Palestine Journal of Mathematics

Vol 13(Special Issue I)(2024) , 170–188

WEIGHTED FRACTIONAL OSTROWSKI, TRAPEZOID AND GRSS TYPE INEQUALITIES ON TIME SCALES

Mohamed Lahrouz and Mohamed El Amri

MSC 2010 Classifications: Primary 26D15, 26A51; Secondary 26A33, 26A42.

Keywords and phrases: Opail's inequality,Hlder's inequality,algebraic inequalities, time scales, conformable fractional calculus, disfocalitiy, disconjugacy, dynamic equations.

Abstract In this work, we first obtain a weighted fractional Montgomery identity on time scales and then derive weighted fractional Ostrowski, Trapezoid and Grss inequalities on time scales, respectively. These results not only give a generalisation of the known results, but also provide other interesting inequalities on time scales as specific cases.

1 Introduction

Dragomir and Wang proved in 1997 (see [1]) that if $f : [a, b] \to \mathbb{R}$, is differentiable with bounded derivative, then for all $t \in [a, b]$,

$$\left| f(t) - \frac{1}{b-a}\int_a^b f(s)ds - \frac{f(b)-f(a)}{b-a}\left(t - \frac{a+b}{2}\right)\right| \le \frac{1}{4}(b-a)(M-m). \tag{1.1}$$

Where $m := \inf_{t\in[a,b]} f'(t)$ and $M := \sup_{t\in[a,b]} f'(t)$. The above inequality is known in the literature as the Ostrowski-Grss type inequality. Ng Liu [2] proved the following Ostrowski-Grss type inequality for time scales which is a combination of both Grss inequality and Ostrowski inequality on time scales due to Bohner and Mathews [3, 4]. For other results along these lines, see [5, 6, 7, 8, 9, 10, 11].

Theorem 1.1. *Let $a, b, s, t \in \mathbb{T}$ with $a < b$ and $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$, be differentibale.If f^Δ is rd-continous (i.e, f^Δ is continous at right-dense point in $[a, b]_{\mathbb{T}}$ and its left-side limits exist at left-dense point in $[a, b]_{\mathbb{T}}$) and $\gamma \le f^\Delta(s) \le \Gamma$ for all $t \in [a, b]$, then we have*

$$\left| f(t) - \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s - \frac{f(b)-f(a)}{(b-a)^2}\left[h_2(t,a) - h_2(t,b)\right]\right| \le \frac{\Gamma-\gamma}{2(b-a)}\int_a^b \left| P(s,t) - \frac{h_2(t,a)-h_2(t,b)}{b-a}\right| \Delta s. \tag{1.2}$$

Where $h_1(t,s) = \int_s^t 1\Delta\tau = t - s$ and $h_2(t,s) = \int_s^t h_1(\tau,s)\Delta\tau$

$$and\ P(s,t) = \begin{cases} s-a, s \in [a,t), \\ s-b, s \in [t,b]. \end{cases}$$

In [4] for general time scales, which unify discrete, continuous and many other cases.By using Montgomery identity on time scales, they established the following Ostrowski inequality on time scales.

Theorem 1.2. *Let $a, b, s, t \in \mathbb{T}$ with $a < b$ and $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$, be differentibale.Then*

$$\left| f(t) - \frac{1}{b-a}\int_a^b f(\sigma(s))\Delta s\right| \le \frac{M}{b-a}\ (h_2(t,a) + h_2(t,b)\,). \tag{1.3}$$

Where $M = \sup_{a \le t \le b} |f^{\Delta}(t)| < \infty$.
This inequality is sharp in the sense that the right side of (1.3) cannot be substituted by a smaller one.

Recently, Karpuz and kan [5] generalized Ostrowski's inequality and Montogomery's identity on arbitrary time scale by the means of generalized polynomials on time scales.By introducing a parameter, Liu, Ng Chen[7] also extended a generalization of the above inequality on time scales.Ng Liu [2] gave a sharp Grss type inequality on time scales and then applied it to the sharp Ostrowski-Grss inequality on time scales.Motivated by the ideas of [4, 5, 7, 12],Tuna and Daghan [13] studied generalizations of Ostrowski and Ostrowski-Grss type inequality on time scales. More recenty, Liu [14] established the following weighted generalization of three point inequality with a parameter for mappings of bounded variation.

