions for our quasilinear parabolic problem.

Lemma 3.1. *(cf. [11]) Assume (3.2) – (3.4) hold true. Let $(u_n)_n$ be a sequence in $L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega))$ such that $\frac{\partial u_n}{\partial t} \in L^{\vec{p}'}(0,T; W^{-1,\vec{p}'}(\Omega))$, and $u_n \rightharpoonup u$ in $L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega))$ with*

$$\sum_{i=1}^{N} \int_{Q_T} (a_i(x,t,u_n,\nabla u_n) - a_i(x,t,u_n,\nabla u))(D^i u_n - D^i u)\,dx\,dt$$
$$+ \int_{Q_T} (|u_n|^{p_0-2}u_n - |u|^{p_0-2}u)(u_n - u)\,dx\,dt \longrightarrow 0$$

then $u_n \longrightarrow u$ in $L^{\vec{p}}(0,T; W_0^{1,\vec{p}}(\Omega))$ for a subsequence.

4 Main result

For all $k > 0$, we define the following function

$$\forall s \in \mathbb{R}, \quad \varphi_k(r) = \int_0^r T_k(s)ds = \begin{cases} \frac{r^2}{2} & \text{if} \quad |r| \leq k, \\ k|r| - \frac{k^2}{2} & \text{if} \quad |r| > k. \end{cases}$$

Definition 4.1. A measurable function u is a renormalized solution of the nonlinear parabolic Drichlet problem (3.6), if $T_k(u) \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, and $|u|^{p_0-2}u \in L^1(Q_T)$, with $u \in C(0,T;L^1(\Omega))$ such that

$$\lim_{h\to\infty} \frac{1}{h}\sum_{i=1}^{N}\int_{\{|u|\le h\}} a_i(x,t,u,\nabla u)D^i u \, dxdt = 0, \tag{4.1}$$

and u verifies the following equality

$$\begin{aligned}&\int_0^T \langle \frac{\partial S(u)}{\partial t}, v\rangle dt + \sum_{i=1}^{N}\int_{Q_T} a_i(x,t,u,\nabla u)(vS''(u)D^i u + S'(u)D^i v) \; dx\, dt \\ &\quad + \int_{Q_T} |u|^{p_0-2}uS'(u)v \, dx\, dt \\ &= \int_{Q_T} fS'(u)v \, dxdt + \sum_{i=1}^{N}\int_{Q_T} \phi_i(x,t,u)(vS''(u)D^i u + S'(u)D^i v)\, dx\, dt,\end{aligned}$$

for all $v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \cap L^\infty(Q_T)$, and for any smooth function $S(\cdot) \in \mathcal{C}^2(\mathbb{R})$, with $S'(\cdot)$ has a compact support.

Theorem 4.2. *Let $f \in L^1(Q_T)$, and $u_0 \in L^1(\Omega)$. Assuming that (3.2) – (3.4) and (3.5) hold true, then the quasilinear parabolic equation (3.6) has at least one renormalized solution u in the anisotropic Sobolev space.*

Proof of Theorem 4.2

Step 1: Approximate problems

Let $f_n = T_n(f)$, $u_{0,n} = T_n(u_0)$ and $\phi_n = (\phi_{1,n}, \phi_{2,n}, \dots, \phi_{N,n})$, with $\phi_{i,n}(x,t,s) = \phi_i(x,t,T_n(s))$ for all $i = 1 \dots N$. We consider the approximate problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \displaystyle\sum_{i=1}^{N} D^i a_i(x,t,T_n(u_n),\nabla u_n) + |u_n|^{p_0-2}u_n = f_n - \sum_{i=1}^{N} D^i\phi_{i,n}(x,t,u_n) & \text{in } Q_T, \\ u_n = 0 & \text{on } \partial\Omega\times[0,T] \\ u_n(x,0) = u_{0,n}(x) & \text{in } \Omega. \end{cases} \tag{4.2}$$

We define the operators $A_n, G_n : L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \longmapsto L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$, by $\forall u,v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$,

$$\int_0^T \langle A_n u, v\rangle \; dt = \sum_{i=1}^{N}\int_{Q_T} a_i(x,t,T_n(u),\nabla u)D^i v \, dx\, dt + \int_{Q_T} |u|^{p_0-2}uv \, dxdt, \tag{4.3}$$

and

$$\int_0^T \langle G_n u, v\rangle \; dt = -\sum_{i=1}^{N}\int_{Q_T} \phi_{i,n}(x,t,u)D^i v \, dx|dt, \tag{4.4}$$

Thanks to Hölder's inequality, we have

$$\begin{aligned}&\left|\int_0^T \langle A_n u, v\rangle \; dt\right| \le \sum_{i=1}^{N}\int_{Q_T} |a_i(x,t,T_n(u),\nabla u)||D^i v| \, dx\, dt + \int_{Q_T} |u|^{p_0-1}|v| \, dxdt \\ &\le \sum_{i=1}^{N}\int_{Q_T} \beta(K_i(x,t) + |T_n(u)|^{p_i-1} + |D^i u|^{p_i-1})|D^i v| \, dx\, dt + \|u\|^{p_0-1}_{L^{p_0}(Q_T)}\|v\|_{L^{p_0}(Q_T)} \\ &\le \beta\sum_{i=1}^{N}(\|K_i(x,t)\|_{L^{p_i'}(Q_T)} + \|n\|^{p_i-1}_{L^{p_i}(Q_T)} + \left\|D^i u\right\|^{p_i-1}_{L^{p_i}(Q_T)})\left\|D^i v\right\|_{L^{p_i}(Q_T)} + \|u\|^{p_0-1}_{L^{p_0}(Q_T)}\|v\|_{L^{p_0}(Q_T)} \\ &\le C\big(1 + \|u\|^{p_0-1}_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}\big)\|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))},\end{aligned}$$

and

$$\left|\int_0^T \langle G_n u, v\rangle \; dt\right| = \sum_{i=1}^{N} \int_{Q_T} |\phi_{i,n}(x,t,u)||D^i v| \, dx \, dt$$
$$\leq \sum_{i=1}^{N} \int_{Q_T} c_i(x,t)(1+n)^{\lambda} |D^i v| \, dx \, dt \tag{4.5}$$
$$\leq C(1+n)^{\lambda} \|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}.$$

Lemma 4.3. *The bounded operator* $B_n = A_n + G_n$ *acted from* $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ *into* $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))$ *is pseudo-monotone, and coercive in the following sense*

$$\frac{\int_0^T \langle B_n u, u\rangle \; dt}{\|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}} \longrightarrow +\infty \qquad \textit{as} \quad \|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \longrightarrow +\infty.$$

Proof of Lemma

For the coercivity, in view of Young's inequality and (3.4) − (3.5), we have

$$\int_0^T \langle B_n u, u\rangle \; dt = \sum_{i=1}^{N} \int_{Q_T} a_i(x,t,T_n(u),\nabla u) D^i u \, dxdt + \int_{Q_T} |u|^{p_0} \, dxdt - \sum_{i=1}^{N} \int_{Q_T} \phi_{i,n}(x,t,u) D^i u \, dxdt$$
$$\geq \sum_{i=1}^{N} \alpha(T_n(u)) \int_{Q_T} |D^i u|^{p_i} \, dx \, dt + \int_{Q_T} |u|^{p_0} \, dx \, dt - (1+n)^{\lambda} \sum_{i=1}^{N} \int_{Q_T} c_i(x,t)|D^i u| \, dxdt$$
$$\geq \frac{\alpha(n)}{2} \sum_{i=1}^{N} \int_{Q_T} |D^i u|^{p_i} \, dx \, dt + \int_{Q_T} |u|^{p_0} \, dx \, dt - C_1(n)$$
$$\geq \min(\frac{\alpha(n)}{2}, 1) \|u\|^{\underline{p}}_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} - C_2(n),$$

for any $u \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$. It follows that

$$\frac{\int_0^T \langle B_n u, u\rangle \; dt}{\|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))}} \longrightarrow +\infty \quad \text{as} \quad \|u\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \to +\infty.$$

Now, we show that B_n is pseudo-monotone. Let $(u_k)_k$ be a sequence in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ such that

$$\begin{cases} u_k \rightharpoonup u \text{ in } L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)), \\ B_n u_k \rightharpoonup \chi_n \quad \text{ in } L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega)), \\ \limsup\limits_{k\to\infty} \int_0^T \langle B_n u_k, u_k\rangle \; dt \leq \int_0^T \langle \chi_n, u\rangle \; dt. \end{cases} \tag{4.6}$$

We prove that

$$\chi_n = B_n u, \quad \text{ and } \quad \int_0^T \langle B_n u_k, u_k\rangle \; dt \longrightarrow \int_0^T \langle \chi_n, u\rangle \; dt \text{ as } k \to +\infty.$$

Firstly, we have $u_k \rightharpoonup u$ in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ and in view of the compact embedding with $L^1(Q_T)$, we obtain $u_k \to u$ strongly in $L^1(Q_T)$, and a.e. in Q_T.