Theorem 1.3. *Let us have* $0 \le k \le 1$*,* $f : [a,b] \to \mathbb{R}$ *be a mapping of bounded variation,* $g : [a,b] \to [0,+\infty)$ *continous and positive on* (a,b) *and let* $h : [a,b] \to \mathbb{R}$ *be differentiable such that* $h'(t) = g(t)$ *on* $[a,b]$*.Then*

$$\left| \int_a^b f(t)g(t)dt - \left\{ (1-k)f(x) + k\left[\frac{\int_a^x g(t)dt}{\int_a^b g(t)dt} f(a) + \frac{\int_x^b g(t)dt}{\int_a^b g(t)dt} f(b) \right] \right\} \int_a^b g(t)dt \right|$$
$$\le \begin{cases} (1-k)\left[\frac{1}{2}\int_a^b g(t)dt + \left| h(x) - \frac{h(a)+h(b)}{2} \right| \right] V_a^b(f); k \in [0; \frac{1}{2}] \\ k\left[\frac{1}{2}\int_a^b g(t)dt + \left| h(x) - \frac{h(a)+h(b)}{2} \right| \right] V_a^b(f); k \in (\frac{1}{2}, 1]. \end{cases}$$

For all $x \in [a,b]$*, where* $V_a^b(f)$ *denotes the total variation of* f *on the interval* $[a,b]$*.*

In the last few years some authors have studied the fractional inequalities by means of the fractional derivatives of Caputo and Riemann-Liouville, we refer you to the papers [15, 16, 17] for these results. In [18, 19] the authors extended the calculus of fractional order to the calculus of conforming fractions. Lately some authors have extended classical inequalities by using conforming fractional calculus, such as Opial'sinequality [20, 21]. Hermite ? Hadamard?s inequality [22, 23]. The purpose of this paper is to use the above research to obtain some weighted fraction Ostrowski, fraction trapezoidal, fraction Grss, and fraction Ostrowski-Grss inequalities, a parameter on time scales, based on a fraction-weighted Montgomery identity on time scales. In Section 2, we briefly present the general definitions and theorems related to fractional calculus on time scales that will be needed throughout the paper. The fractional weighted Montgomery identity, fractional weighted Ostrowski type inequality, fractional weighted Trapezoid type and fractional weighted Grss type inequalities on time scales are derived in subsections 3.1, 3.2, 3.3 and 3.4, respectively.

2 Preliminaries and fundamental lemmas

This section presents the basics of fractional calculus on the time scales we will need throughout the paper. The findings are taken from [24, 25, 26, 27]. A time scale ?$\mathbb{T}$ is an arbitrary non-empty closed subset of real numbers $\mathbb{R}$. We will assume throughout that $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$, and we will define the forward jump operator:$\sigma : \mathbb{T} \to \mathbb{T}, as$ such that $\sigma(t) = inf\,\{s \in \mathbb{T} : s > t\}$, while the backword jump operator $\rho : \mathbb{T} \to \mathbb{T}$, is defined by:$\rho(t) = Sup\,\{s \in \mathbb{T} : s < t\}$, where $Sup\ \emptyset = inf\ \mathbb{T}$ (i.e $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), and $inf\ \emptyset = Sup\ \mathbb{T}$(i.e $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t).
For any $t \in \mathbb{T}$ the notation $x^{\sigma}(t)$ refers to $x\,(\sigma(t))$,i.e.,$x^{\sigma} = x \circ \sigma$.
Finally,the graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$.If $\sigma(t) > t$ then we say that t is right.scattered, while if $\rho(t) < t$ then we say that t is left-scattered .Points that are right-scattered and left-scattered at the same time are called isolated.If $\sigma(t) = t$ then t is called right-dense, and if $\rho(t) = t$ then t is called left-dense.Points are both right-dense and left-dense are called dense.

The set $\mathbb{T}^k$ is defined as follows,if $\mathbb{T}$ has a left-scattered maximum m than $\mathbb{T}^k = \mathbb{T} - \{m\}$ otherwise $\mathbb{T}^k = \mathbb{T}$.
The function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continous (denote $f \in C_{rd}(\mathbb{T}, \mathbb{R})$), if it is continous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left-dense points $t \in \mathbb{T}$.