We have $(u_k)_k$ is a bounded sequence in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$. Using the growth condition (3.2), the Carathéodory function $(a_i(x,t,T_n(u_k),\nabla u_k))_k$ is bounded in $L^{p_i'}(Q_T)$. Therefore there exists a measurable function $\varphi_i \in L^{p_i'}(Q_T)$ such that

$$a_i(x,t,T_n(u_k),\nabla u_k) \rightharpoonup \varphi_i \qquad \text{in} \quad L^{p_i'}(Q_T) \quad \text{for} \quad i = 1,\ldots,N \tag{4.7}$$

Further, we have $\phi_{i,n}(x,t,u_k) \to \phi_{i,n}(x,t,u)$ a.e. in Q_T and $|\phi_{i,n}(x,t,u_k)| \leq C_i(x)(1+n)^\lambda$ in $L^{p_i'}(Q_T)$. Hence, in view of Lebesgue's dominated convergence theorem, we deduce that

$$\phi_{i,n}(x,t,u_k) \longrightarrow \phi_{i,n}(x,t,u) \qquad \text{strongly in} \quad L^{p_i'}(Q_T). \tag{4.8}$$

We also have

$$|u_k|^{p_0-2}u_k \rightharpoonup |u|^{p_0-2}u \qquad \text{weakly in} \quad L^{p_0'}(Q_T). \tag{4.9}$$

Thus, for any $v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, we get

$$\begin{aligned}\int_0^T \langle \chi_n, v\rangle \ dt &= \lim_{k\to\infty} \int_0^T \langle B_n u_k, v\rangle \ dt \\ &= \lim_{k\to\infty} \Big(\sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k) D^i v \, dxdt + \int_{Q_T} \phi_{i,n}(x,t,u_k) D^i v \ dx\, dt \\ &\quad + \int_{Q_T} |u_k|^{p_0-2}u_k v \ dx\, dt\Big) \\ &= \sum_{i=1}^N \int_{Q_T} \varphi_i D^i v \, dxdt + \int_{Q_T} \phi_{i,n}(x,t,u) D^i v \, dxdt + \int_{Q_T} |u|^{p_0-2}uv \ dx\, dt\end{aligned} \tag{4.10}$$

Having in mind (4.6) and (4.10), we obtain

$$\begin{aligned}\limsup_{k\to\infty} \int_0^T \langle B_n(u_k), u_k\rangle \ dt &= \limsup_{k\to\infty} \Big(\sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k) D^i u_k \, dxdt \\ &\quad + \int_{Q_T} \phi_{i,n}(x,t,u_k) D^i u_k \, dxdt + \int_{Q_T} |u_k|^{p_0} \, dxdt\Big) \\ &\leq \sum_{i=1}^N \int_{Q_T} \varphi_i D^i u \, dxdt + \int_{Q_T} \phi_{i,n}(x,t,u) D^i u \, dxdt + \int_{Q_T} |u|^{p_0} \, dxdt\end{aligned}$$

Thanks to (4.8), we have

$$\int_{Q_T} \phi_{i,n}(x,t,u_k) D^i u_k \, dxdt \longrightarrow \int_{Q_T} \phi_{i,n}(x,t,u) D^i u \, dxdt. \tag{4.11}$$

Therefore

$$\begin{aligned}&\limsup_{k\to\infty} \Big(\sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k) D^i u_k \, dxdt + \int_{Q_T} |u_k|^{p_0} \, dxdt\Big) \\ &\quad \leq \sum_{i=1}^N \int_{Q_T} \varphi_i D^i u \, dxdt + \int_{Q_T} |u|^{p_0} \, dxdt\end{aligned} \tag{4.12}$$

On the other hand, using (3.3), we have

$$\begin{aligned}&\sum_{i=1}^N \int_{Q_T} (a_i(x,t,T_n(u_k),\nabla u_k) - a_i(x,t,T_n(u_k),\nabla u))(D^i u_k - D^i u) \, dxdt \\ &\qquad + \int_{Q_T} (|u_k|^{p_0-2}u_k - |u|^{p_0-2}u)(u_k - u) \, dxdt \geq 0.\end{aligned}$$

Then,

$$\sum_{i=1}^{N}\int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k)D^i u_k\,dxdt+\int_{Q_T}|u_k|^{p_0}\,dxdt$$
$$\geq \sum_{i=1}^{N}\int_{Q_T} a_i(x,t,T_n(u_k),\nabla u)(D^i u_k-D^i u)\,dxdt+\sum_{i=1}^{N}\int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k)D^i u\ \,dx\,dt$$
$$+\int_{Q_T}|u|^{p_0-2}u(u_k-u)\ \,dx\,dt+\int_{Q_T}|u_k|^{p_0-2}u_k u\,dxdt.$$

Now, we have $T_n(u_k)\to T_n(u)$ in $L^{p_i}(Q_T)$, then $a_i(x,t,T_n(u_k),\nabla u)\to a_i(x,t,T_n(u),\nabla u)$ strongly in $L^{p_i'}(Q_T)$. Thanks to (4.7)–(4.9) we deduce that

$$\liminf_{k\to\infty}\Big(\sum_{i=1}^{N}\int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k)D^i u_k\,dxdt+\int_{Q_T}|u_k|^{p_0}\,dxdt\Big)$$
$$\geq\sum_{i=1}^{N}\int_{Q_T}\varphi_i D^i u\,dxdt+\int_{Q_T}|u|^{p_0}\,dxdt.$$

Which implies, thanks to (4.12), that

$$\begin{aligned}\lim_{k\to\infty}\Big(&\sum_{i=1}^{N}\int_{Q_T} a_i(x,t,T_n(u_k),\nabla u_k)D^i u_k\,dxdt+\int_{Q_T}|u_k|^{p_0}\ \,dx\,dt\Big)\\&=\sum_{i=1}^{N}\int_{Q_T}\varphi_i D^i u\,dxdt+\int_{Q_T}|u|^{p_0}\,dxdt.\end{aligned}\tag{4.13}$$

By combining (4.10), (4.11) and (4.13), we deduce that

$$\int_0^T\langle B_n u_k,u_k\rangle\ dt\longrightarrow\int_0^T\langle\chi_n,u\rangle\ dt\qquad\text{as}\qquad k\to\infty.$$

Moreover, in view of (4.13), we obtain

$$\sum_{i=1}^{N}\int_{Q_T}(a_i(x,t,T_n(u_k),\nabla u_k)-a_i(x,t,T_n(u_k),\nabla u))(D^i u_k-D^i u)\,dxdt$$
$$+\int_{Q_T}(|u_k|^{p_0-2}u_k-|u|^{p_0-2}u)(u_k-u)\,dx\,dt\longrightarrow 0\qquad\text{as}\quad k\to+\infty.$$

Hence, thanks to Lemma 3.1, we get

$$u_k\longrightarrow u\quad\text{ in }\quad L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)).$$

Thus, $D^i u_k\longrightarrow D^i u$ a.e. in Q_T. It follows that $a_i(x,t,T_n(u_k),\nabla u_k)\longrightarrow a_i(x,t,T_n(u),\nabla u)$ a.e. in Q_T, then

$$a_i(x,t,T_n(u_k),\nabla u_k)\to a_i(x,t,T_n(u),\nabla u)\qquad\text{weakly in}\quad L^{p_i'}(Q_T)\quad\text{ for }\quad i=1,\dots,N.$$

Having in mind (4.8) and (4.9), we deduce that $\chi_n=B_n u$, which completes the proof of Lemma 4.3.

Consequently, In view of Lemma 4.3, there exists at least one weak solution $u_n\in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ for the approximate problem (4.2) (cf. [16], Theorem 2.7).

Step 2 : A priori estimates.

Let $k \geq 1$. By taking $T_k(u_n)$ as a test function for the approximate problem (4.2), we obtain

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, T_k(u_n) \rangle dt - \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n) D^i T_k(u_n)\, dxdt + \int_{Q_T} |u_n|^{p_0-1}|T_k(u_n)| dx\, dt$$
$$= \int_{Q_T} f_n T_k(u_n)\, dx\, dt + \sum_{i=1}^N \int_{Q_T} \phi_{i,n}(x,t,u_n) D^i T_k(u_n)\, dxdt. \tag{4.14}$$

For the first terms on the left-hand side of (4.14), we have

$$\begin{aligned} \int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_k(u_n) \right\rangle dt &= \int_\Omega \int_0^T \frac{\partial u_n}{\partial t} T_k(u_n)\, dt\, dx = \int_\Omega \int_0^T \frac{\partial \varphi_k(u_n)}{\partial t}\, dt\, dx \\ &= \int_\Omega \varphi_k(u_n(s))\, dx - \int_\Omega \varphi_k(u_n(0))\, dx \\ &\geq \int_\Omega \varphi_k(u_n(s))\, dx - k\|u_0\|_{L^1(\Omega)}. \end{aligned} \tag{4.15}$$

Concerning the second and third terms on the left-hand side of (4.14), we have

$$\int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n) D^i T_k(u_n)\, dxdt \geq \frac{\alpha_0}{(1+k)^\delta} \int_{Q_T} |D^i T_k(u_n)|^{p_i}\, dx\, dt, \tag{4.16}$$

and

$$\int_{Q_T} |u_n|^{p_0-2} u_n T_k(u_n)\, dxdt \geq \int_{Q_T} |T_k(u_n)|^{p_0}\, dx\, dt. \tag{4.17}$$

Concerning the two terms on the right-hand side of (4.14), we have

$$\left| \int_{Q_T} f_n T_k(u_n)\, dx\, dt \right| \leq k\|f\|_{L^1(Q_T)}. \tag{4.18}$$

Since $0 < \delta < \underline{p} - 1$, $0 < \lambda < \dfrac{p_0 - \delta(p_i' - 1)}{p_i'}$, and according to (3.5), (3.4) and Young's inequality, we obtain

$$\begin{aligned} &\left| \int_{Q_T} \phi_{i,n}(x,t,u_n) D^i T_k(u_n)\, dx\, dt \right| \\ &\leq \int_{Q_T} |\phi_{i,n}(x,t,u_n)||D^i T_k(u_n)|\, dx\, dt \\ &\leq \int_{Q_T} c_i(x,t)(1+|T_k(u_n)|)^\lambda |D^i T_k(u_n)|\, dxdt \\ &\leq \int_{Q_T} \frac{\alpha_0 |D^i T_k(u_n)|^{p_i}}{2(1+|u_n|)^\delta}\, dx\, dt + C_1 \int_{\{|u_n|\leq k\}} |c_i(x,t)|^{p_i'}(1+|u_n|)^{\lambda p_i' + \frac{\delta p_i'}{p_i}}\, dx\, dt \\ &\leq \int_{Q_T} \frac{\alpha_0 |D^i T_k(u_n)|^{p_i}}{2(1+|u_n|)^\delta}\, dx\, dt + \frac{1}{2N} \int_{\{|u_n|\leq k\}} (1+|u_n|)^{p_0}\, dx\, dt \\ &+ C_2 \int_{\{|u_n|\leq k\}} (c_i(x,t))^{\frac{(p_0-1)p_i'}{p_0-(\lambda+\delta)p_i'+\delta}} (1+|u_n|)\, dx\, dt \\ &\leq \int_{Q_T} \frac{\alpha_0 |D^i T_k(u_n)|^{p_i}}{2(1+|u_n|)^\delta}\, dx\, dt + \frac{1}{2N} \int_{Q_T} |T_k(u_n)|^{p_0}\, dx\, dt + C_3(1+k)\|c_i(x,t)\|^{r_i}_{L^{r_i}(Q_T)} + C_4, \end{aligned} \tag{4.19}$$

with $r_i \geq \frac{(p_0-1)p_i'}{p_0-(\lambda+\delta)p_i'+\delta}$. By combining (4.14)˘(4.19), we deduce that

$$\begin{aligned}\int_\Omega \varphi_k(u_n)(T)\,dx + \frac{1}{2}\int_{Q_T}|T_k(u_n)|^{p_0}\,dx\,dt + \frac{\alpha_0}{2(1+k)^\delta}\sum_{i=1}^N\int_{Q_T}|D^iT_k(u_n)|^{p_i}\,dx\,dt \\ \leq C_5k(1+\|f\|_{L^1(Q_T)}+\|u_0\|_{L^1(\Omega)}) \\ \leq C_6k \qquad \text{for any} \quad k\geq 1,\end{aligned} \tag{4.20}$$

where C_6 is a constant does not depend on n and k. Moreover, we deduce that

$$\begin{aligned}&\|T_k(u_n)\|^{\underline{p}}_{L^{p_0}(Q_T)} + \sum_{i=1}^N \|D^iT_k(u_n)\|^{\underline{p}}_{L^{p_i}(Q_T)} \\ &\leq \int_\Omega \varphi_k(u_n(T))\,dx + \int_{Q_T}|T_k(u_n)|^{p_0}\,dx\,dt + \sum_{i=1}^N\int_{Q_T}|D^iT_k(u_n)|^{p_i}\,dx\,dt + N+1 \\ &\leq C_7k^{1+\delta}.\end{aligned} \tag{4.21}$$

Thus, we get

$$\|T_k(u_n)\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} \leq C_6 k^{\frac{1+\delta}{\underline{p}}}, \tag{4.22}$$

where C_6 a constant that does not depend on n and k.
Now, we will show that $(u_n)_n$ is a Cauchy sequence in measure. Thanks to (4.20), we have

$$\begin{aligned}k^{p_0}\,\text{meas}\,\{u_n\geq k\} &= \int_{\{|u_n|>k\}}|T_k(u_n)|^{p_0}\,dx\,dt \\ &\leq \int_{Q_T}|T_k(u_n)|^{p_0}\,dx\,dt \\ &\leq 2C_6k.\end{aligned} \tag{4.23}$$

Since $1<p_0$, we conclude that

$$\text{meas}(\{|u_n|>k\}) \leq C_8k^{1-p_0} \longrightarrow 0 \qquad \text{as} \quad k\to\infty. \tag{4.24}$$

For all $\sigma>0$, we have

$$\begin{aligned}&\text{meas}\,\{|u_n-u_m|>\sigma\} \\ &\quad\leq \text{meas}\,\{|u_n|>k\} + \text{meas}\,\{|u_m|>k\} + \text{meas}\,\{|T_k(u_n)-T_k(u_m)|>\sigma\}.\end{aligned} \tag{4.25}$$

Let $\varepsilon>0$, using (4.24) we may choose $k=k(\varepsilon)$ large enough such that

$$\text{meas}\,\{|u_n|>k\} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\,\{|u_m|>k\} \leq \frac{\varepsilon}{3}. \tag{4.26}$$

Moreover, since $(T_k(u_n))$ is bounded sequence in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$, then there exists a measurable function $v_k \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ such that $T_k(u_n) \rightharpoonup v_k$ weakly in $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ and by the compact embedding $L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \hookrightarrow\hookrightarrow L^1(Q_T)$, we deduce that $T_k(u_n)\to v_k$ in $L^1(Q_T)$ and a.e. in Q_T.
Thus, we can assume that $(T_k(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure, and for all $k>0$ and $\sigma,\varepsilon>0$, there exists $n_0=n_0(k,\sigma,\varepsilon)$ such that

$$\text{meas}\,\{|T_k(u_n)-T_k(u_m)|>\sigma\} \leq \frac{\varepsilon}{3} \quad \text{for all } m,n\geq n_0(k,\sigma,\varepsilon). \tag{4.27}$$

By combining (4.25) – (4.27), we conclude that $\forall\sigma,\varepsilon>0$ there exists $n_0=n_0(\sigma,\varepsilon)$ such that

$$\text{meas}\ \{|u_n-u_m|>\sigma\} \leq \varepsilon \qquad \text{for any} \quad n,m\geq n_0(\sigma,\varepsilon).$$

It follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u. Consequently, we have

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)), \\ T_k(u_n) \to T_k(u) \text{ strongly in } L^1(Q_T) \text{ and a.e in } \Omega.\end{cases} \tag{4.28}$$

Step 3 : Some regularity results.

Let $h > 0$, by taking $\dfrac{T_h(u_n)}{h}$ as a test function for the approximate problem (4.2), we obtain

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, \frac{T_h(u_n)}{h} \rangle dt + \frac{1}{h}\sum_{i=1}^N \int_{\{|u_n|\le h\}} a_i(x,t,T_n(u_n),\nabla u_n) D^i u_n \, dx\, dt + \int_{Q_T} |u_n|^{p_0-1}\frac{|T_h(u_n)|}{h}\, dx\, dt$$
$$= \int_{Q_T} f_n \frac{T_h(u_n)}{h}\, dx\, dt + \frac{1}{h}\sum_{i=1}^N \int_{\{|u_n|\le h\}} \phi_{i,n}(x,t,u_n) D^i u_n \, dx dt. \tag{4.29}$$

On the one hand, we have

$$\begin{aligned}\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \frac{T_h(u_n)}{h}\right\rangle dt &= \frac{1}{h}\int_\Omega \int_0^T \frac{\partial \varphi_h(u_n)}{\partial t} dt dx \\ &= \frac{1}{h}\int_\Omega \varphi_h(u_n(T))\, dx - \frac{1}{h}\int_\Omega \varphi_h(u_n(0))\, dx \\ &= \frac{1}{h}\int_\Omega \varphi_h(u_n(T))\, dx - \frac{1}{h}\int_\Omega \varphi_h(u_{0,n})\, dx,\end{aligned}$$

for $n > h$, we obtain

$$\begin{aligned}\frac{1}{h}\int_\Omega \varphi_h(u_{0,n})\, dx &= \frac{1}{h}\int_{\{|u_{0,n}|\le h\}} \varphi_h(u_{0,n})\, dx \frac{1}{h}\int_{\{|u_{0,n}|>h\}} \varphi_h(u_{0,n})\, dx \\ &= \frac{1}{h}\int_{\{|u_0|\le h\}} \varphi_h(u_0)\, dx + \frac{1}{h}\int_{\{|u_{0,n}|>h\}} \varphi_h(u_{0,n})\, dx \\ &= \frac{1}{h}\int_{\{|u_0|\le h\}} \frac{u_0^2}{2}\, dx + \int_{\{|u_{0,n}|>h\}} |u_{0,n}| - \frac{h}{2}\, dx \\ &\le \int_\Omega u_0 \frac{T_h(u_0)}{h}\, dx + \int_{\{|u_{0,n}|>h\}} |u_{0,n}|\, dx,\end{aligned}$$

then

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \frac{T_h(u_n)}{h}\right\rangle dt \ge \frac{1}{h}\int_\Omega \varphi_h(u_n(T))\, dx - \int_\Omega u_0 \frac{T_h(u_0)}{h}\, dx - \int_{\{|u_{0,n}|>h\}} |u_{0,n}|\, dx. \tag{4.30}$$