Definition 2.1 [24](Conformable $\alpha-$fractional derivative)
Let the function $f : \mathbb{T} \to \mathbb{R}$ and $\alpha \in (0, 1]$. Then, for $t > 0$, we define $T_\alpha (f) (t)$ to be the number (provided it exists) with the proprety that,
given any $\epsilon > 0$, there is a neighborhood U of t such that for all $s \in U$,

$$\left| [f^\sigma (t) - f(s)]\, t^{1-\alpha} - T_\alpha (f) (t) [\sigma(t) - s] \right| \leq \epsilon \left|\sigma(t) - s\right|$$

$T_\alpha (f) (t)$ is called the conformable $\alpha-$fractional derivative of f of order α at t on $\mathbb{T}$, and we define the conformable fractional derivative on $\mathbb{T}$ at 0 as $T_\alpha (f) (0) = \lim_{t\to 0^+} T_\alpha (f) (t)$. If $\alpha = 1$, then we obtain from **Definition 2.1** the Hilger delta derivative of time scales [25]. The $\alpha-$fractional derivative of order zero is defined by the identity operator $T_0 (f) := f$. The basic properties of the $\alpha-$ fractional derivative on time scales are given in[24], together with a number of illustrative examples.

Theorem 2.2 [24]
Let $\alpha \in (0, 1]$ and $\mathbb{T}$ be a time scales.Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^k$.If f is $\alpha-$ fractional differentiable of order α at t, then

$$f^\sigma (t) = f(t) + \mu(t)t^{\alpha-1}T_\alpha f(t).$$

The characteristics of the conformable fractional derivative are as follows.
Theorem 2.3 [24]
Let $f, g : \mathbb{T} \to \mathbb{R}$ be conformable fractional derivative of order $\alpha \in (0, 1]$. Then the following properties are hold:
(i) The $f+g : \mathbb{T} \to \mathbb{R}$ is conformable fractional derivative of order α and,$T_\alpha (f + g) = T_\alpha (f) + T_\alpha (g)$.
(ii) For any $\lambda \in \mathbb{R}$, then $\lambda f : \mathbb{T} \to \mathbb{R}$ is $\alpha-$ fractional differentiable and $T_\alpha (\lambda f) = \lambda T_\alpha (f)$.
(iii) If f and g are $\alpha-$ fractional differentiable , then the product $fg : \mathbb{T} \to \mathbb{R}$ is $\alpha-$ fractional differentiable and $T_\alpha (fg) = T_\alpha (f) g + f^\sigma T_\alpha (g) = T_\alpha (f) g^\sigma + fT_\alpha (g)$.
(iv) If f is $\alpha-$ fractional differentiable , then $\frac{1}{f}$ is $\alpha-$ fractional differentiable and $T_\alpha \left(\frac{1}{f}\right) = -\frac{T_\alpha(f)}{ff^\sigma}$, valid at all points $t \in \mathbb{T}^k$ for which $f(t)f (\sigma(t)) \neq 0$.
(v) If f and g are $\alpha-$ fractional differentiable , then f/g is $\alpha-$ fractional differentiable with $T_\alpha \left(\frac{f}{g}\right) = \frac{T_\alpha(f)g-fT\alpha(g)}{gg^\sigma}$,valid at all points $t \in \mathbb{T}^k$ for which $g(t)g (\sigma(t)) \neq 0$.

Definition 2.4[24](Conformable fractional integral)
For $0 < \alpha \leq 1$, then the $\alpha-$ fractional integral of f, is defined by

$$\int f(s)\Delta^\alpha s = \int f(s)s^{\alpha-1}\Delta s.$$

The following properties are satisfied by the conformal fractional integral.
Theorem 2.5
Let $\alpha \in (0, 1]$,$a, b, c \in \mathbb{T}$, and $\lambda \in \mathbb{R}$, and f, g be two rd-continuous functions, then
(i) $\int_a^b [f(t) + g(t)]\, \Delta^\alpha t = \int_a^b f(t)\, \Delta^\alpha t + \int_a^b g(t)\, \Delta^\alpha t.$
(ii) $\int_a^b (\lambda f(t))\, \Delta^\alpha t\lambda \int_a^b f(t)\, \Delta^\alpha t.$
(iii) $\int_a^b f(t)\, \Delta^\alpha t = -\int_b^a f(t)\, \Delta^\alpha t.$
(iv) $\int_a^b f(t)\, \Delta^\alpha t \int_a^c f(t)\, \Delta^\alpha t + \int_c^b f(t)\, \Delta^\alpha t.$
(v) $\left|\int_a^b f(t)\, \Delta^\alpha t - vert \leq \int_a^b |f(t)|\, \Delta^\alpha t.\right.$
The fractional integration by parts formula is given by