Concerning the second term on the right-hand side of (4.29). We have $0 < \delta < \underline{p} - 1$ and $0 < \lambda < \dfrac{p_0 - \delta(p_i' - 1)}{p_i'}$, then by (3.5), (3.4) and Young's inequality, we get

$$\begin{aligned}&\left|\frac{1}{h}\int_{Q_T} \phi_{i,n}(x,t,u_n) D^i T_h(u_n)\, dx\, dt\right| \\ &\le \frac{1}{h}\int_{\{|u_n|\le h\}} c_i(x,t)(1+|u_n|)^\lambda |D^i u_n|\, dx\, dt \\ &\le \frac{1}{h}\int_{\{|u_n|\le h\}} \frac{\alpha_0 |D^i T_h(u_n)|^{p_i}}{2(1+|u_n|)^\delta}\, dx\, dt + \frac{C_1}{h}\int_{\{|u_n|\le k\}} (c_i(x,t))^{p_i'}(1+|u_n|)^{\lambda p_i' + \frac{\delta p_i'}{p_i}}\, dx\, dt \\ &\le \frac{1}{2h}\int_{\{|u_n|\le h\}} a_i(x,t,T_n(u_n),\nabla u_n) D^i u_n\, dx\, dt + \frac{1}{2Nh}\int_{\{|u_n|\le h\}} (1+|u_n|)^{p_0}\, dx\, dt \\ &\quad + \frac{C_2}{h}\int_{\{|u_n|\le h\}} (c_i(x,t))^{\frac{(p_0-1)p_i'}{p_0-(\lambda+\delta)p_i'+\delta}}(1+|u_n|)\, dx\, dt \\ &\le \frac{1}{2h}\int_{\{|u_n|\le h\}} a_i(x,t,T_n(u_n),\nabla u_n) D^i u_n\, dx\, dt + \frac{1}{2Nh}\int_{\{|u_n|\le h\}} |u_n|^{p_0}\, dx\, dt \\ &\quad + C_3\int_{Q_T} (c_i(x,t))^{r_i}\frac{|T_h(u_n)|}{h}\, dx\, dt + \frac{C_4}{h},\end{aligned} \tag{4.31}$$

with $r_i \geq \dfrac{(p_0-1)p_i'}{p_0-(\lambda+\delta)p_i'+\delta}$.

Since meas$\{|u_n|>h\} \to 0$, then $\dfrac{|T_h(u_n)|}{h} \rightharpoonup 0$ weak$-*$ in $L^\infty(Q_T)$, as $n, h \to \infty$. By combining (4.29) and (4.30) – (4.31), we obtain

$$\begin{aligned}&\frac{1}{2h}\sum_{i=1}^{N}\int_{\{|u_n|\leq h\}} a_i(x,t,T_n(u_n),\nabla u_n)D^i u_n\,dx\,dt + \frac{1}{2}\int_{Q_T}|u_n|^{p_0-1}\frac{|T_h(u_n)|}{h}\,dx\,dt\\&\leq \int_{Q_T}|f_n|\frac{|T_h(u_n)|}{h}\,dx\,dt + \int_\Omega |u_0|\frac{|T_h(u_0)|}{h}\,dx + \int_{\{|u_{0,n}|>h\}}|u_{0,n}|\,dx\\&\quad +C_3\int_{Q_T}|c_i(x,t)|^{r_i}\frac{|T_h(u_n)|}{h}\,dx\,dt + \frac{C_4}{h} \longrightarrow 0 \quad \text{as} \quad n,h\to\infty.\end{aligned} \tag{4.32}$$

We deduce that

$$\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\leq h\}} a_i(x,t,T_n(u_n),\nabla u_n)D^i u_n\,dx\,dt \longrightarrow 0 \quad \text{as} \quad h\to\infty, \tag{4.33}$$

$$\limsup_{n\to\infty}\frac{1}{h}\int_{\{|u_n|\leq h\}}|u_n|^{p_0}\,dx\,dt \longrightarrow 0 \quad \text{as} \quad h\to\infty, \tag{4.34}$$

and

$$\limsup_{n\to\infty}\int_{\{|u_n|> h\}}|u_n|^{p_0-1}\,dx\,dt \longrightarrow 0 \quad \text{as} \quad h\to\infty. \tag{4.35}$$

Moreover, in view of (4.31) we conclude that

$$\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{Q_T}\left|\phi_{i,n}(x,t,u_n)D^iT_h(u_n)\right|\,dx\,dt \longrightarrow 0 \quad \text{as} \quad h\to\infty. \tag{4.36}$$

Now, we prove that $u_n^{p_0-1} \longrightarrow u^{p_0-1}$ strongly in $L^1(Q_T)$ by using Vitali's theorem. In view of (4.35) we have, for any $\eta>0$, there exists $h(\eta)>0$ such that

$$\int_{\{|u_n|>h(\eta)\}}|u_n|^{p_0-1}\,dx\,dt \leq \frac{\eta}{2}. \tag{4.37}$$

On the other hand, for any measurable subset $E\subset Q_T$, there exists $\beta(\eta)>0$ such that

$$\int_E |T_{h(\eta)}(u_n)|^{p_0-1}\,dx\,dt \leq \frac{\eta}{2} \qquad \text{for} \quad meas(E)\leq\beta(\eta). \tag{4.38}$$

By combining (4.37) and (4.38), we conclude that, for any $\eta>0$ there exists $\beta(\eta)>0$ such that

$$\int_E |u_n|^{p_0-1}dx\,dt \leq \int_E |T_{h(\eta)}(u_n)|^{p_0-1}\,dx\,dt + \int_{\{|u_n|>h(\eta)\}}|u_n|^{p_0-1}\,dx\,dt \leq \eta \quad \text{for meas}(E)\leq\beta(\eta), \tag{4.39}$$

which implies that the sequence $(|u_n|^{p_0-2}u_n)_n$ is uniformly equi-integrable. Then, in view of Vitali's theorem, we deduce that

$$|u_n|^{p_0-2}u_n \longrightarrow |u|^{p_0-2}u \qquad \text{strongly in} \quad L^1(Q_T). \tag{4.40}$$

Step 4 : The weak convergence of $(S_h(u_n))_t$ in $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))+L^1(Q_T)$.

For $h>0$, let $S_h(\cdot)$ a function in $C^2(\mathbb{R})$, with supp$(S_h')\subset[-h,h]$ and $v\in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))$ $\cap L^\infty(Q_T)$.

By taking $S'_h(u_n)v$ as a test function for the approximate problem (4.2), we have

$$\begin{aligned} &\int_0^T \langle \frac{\partial u_n}{\partial t}, S'_h(u_n)v\rangle dt + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n)(S'_h(u_n)D^i v + S''_h(u_n)vD^i u_n)\ dx\, dt \\ &\quad + \int_{Q_T} |u_n|^{p_0-2}u_n S'_h(u_n)v\ dxdt \\ &= \int_{Q_T} f_n S'_h(u_n)v\ \ dx\, dt + \sum_{i=1}^N \int_{Q_T} \phi_{i,n}(x,t,u_n)(S'_h(u_n)D^i v + S''_h(u_n)vD^i u_n)\, dx\, dt. \end{aligned} \tag{4.41}$$

For the first term on the right-hand side of (4.41), we have

$$\begin{aligned} \Big|\int_0^T \langle \frac{\partial u_n}{\partial t}, S'_h(u_n)v\rangle dt\Big| &= \Big|\int_0^T\int_\Omega \frac{\partial S_h(u_n)}{\partial t} v\ dx\ dt\Big| \\ &\le \sum_{i=1}^N \int_{Q_T} |a_i(x,t,T_n(u_n),\nabla u_n)||S'_h(u_n)D^i v + S''_h(u_n)vD^i u_n|\ dx\ dt \\ &\quad + \sum_{i=1}^N \int_{Q_T} |\phi_{i,n}(x,t,u_n)||S'_h(u_n)D^i v + S''_h(u_n)vD^i u_n|\ dx\ dt \\ &\quad + \int_{Q_T} |u_n|^{p_0-1}|S'_h(u_n)v|\ dx\ dt + \int_{Q_T} |f_n||S'_h(u_n)v|\ dx\ dt. \end{aligned} \tag{4.42}$$

For the first term on the right-hand side of (4.42), we have

$$\begin{aligned} &\int_{Q_T} |a_i(x,t,T_n(u_n),\nabla u_n)||S'_h(u_n)D^i v + S''_h(u_n)vD^i u_n|\ dx\ dt \\ &\le \int_{\{|u_n|\le h\}} \beta(K_i(x,t) + |T_n(u_n)|^{p_i-1} + |D^i u_n|^{p_i-1}) \\ &\qquad \times (|S'_h(u_n)\,\|D^i v| + |S''_h(u_n)\|\, v\|D^i u_n|)\ dx\ dt \\ &\le 3\beta(\|K_i(x,t)\|_{L^{p_i}(Q_T)} + \|T_h(u_n)\|^{p_i-1}_{L^{p_i}(Q_T)} + \left\|D^i T_h(u_n)\right\|^{p_i-1}_{L^{p_i}(Q_T)}) \\ &\qquad \times (\|S'_h(\cdot)\|_{L^\infty(\mathbb{R})}\left\|D^i v\right\|_{L^{p_i}(Q_T)} + \|S''_h(\cdot)\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(Q_T)}\left\|D^i T_h(u_n)\right\|_{L^{p_i}(Q_T)}) \\ &\le C_0(\|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} + \|v\|_{L^\infty(Q_T)}), \end{aligned} \tag{4.43}$$

with C_0 is a constant that doesn't depends on n. Concerning the second term on the right-hand side of (4.42), we have $r_i \ge p'_i$, then by Holder's inequality, we get

$$\begin{aligned} &\int_{Q_T} |\phi_{i,n}(x,t,u_n)||S'_h(u_n)D^i v + S''_h(u_n)vD^i u_n|\ dx\ dt \\ &\le \int_{\{|u_n|\le h\}} c_i(x,t)(1+|u_n|)^\lambda \times (|S'_h(u_n)\,\|D^i v| + |S''_h(u_n)\|\, v\|D^i u_n|)\ dx\ dt \\ &\le \|c_i(x,t)\|_{L^{\frac{p'_i(p_0-1)}{p_0-1-\lambda p'_i}}(Q_T)} \|(1+|u_n|)^{p_0-1}\|^{\frac{\lambda}{p_0-1}}_{L^1(Q_T)} \\ &\times \Big(\|S'_h(\cdot)\|_{L^\infty(\mathbb{R})}\left\|D^i v\right\|_{L^{p_i}(Q_T)} + \|S''_h(\cdot)\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(Q_T)}\left\|D^i T_h(u_n)\right\|_{L^{p_i}(Q_T)}) \\ &\le C_1(\|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} + \|v\|_{L^\infty(Q_T)}). \end{aligned} \tag{4.44}$$