Theorem 2.6
Let f, g be rd-continuous $,a, b \in \mathbb{T}$, then

$$\int_a^b f(t)T_\alpha g(t)\Delta^\alpha t = [f(t)g(t)]_a^b - \int_a^b T_\alpha(f)(t)g^\sigma(t)\Delta^\alpha t, \tag{2.1}$$

$$\int_a^b f^\sigma(t)T_\alpha g(t)\Delta^\alpha t = [f(t)g(t)]_a^b - \int_a^b T_\alpha(f)(t)g(t)\Delta^\alpha t. \tag{2.2}$$

The following useful relationships are often used beetween the time scale fractional calculus and the $\alpha-$differential caculus $\mathbb{R}$, the α=difference calculus $h\mathbb{Z}$ $(h > 0)$ and fractional q-difference calculus $\overline{q^{\mathbb{Z}}} = \{q^n, n \in \mathbb{Z}\} \cup \{0\}$ $(q > 1)$.

(i) If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t,\ \mu(t) = 0,\ T_\alpha f(t) := D^\alpha f(t) = t^{1-\alpha} f'(t)$ *and* $\int_a^b f(t)\ \Delta^\alpha t := \int_a^b f(t)\ d_\alpha t = \int_a^b \frac{f(t)}{t^{1-\alpha}}\ dt.$

(ii) If $\mathbb{T} = h\mathbb{Z}$ $(h > 0)$, then $\sigma(t) = t + h,\ \mu(t) = h,\ T_\alpha f(t) := \Delta^\alpha f(t) = t^{1-\alpha}\frac{f(t+h)-f(t)}{h}$ *and* $\int_a^b f(t)\ \Delta^\alpha t = \sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} t^{\alpha-1} f(ht)h^\alpha.$

(iii) If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ $(q > 1)$, then $\sigma(t) = qt,\ \mu(t) = (q-1)t,\ T_\alpha f(t) := D_q^\alpha f(t) = \dfrac{f(qt) - f(t)}{(q-1)t^\alpha}$ *and* $\int_a^b f(t)\ \Delta^\alpha t = (q-1)\sum_{t=\log_q a}^{(\log_q b)-1} q^{\alpha t} f(q^t).$

Definition 2.7
Let $h_{\alpha,k} : \mathbb{T}^2 \to \mathbb{R}\ ; k \in \mathbb{N}_0$ *and* $\alpha \in (0, 1]$ be defined by

$$h_{\alpha,0}(t, s) = 1 \ \ for\ all\ \ s, t \in \mathbb{T}$$

and then recursively by

$$h_{\alpha,k+1}(t, s) = \int_s^t h_{\alpha,k}(\tau, s)\Delta^\alpha \tau \ \ for\ all\ \ s, t \in \mathbb{T}$$

If we let $T_\alpha^t\,(h_{\alpha,k}(t, s))$ denote for each fixed s the $\alpha-$conformable derivative of $h_{\alpha,k}(t, s)$ with respect to t, then , by induction, we have

$$T_\alpha^t\,(h_{\alpha,k}(t, s)) = h_{\alpha,k-1}(t, s) \ \ for\ all\ \ k \in \mathbb{N},\ t \in \mathbb{T}^k.$$

If $\mathbb{T} = \mathbb{R}$; $\alpha = 1$, then

$$h_{1,k}(t, s) := h_k(t, s) = \frac{(t-s)^k}{k!}\ for\ all\ s, t \in \mathbb{R}.$$

If $\mathbb{T} = \mathbb{Z}$; $\alpha = 1$, then

$$h_{1,k}(t, s) := h_k(t, s) = \binom{t-s}{k} for\ all\ s, t \in \mathbb{Z},$$

$$where\ \binom{t-s}{k} = \frac{(t-s)(t-s-1)...(t-s-k+1)}{k!}.$$

If $\mathbb{T} = q^{\mathbb{N}_0}$; $q > 1$, $\alpha = 1$, then

$$h_{1,k}(t, s) := h_k(t, s) = \prod_{j=0}^{k-1}\left(\frac{(t-q^j s)}{\sum_{i=0}^{j} q^i}\right),\ for\ all\ s, t \in q^{\mathbb{N}_0}.$$