For the other terms in (4.42), we have

$$\begin{aligned}
&\int_{Q_T} |u_n|^{p_0-1}|S_h'(u_n)v|\,dx\,dt + \int_{Q_T} |f_n||S_h'(u_n)v|\,dx\,dt \\
&\le \int_{Q_T} |u_n|^{p_0-1}|S_h'(u_n)v|\,dx\,dt + \int_{Q_T} |f_n||S_h'(u_n)v|\,dx\,dt \\
&\le \|T_h(u_n)\|_{L^{p_0}(Q_T)}^{p_0-1}\|S_h'(\cdot)\|_{L^\infty(\mathbb{R})}\|v\|_{L^{p_0}(Q_T)} + \|f\|_{L^1(Q_T)}\|S_h'(\cdot)\|_{L^\infty(R)}\|v\|_{L^\infty(Q_T)} \\
&\le C_2(\|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} + \|v\|_{L^\infty(Q_T)}).
\end{aligned} \tag{4.45}$$

Using (4.42) – (4.45), we concludes that

$$\left|\int_0^T \langle \frac{\partial S_h(u_n)}{\partial t}, v\rangle dt\right| \le C_3\left(\|v\|_{L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega))} + \|v\|_{L^\infty(Q_T)}\right), \tag{4.46}$$

for any $v \in L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \cap L^\infty(Q_T)$, with C_3 is a constant that does not depend on n. Hence $\left(\frac{\partial S_h(u_n)}{\partial t}\right)_n$ is bounded in $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega)) + L^1(Q_T)$ and

$$\frac{\partial S_h(u_n)}{\partial t} \rightharpoonup \frac{\partial S_h(u)}{\partial t} \quad \text{in} \quad L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega)) + L^1(Q_T). \tag{4.47}$$

Step 5: Convergence of the gradient.

Let $0 < k < h < n$. By taking $S_h(\cdot)$ be an increasing function of $C^2(\mathbb{R})$, such that $S_h(s) = s$ for $|s| \le k$ and $\text{supp}(S_h') \subset [-h,h]$.
By taking $S_h'(u_n)(T_k(u_n) - (T_k(u))_\mu)$ as a test function in (4.2), we obtain

$$J^1_{n,\mu,h} + J^2_{n,\mu,h} + J^3_{n,\mu,h} + J^4_{n,\mu,h} = J^5_{n,\mu,h} + J^6_{n,\mu,h} + J^7_{n,\mu,h}; \tag{4.48}$$

where

$$\begin{aligned}
J^1_{n,\mu,h} &= \int_0^T \langle \frac{\partial u_n}{\partial t}, S_h'(u_n)(T_k(u_n) - (T_k(u))_\mu)\rangle\,dt, \\
J^2_{n,\mu,h} &= \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n)S_h'(u_n)D^i(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt, \\
J^3_{n,\mu,h} &= \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n)S_h''(u_n)D^i u_n(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt, \\
J^4_{n,\mu,h} &= \int_{Q_T} |u_n|^{p_0-2}u_n S_h'(u_n)(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt, \\
J^5_{n,\mu,h} &= \int_{Q_T} f_n S_h'(u_n)(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt, \\
J^6_{n,\mu,h} &= \sum_{i=1}^N \int_{Q_T} \phi_{i,n}(x,t,u_n)S_h'(u_n)D^i(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt, \\
J^7_{n,\mu,h} &= \sum_{i=1}^N \int_{Q_T} \phi_{i,n}(x,t,u_n)S_h''(u_n)D^i u_n(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt.
\end{aligned} \tag{4.49}$$

For the first term $J^1_{n,\mu,h}$, we have

$$\begin{aligned} J^1_{n,\mu,h} &= \int_{Q_T} \frac{\partial S_h(u_n)}{\partial t}(T_k(u_n)-(T_k(u))_\mu)\,dx\,dt \\ &= \int_{Q_T} \frac{\partial (S_h(u_n)-T_k(u_n))}{\partial t}(T_k(u_n)-(T_k(u))_\mu)\,dx\,dt + \int_{Q_T} \frac{\partial T_k(u_n)}{\partial t}(T_k(u_n)-(T_k(u))_\mu)\,dx\,dt \\ &= \Big[\int_\Omega (S_h(u_n)-T_k(u_n))(T_k(u_n)-(T_k(u))_\mu)dx\Big]_0^T \\ &\quad - \int_{Q_T} (S_h(u_n)-T_k(u_n))(\frac{\partial T_k(u_n)}{\partial t} - \frac{\partial (T_k(u))_\mu}{\partial t})\,dx\,dt \\ &\quad + \int_{Q_T} \frac{\partial T_k(u_n)}{\partial t}(T_k(u_n)-(T_k(u))_\mu)\,dx\,dt \\ &= I_1+I_2+I_3. \end{aligned} \tag{4.50}$$

Concerning the first term on the right hand side of (4.50), we have $S_h(u_n)=T_k(u_n)=u_n$ on the set $\{|u_n|\le k\}$, and since $|S_h(u_n)|\ge |T_k(u_n)|$ on the set $\{|u_n|>k\}$. Moreover, $S_h(u_n)$ and $T_k(u_n)$ have the same sign of u_n, then

$$\begin{aligned} I_1 &= \left[\int_{\{|u_n|>k\}} (S_h(u_n)-T_k(u_n))(T_k(u_n)-(T_k(u))_\mu)dx\right]_0^T \\ &\ge -\int_{\{|u_{0,n}|>k\}} (S_h(u_{0,n})-T_k(u_{0,n}))(T_k(u_{0,n})-(T_k(u_0))_\mu)dx. \end{aligned}$$

Since $(T_k(u_0))_\mu = T_k(u_0)$, we deduce that $I_1\ge \varepsilon_1(n)$, with

$$\varepsilon_1(n) = -\int_{\{|u_{0,n}|>k\}} (S_h(u_{0,n})-T_k(u_{0,n}))(T_k(u_{0,n})-T_k(u_0))dx \longrightarrow 0 \quad \text{as} \quad n\to\infty. \tag{4.51}$$

For the second term on the right-hand side of (4.50), we have $(S_h(u_n)-T_k(u_n))\dfrac{\partial T_k(u_n)}{\partial t}=0.$ Hence

$$\begin{aligned} I_2 &= \int_{\{|u_n|>k\}} (S_h(u_n)-T_k(u_n))\frac{\partial (T_k(u))_\mu}{\partial t}\,dx\,dt \\ &=\mu \int_{\{\{|u_n|>k\}} (S_h(u_n)-T_k(u_n))(T_k(u)-(T_k(u))_\mu)\,dx\,dt \\ &=\mu \int_{\{|u_n|>k\}} (S_h(u_n)-T_k(u_n))(T_k(u)-T_k(u_n))\,dx\,dt \\ &\quad + \mu \int_{\{|u_n|>k\}} (S_h(u_n)-T_k(u_n))(T_k(u_n)-(T_k(u))_\mu)\,dx\,dt \\ &\ge\mu \int_{\{|u_n|>k\}} (S_h(u_n)-T_k(u_n))(T_k(u)-T_k(u_n))\,dx\,dt \longrightarrow 0 \quad \text{as} \quad n\to\infty. \end{aligned} \tag{4.52}$$

It follows that $I_2 \geq \varepsilon_2(n)$. Concerning the last term I_3, we have

$$\begin{aligned} I_3 &= \int_{Q_T} \frac{\partial(T_k(u_n) - (T_k(u))_\mu)}{\partial t}(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt + \int_{Q_T} \frac{\partial(T_k(u))_\mu}{\partial t}(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\ &= \left[\frac{1}{2}\int_\Omega (T_k(u_n) - (T_k(u))_\mu)^2 dx\right]_0^T + \mu\int_{Q_T}(T_k(u) - (T_k(u))_\mu)(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\ &\geq -\frac{1}{2}\int_\Omega (T_k(u_{0,n}) - T_k(u_0))^2 dx + \mu\int_{Q_T}(T_k(u) - (T_k(u))_\mu)(T_k(u_n) - (T_k(u))_\mu)\,dx\,dt \\ &\geq \varepsilon_3(n) + \mu\int_{Q_T}(T_k(u) - (T_k(u))_\mu)^2\,dx\,dt \\ &\geq \varepsilon_3(n). \end{aligned} \tag{4.53}$$