If $\mathbb{T} = \mathbb{R}$; $\alpha \in (0,1]$, then

$$h_{\alpha,k}(t,s) = \frac{(t^\alpha - s^\alpha)^k}{\alpha^k.k!} \; for\ all\ s,t \in \mathbb{R}.$$

If $\mathbb{T} = k\mathbb{Z}$; $h > 0$, $\alpha \in (0,1]$, then

$$h_{\alpha,1}(t,s) = h^\alpha \sum_{k=\frac{s}{h}}^{\frac{t}{h}-1} k^{\alpha-1}$$

$$h_{\alpha,2}(t,s) = h^{2\alpha} \sum_{k_1=\frac{s}{h}}^{\frac{t}{h}-1} \sum_{k_2=\frac{s}{h}}^{k_1-1} (k_1k_2)^{\alpha-1}$$

$$h_{\alpha,n}(t,s) = h^{n\alpha} \sum_{k_1=\frac{s}{h}}^{\frac{t}{h}-1} \sum_{k_2=\frac{s}{h}}^{k_1-1} ... \sum_{k_n=\frac{s}{h}}^{k_{n-1}-1} (k_1k_2...k_n)^{\alpha-1}$$

If $\mathbb{T} = q^{\mathbb{N}_0}$; $q > 1$, $\alpha \in (0,1]$, then

$$h_{\alpha,1}(t,s) = \frac{q-1}{q^\alpha - 1}(t^\alpha - s^\alpha)$$

$$h_{\alpha,2}(t,s) = \frac{(q-1)^2}{(q^\alpha-1)^2(q^\alpha+1)} \left(t^\alpha - s^\alpha\right)\left(t^\alpha - (qs)^\alpha\right).$$

3 Main Results

We suppose that $\mathbb{T}$ is a time scale, and also that in $\mathbb{T}$ the interval $[a,b]_\mathbb{T}$ means the set $\{t \in \mathbb{T} : a \leq t \leq b\}$ for the points $a < b$ in $\mathbb{T}$. In this section, we first derive a fractional weighted Montgomery identity on the time scales, and then find the fractional weighted Ostrovsky-type, fractional trapezoidal, and fractional salute inequalities on the time scale.

3.1 Fractional Weighted Montgomery identity on time scales

Lemma 3.1.1
Let $0 \leq \lambda \leq 1$, $g : [a,b]_\mathbb{T} \to [0,+\infty)$ be rd-continous and positive and $\varphi : [a,b]_\mathbb{T} \to \mathbb{R}$ be conformable differentiable of order $\alpha \in (0,1]$ such that $T_\alpha\varphi(t) = g(t)\ on\ [a,b]_\mathbb{T}$. If $f : [a,b]_\mathbb{T} \to \mathbb{R}$ is conformable differentiable of order$\alpha \in (0,1]$.Then for all $t \in [a,b]_\mathbb{T}$, we have

$$\int_a^b Q(t,s).T_\alpha f(s)\Delta^\alpha s = \left\{(1-\lambda)f(t) + \lambda\left[\frac{\int_a^t g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}f(a) + \frac{\int_t^b g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}f(b)\right]\right\}.\int_a^b g(s)\Delta^\alpha s - \int_a^b g(s)f(\sigma(s))\Delta^\alpha s \quad (3.1)$$

Where

$$Q(t,s) = \begin{cases} \varphi(s) - [(1-\lambda)\varphi(a) + \lambda\varphi(t)] \ ; \ a \leq s < t, \\ \varphi(s) - [\lambda\varphi(t) + (1-\lambda)\varphi(b)] \ ; \ t \leq s \leq b. \end{cases} \quad (3.2)$$

Proof
Using item (i) of **Theorem 2.6**, we have

$$\int_a^t Q(t,s)T_\alpha f(s)\Delta^\alpha s = \int_a^t \left[\varphi(s) - \left[(1-\lambda)\varphi(a) + \lambda\varphi(t)\right]\right].T_\alpha f(s)\Delta^\alpha s$$
$$= \left[\left[\varphi(s) - \left[(1-\lambda)\varphi(a) + \lambda\varphi(t)\right]\right] f(s)\right]_a^t - \int_a^t g(s)f^\sigma(s)\Delta^\alpha s$$
$$= \left[\varphi(t) - \left[(1-\lambda)\varphi(a) + \lambda\varphi(t)\right]\right] f(t) - \left[\varphi(a) - \left[(1-\lambda)\varphi(a) + \lambda\varphi(t)\right]\right] f(a)$$
$$- \int_a^t g(s)f^\sigma(s)\Delta^\alpha s. \tag{*}$$

and

$$\int_t^b Q(t,s)T_\alpha f(s)\Delta^\alpha s = \int_t^b \left[\varphi(s) - \left[\lambda\varphi(t) + (1-\lambda)\varphi(b)\right]\right].T_\alpha f(s)\Delta^\alpha s$$
$$= \left[\left[\varphi(s) - \left[\lambda\varphi(t) + (1-\lambda)\varphi(b)\right]\right] f(s)\right]_t^b - \int_t^b g(s)f^\sigma(s)\Delta^\alpha s$$
$$= \left[\varphi(b) - \left[\lambda\varphi(a) + (1-\lambda)\varphi(b)\right]\right] f(b) - \left[\varphi(t) - \left[\lambda\varphi(t) + (1-\lambda)\varphi(b)\right]\right] f(t)$$
$$- \int_t^b g(s)f^\sigma(s)\Delta^\alpha s. \tag{**}$$

There for the inequality (3.1) is proved by adding the above two identities (*) and (**).

Corollary 3.1.2

If $\mathbb{T} = \mathbb{R}$ then **Lemma 3.1.1** holds.

$$\int_a^b Q(t,s)D^\alpha f(s)d_\alpha s = \left\{(1-\lambda)f(t) + \lambda\left[\frac{\int_a^t g(s)d_\alpha s}{\int_a^b g(s)d_\alpha s}f(a) + \frac{\int_t^b g(s)d_\alpha s}{\int_a^b g(s)d_\alpha s}f(b)\right]\right\}\int_a^b g(s)d_\alpha s - \int_a^b g(s)f(s)\, d_\alpha s$$

Where $g(t) = D^\alpha\varphi(t)$ *on* $[a,b]$ and $Q(t,s) = \begin{cases} \varphi(s) - [(1-\lambda)\varphi(a) + \lambda\varphi(t)] & if\ a \le s < t, \\ \varphi(s) - [\lambda\varphi(t) + (1-\lambda)\varphi(b)] & if\ t \le s \le b. \end{cases}$.

Corollary 3.1.3

In the case of $\mathbb{T} = h\mathbb{Z}$ $(h > 0)$ in Lemma 3.1.1 we have

$$\sum_{s=\frac{a}{h}}^{\frac{b}{h}-1} s^{\alpha-1}Q(t,hs)\Delta^\alpha f(sh) =$$
$$\left\{(1-\lambda)f(t) + \lambda\left[\frac{\sum_{s=\frac{a}{h}}^{\frac{t}{h}-1} s^{\alpha-1}g(hs)}{\sum_{s=\frac{a}{h}}^{\frac{b}{h}-1} s^{\alpha-1}g(hs)}f(a) + \frac{\sum_{s=\frac{t}{h}}^{\frac{b}{h}-1} s^{\alpha-1}g(hs)}{\sum_{s=\frac{a}{h}}^{\frac{b}{h}-1} s^{\alpha-1}g(hs)}f(b)\right]\right\}\sum_{s=\frac{a}{h}}^{\frac{b}{h}-1} s^{\alpha-1}g(hs) - \sum_{s=\frac{a}{h}}^{\frac{b}{h}-1} s^{\alpha-1}g(hs)f(h(s+1)),$$

where g(t)=$t^{1-\alpha}\dfrac{\varphi(t+h)-\varphi(t)}{h}$ on $[a,b]_{h\mathbb{Z}}$ and

$$Q(t,s) = \begin{cases} \varphi(s) - [(1-\lambda)\varphi(a) + \lambda\varphi(t)] & if\ s \in \{a, a+h, \cdots, t-h\}, \\ \varphi(s) - [\lambda\varphi(t) + (1-\lambda)\varphi(b)] & if\ s \in \{t, t+h, \cdots, b\}. \end{cases}$$