By combining (4.50) and (4.51) – (4.53), we conclude that

$$\liminf_{n\to\infty} J^1_{n,\mu,h} \geq 0. \tag{4.54}$$

The second term of (4.49), we have $S'_h(s) \geq 0$ and $S'_h(s) = 1$ for $|s| \leq k$, with $\mathrm{supp}(S'_h) \subset [-h, h]$, then

$$\begin{aligned} J^2_{n,\mu,h} &= \sum_{i=1}^N \int_{\{|u_n|\leq k\}} a_i(x,t,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n) - D^i(T_k(u))_\mu)\,dx\,dt \\ &\quad - \sum_{i=1}^N \int_{\{k<|u_n|\leq h\}} S'_h(u_n)a_i(x,t,T_h(u_n),\nabla T_h(u_n))D^i(T_k(u))_\mu\,dx\,dt \\ &\geq \sum_{i=1}^N \int_{Q_T} (a_i(x,t,T_k(u_n),\nabla T_k(u_n)) - a_i(x,t,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n) - D^iT_k(u))\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u) - D^i(T_k(u))_\mu)\,dx\,dt \\ &\quad + \sum_{i=1}^N \int_{Q_T} a_i(x,t,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n) - D^iT_k(u))\,dx\,dt \\ &\quad - \sum_{i=1}^N \|S'_h(\cdot)\|_{L^\infty(\mathbb{R})}\int_{\{k<|u_n|\leq h\}} |a_i(x,t,T_h(u_n),\nabla T_h(u_n))||D^i(T_k(u))_\mu|\,dx\,dt. \end{aligned} \tag{4.55}$$

For second term of the right-hand side of (4.55), we have $(a_i(x,t,T_h(u_n),\nabla T_h(u_n)))_n$ is bounded in $L^{p'_i}(Q_T)$, then there exists a measurable function $\eta_{i,k} \in L^{p'_i}(Q_T)$ such that $a_i(x,t,T_h(u_n),\nabla T_h(u_n)) \rightharpoonup \eta_{i,k}$ in $L^{p'_i}(Q_T)$. and we have $D^i(T_k(u))_\mu \to D^iT_k(u)$ strongly in $L^{p_i}(Q_T)$, we conclude that

$$\begin{aligned} \lim_{n\to\infty}\int_{Q_T} & a_i(x,t,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u) - D^i(T_k(u))_\mu)\,dx\,dt \\ &= \int_{Q_T} \eta_{i,k}(D^iT_k(u) - D^i(T_k(u))_\mu)\,dx\,dt \longrightarrow 0 \qquad \text{as} \quad \mu\to\infty. \end{aligned} \tag{4.56}$$

Similarly, we show that

$$\begin{aligned} \lim_{n\to\infty}\int_{\{k<|u_n|\leq h\}} & |a_i(x,t,T_h(u_n),\nabla T_h(u_n))||D^iT_k(u)|\,dx\,dt \\ &= \int_{\{k<|u|\leq h\}} |\eta_{i,h}|\,|D^i(T_k(u))_\mu|\,dx\,dt \longrightarrow 0 \qquad \text{as} \quad \mu\to\infty. \end{aligned} \tag{4.57}$$

Concerning the third term on the right-hand side of (4.55). We have $a_i(x,t,T_k(u_n),\nabla T_k(u)) \to a_i(x,t,T_k(u),\nabla T_k(u))$ strongly in $L^{p'_i}(Q_T)$. and since $D^iT_k(u_n) \rightharpoonup D^iT_k(u)$ weakly in

$L^{p_i}(Q_T)$, it follows that

$$\int_{Q_T} a_i(x,t,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\,dx\,dt \longrightarrow 0 \qquad \text{as} \quad n\to\infty. \tag{4.58}$$

By combining (4.55) and (4.56) – (4.58), we deduce that

$$\begin{aligned} J^2_{n,\mu,h} \geq \sum_{i=1}^{N}\int_{Q_T} &(a_i(x,t,T_k(u_n),\nabla T_k(u_n)) - a_i(x,t,T_k(u_n),\nabla T_k(u))) \\ &\times (D^iT_k(u_n)-D^iT_k(u))\,dx\,dt + \varepsilon_2(n,\mu). \end{aligned} \tag{4.59}$$

The third term of (4.49), we have $\operatorname{supp}(S''_h) \subset [-h,h]$, we assume that $\|S''_h(\cdot)\|_{L^\infty(\mathbb{R})} \leq \dfrac{1}{2h}$. Thus, in view of Hölder's inequality, we have

$$\begin{aligned} |J^3_{n,\mu,h}| &\leq \|S''_h(\cdot)\|_{L^\infty(\mathbb{R})}\sum_{i=1}^{N}\int_{\{|u_n|\leq h\}} a_i(x,t,T_h(u_n),\nabla T_h(u_n))D^iT_h(u_n)|T_k(u_n)-(T_k(u))_\mu|\,dx\,dt \\ &\leq \frac{k}{h}\sum_{i=1}^{N}\int_{\{|u_n|\leq h\}} a_i(x,t,T_h(u_n),\nabla T_h(u_n))D^iT_h(u_n)\,dx\,dt \longrightarrow 0 \qquad \text{as} \quad n,h\to\infty. \end{aligned}$$

Thanks to (4.33), we get

$$|J^3_{n,\mu,h}| \leq \varepsilon_3(n,h). \tag{4.60}$$

The fourth and fifth terms of (4.49). We have $|u_n|^{p_0-2}u_n \to |u|^{p_0-2}u$ strongly in $L^1(Q_T)$, and since $T_k(u_n)-(T_k(u))_\mu \rightharpoonup 0$ weak$-*$ in $L^\infty(Q_T)$ as n and μ tend to infinity, then

$$\begin{aligned} |J^4_{n,\mu,h}| &= \Big|\int_{Q_T} |u_n|^{p_0-2}u_nS'_h(u_n)(T_k(u_n)-(T_k(u))_\mu)\,dx\,dt\Big| \\ &\leq \|S'_h(\cdot)\|_{L^\infty(\mathbb{R})}\int_{Q_T}|u_n|^{p_0-1}|T_k(u_n)-(T_k(u))_\mu|\,dx\,dt = \varepsilon_4(n,\mu)\to 0 \quad \text{as} \ n,\mu\to\infty. \end{aligned} \tag{4.61}$$

Similarly, we have $f_n \to f$ strongly in $L^1(Q_T)$, then

$$|J^5_{n,\mu,h}| \leq \|S'_h(\cdot)\|_{L^\infty(\mathbb{R})}\int_{Q_T}|f_n||T_k(u_n)-(T_k(u))_\mu|dxdt = \varepsilon_5(n,\mu) \longrightarrow 0 \qquad \text{as} \quad n,\mu\to\infty. \tag{4.62}$$

For The sixth term of (4.49). We have $c_i(x,t)\in L^{p'_i}(Q_T)$ and since $D^iT_k(u_n)-D^i(T_k(u))_\mu \rightharpoonup 0$ weakly in $L^{p_i}(Q_T)$, as $n,\mu\longrightarrow\infty$, then

$$\begin{aligned} |J^6_{n,\mu,h}| &\leq \sum_{i=1}^{N}\int_{Q_T}|\phi_{i,n}(x,t,u_n)|S'_h(u_n)|D^iT_k(u_n)-D^i(T_k(u))_\mu|\,dx\,dt \\ &\leq \|S'_h(\cdot)\|_{L^\infty(\mathbb{R})}\sum_{i=1}^{N}\int_{Q_T} c_i(x,t)(1+|T_h(u_n)|)^\lambda S'_h(u_n)|D^iT_k(u_n)-D^i(T_k(u))_\mu|\,dx\,dt \\ &= \varepsilon_6(n,\mu) \longrightarrow 0 \qquad \text{as} \quad n,\mu\to\infty. \end{aligned} \tag{4.63}$$

Concerning the last term of (4.49). Thanks to (4.36), we have

$$\begin{aligned} |J^7_{n,h}| &\leq \sum_{i=1}^{N}\int_{Q_T}|\phi_{i,n}(x,t,u_n)||S''_h(u_n)|D^iT_h(u_n)|T_k(u_n)-(T_k(u))_\mu|\,dx\,dt \\ &\leq \frac{k}{h}\sum_{i=1}^{N}\int_{Q_T}|\phi_{i,n}(x,t,u_n)|\,|D^iT_h(u_n)|\,dx\,dt = \varepsilon_7(n,h) \longrightarrow 0 \qquad \text{as} \quad n,h\to\infty. \end{aligned} \tag{4.64}$$

By Combining (4.48), (4.54), and (4.59) – (4.64), we deduce that

$$0 \le \sum_{i=1}^{N} \int_{Q_T} (a_i(x,t,T_k(u_n),\nabla T_k(u_n)) - a_i(x,t,T_k(u_n),\nabla T_k(u)))(D^i T_k(u_n) - D^i T_k(u))\,dx\,dt \\ \le \varepsilon_8(n,\mu,h). \tag{4.65}$$

Since $T_k(u_n) \longrightarrow T_k(u)$ strongly in $L^{p_0}(Q_T)$. Thus, by letting $n,\ \mu$ and h tend to infinity, we conclude that

$$\lim_{n\to\infty}\Big(\sum_{i=1}^{N} \int_{Q_T} (a_i(x,t,T_k(u_n),\nabla T_k(u_n)) - a_i(x,t,T_k(u_n),\nabla T_k(u)))(D^i T_k(u_n) - D^i T_k(u))\,dx\,dt \\ + \int_{Q_T} (|T_k(u_n)|^{p_0-2}T_k(u_n) - |T_k(u)|^{p_0-2}T_k(u))(T_k(u_n) - T_k(u))\,dx\,dt\Big) = 0. \tag{4.66}$$

Consequently, in view of Lemma 3.1, we deduce that

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{in} \quad L^{\vec{p}}(0,T;W_0^{1,\vec{p}}(\Omega)) \quad \forall k > 0. \tag{4.67}$$

Therefore, $\nabla u_n \longrightarrow \nabla u$ a.e in Q_T.