Corollary 3.1.4

In the case of $\mathbb{T} = q^{\mathbb{N}}$ $(q > 1)$, $a = q^m$ *and* $b = q^n$ *with* $m < n$ in Lemma 3.1.1 we have

$$\sum_{s=m}^{n-1} Q(q^k, q^j)\frac{f(q^{j+1}) - f(q^j)}{q-1} =$$
$$\left\{(1-\lambda)f(q^k) + \lambda\left[\frac{\sum_{j=m}^{k-1} q^{\alpha j}g(q^j)}{\sum_{j=m}^{n-1} q^{\alpha j}g(q^j)}f(q^m) + \frac{\sum_{j=k}^{n-1} q^{\alpha j}g(q^j)}{\sum_{j=m}^{n-1} q^{\alpha j}g(q^j)}f(q^n)\right]\right\}\sum_{j=m}^{n-1} q^{\alpha j}g(q^j) - \sum_{j=m}^{n-1} q^{\alpha j}g(q^j)f(q^{j+1})$$

where $Q(q^k, q^j) = \begin{cases} \varphi(q^j) - [(1-\lambda)\varphi(q^m) + \lambda\varphi(q^k)] & if\ m \le j < k, \\ \varphi(q^j) - [\lambda\varphi(q^k) + (1-\lambda)\varphi(q^n)] & if\ k \le j \le n. \end{cases}$

Corollary 3.1.5
In the case of $\varphi(t) = h_{\alpha,1}(t,a)$ in Lemma 3.1.1 we have $(1-\lambda)f(t)$ is equal to

$$\frac{1}{h_{\alpha,1}(b,a)}\int_a^b Q(t,s)T_\alpha f(s)\Delta^\alpha s - \lambda\left[\frac{h_{\alpha,1}(t,a)}{h_{\alpha,1}(b,a)}f(a) + \frac{h_{\alpha,1}(b,t)}{h_{\alpha,1}(b,a)}f(b)\right] + \frac{1}{h_{\alpha,1}(b,a)}\int_a^b f^\sigma(s)\Delta^\alpha s$$

where

$$Q(t,s) = \begin{cases} h_{\alpha,1}(s,a) - \lambda h_{\alpha,1}(t,a)\ ;\ a \le s < t, \\ h_{\alpha,1}(s,a) - \left[\lambda h_{\alpha,1}(t,a) + (1-\lambda)h_{\alpha,1}(b,a)\right]\ ;\ t \le s \le b. \end{cases}$$

Corollary 3.1.6
In the case of $\mathbb{T} = \mathbb{R}$ in Corollary 3.1.5 we have $(1-\lambda)f(t)$ is equal to

$$\frac{\alpha}{b^\alpha - a^\alpha}\int_a^b Q(t,s)D^\alpha f(s)d_\alpha s - \lambda\left[\frac{t^\alpha - a^\alpha}{b^\alpha - a^\alpha}f(a) + \frac{b^\alpha - t^\alpha}{b^\alpha - a^\alpha}f(b)\right] + \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(s)d_\alpha s$$

Where $Q(t,s) = \begin{cases} \frac{s^\alpha - [(1-\lambda)a^\alpha + \lambda t^\alpha]}{\alpha} & if\ a \le s < t, \\ \frac{s^\alpha - [\lambda t^\alpha + (1-\lambda)b^\alpha]}{\alpha} & if\ t \le s \le b. \end{cases}$

Corollary 3.1.7
In the case of $\mathbb{T} = h\mathbb{Z}, h > 0$ in Corollary 3.1.5 we have $(1-\lambda)f(t)$ is equal to

$$A(a,b,h,\alpha)\,.\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1}.Q(t;hk)\Delta^\alpha f(kh) - \lambda.B(a,b,t,h,\alpha) + A(a,b,h,\alpha)\,.\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1}.f(h(k+1))\,.$$