Step 6 :The convergence of $(u_n)_n$ in $C(0,T;L^1(\Omega))$.

Let $0 < s \le T$, by taking $T_1(u_n - (T_h(u))_\mu)\cdot\chi_{[0,s]}(t)$ as a test function for the approximate problem (3.6), we have

$$\int_\Omega\int_0^s \frac{\partial u_n}{\partial t} T_1(u_n - (T_h(u))_\mu)\,dx\,dt + \sum_{i=1}^{N}\int_0^s\int_\Omega a_i(x,t,T_n(u_n),\nabla u_n)D^iT_1(u_n - (T_h(u))_\mu)\,dx\,dt \\ + \int_0^s\int_\Omega |u_n|^{p_0-2}\,u_n T_1(u_n - (T_h(u))_\mu)\,dx\,dt = \int_0^s\int_\Omega f_n T_1(u_n - (T_h(u))_\mu)\,dx\,dt \\ + \sum_{i=1}^{N}\int_0^s\int_\Omega \phi_{i,n}(x,t,u_n)D^iT_1(u_n - (T_h(u))_\mu)\,dx\,dt.$$

We have $\{|u_n - (T_h(u)| \le 1\} \subset \{|u_n| \le h+1\}$, then

$$\int_\Omega\int_0^s \frac{\partial(u_n - (T_h(u))_\mu)}{\partial t} T_1(u_n - (T_h(u))_\mu)\,dx\,dt + \int_\Omega\int_0^s \frac{\partial(T_h(u))_\mu}{\partial t} T_1(u_n - (T_h(u))_\mu)\,dx\,dt \\ + \sum_{i=1}^{N}\int_0^s\int_{\{|u_n|\le h+1\}} a_i(x,t,T_n(u_n),\nabla u_n)D^iT_1(u_n - (T_h(u))_\mu)\,dx\,dt \\ + \int_0^s\int_\Omega |u_n|^{p_0-2}\,u_n T_1(u_n - (T_h(u))_\mu)\,dx\,dt = \int_0^s\int_\Omega f_n T_1(u_n - (T_h(u))_\mu)\,dx\,dt \\ + \sum_{i=1}^{N}\int_0^s\int_{\{|u_n|\le h+1\}} \phi_{i,n}(x,t,u_n)D^iT_1(u_n - (T_h(u))_\mu)\,dx\,dt. \tag{4.68}$$

For the two firsts terms on the left-hand side of (4.68), we have

$$\int_\Omega\int_0^s \frac{\partial(u_n - (T_h(u))_\mu)}{\partial t} T_1(u_n - (T_h(u))_\mu)\,dx\,dt \\ = \int_\Omega\int_0^s \frac{\partial\varphi_1(u_n - (T_h(u))_\mu)}{\partial t}\,dx\,dt \\ = \int_\Omega \varphi_1(u_n(s) - (T_h(u(s)))_\mu)\,dx - \int_\Omega \varphi_1(u_{0,n} - (T_h(u_0))_\mu)\,dx \\ = \int_\Omega \varphi_1(u_n(s) - (T_h(u(s)))_\mu)\,dx - \int_\Omega \varphi_1(u_{0,n} - T_h(u_0))\,dx, \tag{4.69}$$

and

$$\begin{aligned}&\int_\Omega\int_0^s \frac{\partial (T_h(u))_\mu}{\partial t} T_1(u_n-(T_h(u))_\mu)\,dx\,dt\\&=\mu\int_\Omega\int_0^s (T_h(u)-(T_h(u))_\mu)T_1(u_n-(T_h(u))_\mu)\,dx\,dt\\&\longrightarrow \mu\int_\Omega\int_0^s (T_h(u)-(T_h(u))_\mu)T_1(u-(T_h(u))_\mu)\,dx\,dt\geq 0 \quad \text{as} \quad n\to\infty.\end{aligned}\tag{4.70}$$

Concerning the third term on the left-hand side and second term on the right-hand side of (4.68). In view of (3.2), (3.5) and (4.67) we have $a_i(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n)) \rightharpoonup a_i(x,t,T_{h+1}(u),\nabla T_{h+1}(u))$, and $\phi_{i,n}(x,t,T_{h+1}(u_n)) \longrightarrow \phi_i(x,t,T_{h+1}(u))$, in $L^{p_i'}(Q_T)$, and since $T_1(u_n-(T_h(u))_\mu) \longrightarrow T_1(u-(T_h(u)))$, strongly in $L^{p_i}(Q_T)$ as n and μ tends to infinity, we deduce that

$$\begin{aligned}&\sum_{i=1}^N\int_0^s\int_{\{|u|\leq h+1\}} a_i(x,t,T_n(u_n),\nabla u_n)D^iT_1(u_n-(T_h(u))_\mu)\,dx\,dt\\&\longrightarrow \sum_{i=1}^N\int_0^s\int_{\{h\leq|u|\leq h+1\}} a_i(x,t,u,\nabla u)D^iu\,dx\,dt \quad \text{as} \quad n,\mu\to\infty,\end{aligned}\tag{4.71}$$

and

$$\begin{aligned}&\sum_{i=1}^N\int_0^s\int_{\{|u|\leq h+1\}} \phi_{i,n}(x,t,u_n)D^iT_1(u_n-(T_h(u))_\mu)\,dx\,dt\\&\longrightarrow \sum_{i=1}^N\int_0^s\int_{\{h\leq|u|\leq h+1\}} \phi_i(x,t,u)D^iu\,dx\,dt \quad \text{as} \quad n,\mu\to\infty.\end{aligned}\tag{4.72}$$

On other hand, thanks to (3.3), (3.5) and Young's inequality, by taking h large enough, we obtain

$$\begin{aligned}&\left|\int_0^s\int_{\{h\leq|u|\leq h+1\}}\phi_i(x,t,u)D^iu\,dxdt\right|\\&\leq \int_0^s\int_{\{h\leq|u|\leq h+1\}} c_i(x,t)(1+|u|)^\lambda|D^iu|\,dxdt\\&\leq \int_0^s\int_{\{h\leq|u|\leq h+1\}} \frac{\alpha_0|D^iT_h(u)|^{p_i}}{2(1+|u|)^\delta}\,dx\,dt + C_1\int_0^s\int_{\{h\leq|u|\leq h+1\}} (c_i(x,t))^{p_i'}|u|^{\lambda p_i'+\frac{\delta p_i'}{p_i}}\,dx\,dt\\&\leq \frac12\int_0^s\int_{\{h\leq|u|\leq h+1\}} a_i(x,t,u,\nabla u)D^iu\,dx\,dt + \int_{\{h\leq|u|\leq h+1\}}|u|^{p_0}\,dx\,dt\\&\quad +C_2\int_{\{h\leq|u|\leq h+1\}} (c_i(x,t))^{\frac{(p_0-1)p_i'}{p_0-(\lambda+\delta)p_i'+\delta}}\,dx\,dt\\&\leq \frac12\int_0^s\int_{\{h\leq|u|\leq h+1\}} a_i(x,t,u,\nabla u)D^iu\,dx\,dt + \int_{\{h\leq|u|\leq h+1\}}|u|^{p_0}\,dx\,dt\\&\quad +C_3\int_{\{h\leq|u|\leq h+1\}} (c_i(x,t))^{r_i}\,dx\,dt.\end{aligned}$$

Since $\text{meas}\{h\leq|u|\leq h+1\}\longrightarrow 0 \quad \text{as} \quad h\to\infty$, and $c_i(x,t)\in L^{r_i}(Q_T)$, in view of (4.34), we deduce that

$$\int_{\{h\leq|u|\leq h+1\}}|u|^{p_0}\,dx\,dt + C_3\int_{\{h\leq|u|\leq h+1\}}(c_i(x,t))^{r_i}\,dx\,dt=\varepsilon(h)\longrightarrow 0,\ \text{as } h\to\infty.$$

It follows that

$$\left|\int_0^s\int_{\{h\leq|u|\leq h+1\}}\phi_i(x,t,u)D^iu\,dxdt\right| \leq \frac12\int_0^s\int_{\{h\leq|u|\leq h+1\}} a_i(x,t,u,\nabla u)D^iu\,dx\,dt+\varepsilon(h).\tag{4.73}$$

Moreover, we have $T_1(u_n-(T_h(u))_\mu)\rightharpoonup T_1(u-(T_h(u)))$ and $T_1(u-(T_h(u)))\rightharpoonup 0$ weak$-*$ in $L^\infty(Q_T)$ as n, μ and h tends to infinity. In view of (4.40), we conclude that

$$\begin{aligned}\int_0^s\int_\Omega |u_n|^{p_0-2}u_nT_1(u_n-(T_h(u))_\mu)\,dx\,dt \quad &\longrightarrow \int_0^s\int_\Omega|u|^{p_0-2}uT_1(u-T_h(u))\,dx\,dt \quad \text{as} \quad n,\mu\to\infty\\&\longrightarrow 0 \quad \text{as} \quad h\to\infty,\end{aligned}\tag{4.74}$$

and

$$\int_0^s\int_\Omega f_n T_1(u_n-(T_h(u))_\mu)\,dx\,dt \quad \longrightarrow \int_0^s\int_\Omega fT_1(u-T_h(u))\,dx\,dt \quad \text{as} \quad n,\mu\to\infty$$
$$\longrightarrow 0 \quad \text{as} \quad h\to\infty. \tag{4.75}$$

By combining (4.68) – (4.75) we conclude that

$$0\le \int_\Omega \varphi_1(u_n(s)-(T_h(u(s)))_\mu)\,dx \le \int_\Omega \varphi_1(u_0-T_h(u_0))\,dx+\varepsilon(n,\mu,h).$$

And since u_0 belongs to $L^1(\Omega)$, then

$$\int_\Omega \varphi_1(u_n(s)-(T_h(u(s)))_\mu)\,dx \longrightarrow 0 \quad \text{as} \quad n,\mu,h\to\infty.$$

On other hand, thanks to Hölder's inequality, we have

$$\int_{\{|u_n(s)-(T_h(u(s)))_\mu|\le 1\}} |u_n(s)-(T_h(u(s)))_\mu|\ dx$$
$$\le meas(\Omega)^{\frac12}\left(\int_{\{|u_n(s)-(T_h(u(s)))_\mu|\le 1\}} (u_n(s)-(T_h(u(s)))_\mu)^2\ dx\right)^{\frac12}.$$

Having in mind that

$$\int_{\{|u_n(s)-(T_h(u(s)))_\mu|\le 1\}} |u_n(s)-(T_h(u(s)))_\mu|^2\ dx+\int_{\{|u_n(s)-(T_h(u(s)))_\mu|>1\}} |u_n(s)-(T_h(u(s)))_\mu|\ dx$$
$$\le 2\int_\Omega \varphi_1(u_n(s)-(T_h(u(s)))_\mu)\,dx,$$

we deduce that

$$\int_\Omega |u_n(s)-(T_h(u(s)))_\mu|\ dx \longrightarrow 0 \quad \text{as} \quad n,\mu,h\to\infty. \tag{4.76}$$

Thus, thanks to (4.76), for any n and m in $I\!N^*$, we have

$$\int_\Omega |u_n(s)-u_m(s)|\,dx \le \int_\Omega |u_n(s)-(T_h(u(s)))_\mu|\,dx+\int_\Omega |u_m(s)-(T_h(u(s)))_\mu|\,dx \longrightarrow 0,$$

as $n,\ m,\ \mu$ and h tends to infinity. Thus, the sequence $(u_n)_n$ is a Cauchy sequence in $C([0,T];L^1(\Omega))$, thus $u\in C([0,T];L^1(\Omega))$, hence $u_n(s)\to u(s)$ in $L^1(\Omega)$, for any $0\le s\le T$.

Step 7 : Passage to the limit.

Let $\varphi\in L^{\vec p}(0,T;W_0^{1,\vec p}(\Omega))\cap L^\infty(Q_T)$, and $S(\cdot)\in C^2(\mathbb{R})$, with supp $S'(\cdot)\subset[-M,M]$ for some $M>0$. By taking $S'(u_n)\varphi$ as a test function for the approximate problem (4.2), we obtain

$$\int_0^T\langle\frac{\partial u_n}{\partial t},S'(u_n)\varphi\rangle dt+\sum_{i=1}^N\int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n)D^i(S'(u_n)\varphi)\,dx\,dt+\int_{Q_T}|u_n|^{p_0-2}u_nS'(u_n)\varphi\,dxdt$$
$$=\int_{Q_T} f_nS'(u_n)\varphi\ \,dx\,dt+\sum_{i=1}^N\int_{Q_T}\phi_{i,n}(x,t,u_n)D^i(S'(u_n)\varphi)\ \,dx\,dt. \tag{4.77}$$

Firstly, in view of (4.47), we have $\frac{\partial S(u_n)}{\partial t}\rightharpoonup\frac{\partial S(u)}{\partial t}$ weakly in $L^{\vec{p}'}(0,T;W^{-1,\vec{p}'}(\Omega))+L^1(Q_T)$, then

$$\lim_{n\to\infty}\int_0^T\langle\frac{\partial u_n}{\partial t},S'(u_n)\varphi\rangle dt=\lim_{n\to\infty}\int_{Q_T}\frac{\partial S(u_n)}{\partial t}\varphi\ \,dx\,dt=\int_{Q_T}\frac{\partial S(u)}{\partial t}\varphi\ \,dx\,dt. \tag{4.78}$$

Concerning the second term on the left-hand side of (4.77), we have

$$\begin{aligned}
&\int_{Q_T} a_i(x,t,T_n(u_n),\nabla u_n) D^i(S'(u_n)\varphi)\,dx\,dt \\
&= \int_{Q_T} a_i(x,t,T_M(u_n),\nabla T_M(u_n))(S''(u_n)\varphi D^i T_M(u_n) + S'(u_n)D^i\varphi)\,dx\,dt.
\end{aligned}$$

In view of (3.2) we have $(a_i(x,t,T_M(u_n),\nabla T_M(u_n)))_n$ is bounded in $L^{p_i'}(Q_T)$, and $a_i(x,t, T_M(u_n),\nabla T_M(u_n)) \to a_i(x,t,T_M(u),\nabla T_M(u))$ a.e. in Q_T. Thus, we obtain

$$a_i(x,t,T_M(u_n),\nabla T_M(u_n)) \rightharpoonup a_i(x,t,T_M(u),\nabla T_M(u)) \quad \text{in } L^{p_i'}(Q_T),$$

and since

$$\phi_{i,n}(x,t,T_M(u_n)) \longrightarrow \phi_i(x,t,T_M(u)) \quad \text{in } L^{p_i'}(Q_T),$$

and

$$S''(u_n)\varphi D^i T_M(u_n) + S'(u_n)D^i\varphi \longrightarrow S''(u)\varphi D^i T_M(u) + S'(u)D^i\varphi \quad \text{strongly in } L^{p_i}(Q_T),$$

we conclude that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{Q_T} a_i(x,t,T_M(u_n),\nabla T_M(u_n))(S''(u_n)\varphi D^i T_M(u_n) + S'(u_n)D^i\varphi)\,dx\,dt \\
&= \int_{Q_T} a_i(x,t,T_M(u),\nabla T_M(u))(S''(u)\varphi D^i T_M(u) + S'(u)D^i\varphi)\,dx\,dt \qquad (4.79)\\
&= \int_{Q_T} a_i(x,t,u,\nabla u)(S''(u)\varphi D^i u + S'(u)D^i\varphi)\,dx\,dt.
\end{aligned}$$

Similarly, we have $\phi_{i,n}(x,t,T_M(u_n)) \longrightarrow \phi_i(x,t,T_M(u))$ strongly in $L^{p_i'}(Q_T)$, then

$$\begin{aligned}
&\lim_{n\to\infty}\int_{Q_T} \phi_{i,n}(x,t,u_n) D^i(S'(u_n)\varphi)\,dx\,dt \\
&= \lim_{n\to\infty}\int_{Q_T} \phi_{i,n}(x,t,T_M(u_n))(S''(u_n)\varphi D^i T_M(u_n) + S'(u_n)D^i\varphi)\,dx\,dt \\
&= \int_{Q_T} \phi_i(x,t,T_M(u))(S''(u)\varphi D^i T_M(u) + S'(u)D^i\varphi)\,dx\,dt \qquad (4.80)\\
&= \int_{Q_T} \phi_i(x,t,u)(S''(u)\varphi D^i u + S'(u)D^i\varphi)\,dx\,dt.
\end{aligned}$$

Moreover, since $S(u_n)\varphi \to S(u)\varphi$ weak $-\star$ in $L^\infty(Q_T)$, then

$$\int_{Q_T} |u_n|^{p_0-2}u_n S'(u_n)\varphi\,dx\,dt \longrightarrow \int_{Q_T} |u|^{p_0-2}uS'(u)\varphi\,dx\,dt, \qquad (4.81)$$

and

$$\int_\Omega f_n S'(u_n)\varphi\,dx\,dt \longrightarrow \int_\Omega fS'(u)\varphi\,dx\,dt. \qquad (4.82)$$

By combining (4.77) – (4.82), we deduce that

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial S(u)}{\partial t},\varphi\right\rangle dt + \sum_{i=1}^N \int_{Q_T} a_i(x,t,u,\nabla u)\cdot(S''(u)\varphi D^i u + S'(u)D^i\varphi)\,dx\,dt + \int_{Q_T}|u|^{p_0-2}uS'(u)\varphi dx \\
&= \int_{Q_T} fS'(u)\varphi\,dx\,dt + \int_{Q_T}\phi_i(x,t,u,u)(S''(u)\varphi D^i u + S'(u)D^i\varphi)\,dx\,dt
\end{aligned}$$

Therefore, u is a renormalized solution to problem (3.6).

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Author information

Moussa Chrif, Hassane Hjiaj and Mohamed Sasy,
Regional Center of Education and Training Professions (CRMEF) Scientific Research and Pedagogical Development Laboratory, Research Team in Mathematics and Didactics, Meknes, Morocco.
Department of Mathematics, Faculty of Sciences, University of Abdelmalek Essaadi, Tetouan, Morocco.
Department of Mathematics, Faculty of Sciences, University of Abdelmalek Essaadi, Tetouan, Morocco., Morocco.
E-mail: `moussachrif1424@gmail.com`, `hjiajhassane@yahoo.fr`, `mohamed.sasy@etu.uae.ac.ma`