Where

$$Q(t,s) = \begin{cases} \left(\sum_{k=\frac{a}{h}}^{\frac{s}{h}-1} k^{\alpha-1} - \lambda\sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} k^{\alpha-1}\right)h^\alpha & , s \in \{a, a+h, ...t-h\}, \\ \left(\sum_{k=\frac{a}{h}}^{\frac{s}{h}-1} k^{\alpha-1} - \left[\lambda\sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} k^{\alpha-1} + (1-\lambda)\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1}\right]\right)h^\alpha & , s \in \{t, t+h, ...b\}. \end{cases}$$

and

$$B(a,b,t,h,\alpha) = \left[\frac{\sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} k^{\alpha-1}}{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1}}f(a) + \frac{\sum_{k=\frac{t}{h}}^{\frac{b}{h}-1} k^{\alpha-1}}{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1}}f(b)\right] \text{ and } A(a,b,h,\alpha) = \left(\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1}\right)^{-1}$$

Corollary 3.1.8
In the case of $\lambda = 0$ in Corollary 3.1.5 we have

$$f(t) = \frac{1}{h_{\alpha,1}(b,a)}\int_a^b Q(t,s)T_\alpha f(s)\Delta^\alpha s + \frac{1}{h_{\alpha,1}(b,a)}\int_a^b f^\sigma(s)\Delta^\alpha s$$

Where $Q(t,s) = \begin{cases} h_{\alpha,1}(s,a) & if\ a \le s < t, \\ h_{\alpha,1}(s,b) & if\ t \le s \le b. \end{cases}$

3.2 Weighted Fractional Ostrowski type inequality on time scales

Theorem 3.2.1
Let $0 \le \lambda \le 1$, $g : [a,b]_\mathbb{T} \to [0,+\infty)$ be rd-continous and positive and $\varphi : [a,b]_\mathbb{T} \to \mathbb{R}$ be conformable differentiable of order $\alpha \in (0,1]$ such that $T_\alpha\varphi(t) = g(t)\ on\ [a,b]_\mathbb{T}$. Let $f : [a,b] \to \mathbb{R}$ be conformable differentiable of order $\alpha \in (0,1]$.Then for all $t \in [a,b]_\mathbb{T}$, we have

$$\left|\left\{(1-\lambda)f(t) + \lambda\left[\frac{\int_a^t g(s)\,\Delta^\alpha s}{\int_a^b g(s)\,\Delta^\alpha s}f(a) + \frac{\int_t^b g(s)\,\Delta^\alpha s}{\int_a^b g(s)\,\Delta^\alpha s}f(b)\right]\right\}\int_a^b g(s)\,\Delta^\alpha s - \int_a^b g(s)f^\sigma(s)\,\Delta^\alpha s\right| \le K\int_a^b |Q(t,s)|\,\Delta^\alpha s,$$

Where

$$Q(t,s) = \begin{cases} \varphi(s) - [(1-\lambda)\varphi(a) + \lambda\varphi(t)]\ ;\ a \le s < t, \\ \varphi(s) - [\lambda\varphi(t) + (1-\lambda)\varphi(b)]\ ;\ t \le s \le b. \end{cases} \quad and\ K = \sup_{t\in[a,b]_\mathbb{T}}\left|T_\alpha f(t)\right| < +\infty.$$

Proof
The proof of the theorem 3.2.1 can be done easily from **Lemma 3.1.1**, since it is sufficient to proceed by majoration.
Corollary 3.2.2 If $\mathbb{T} = \mathbb{R}$ then Theorem 3.2.1 holds

$$\left|\left\{(1-\lambda)f(t) + \lambda\left[\frac{\int_a^t g(s)\,d_\alpha s}{\int_a^b g(s)\,d_\alpha s}f(a) + \frac{\int_t^b g(s)\,d_\alpha s}{\int_a^b g(s)\,d_\alpha s}f(b)\right]\right\}\int_a^b g(s)\,d_\alpha s - \int_a^b g(s)f(s)\,d_\alpha s\right| \le K\int_a^b |Q(t,s)|\,d_\alpha s.$$

Where $g(t) = D^{\alpha}\varphi(t)$ *on* $[a,b]$

$$Q(t,s) = \begin{cases} \varphi(s) - [(1-\lambda)\varphi(a) + \lambda\varphi(t)] \ ; \ a \le s < t, \\ \varphi(s) - [\lambda\varphi(t) + (1-\lambda)\varphi(b)] \ ; \ t \le s \le b. \end{cases}$$

and $K = \sup_{t\in[a,b]_{\mathbb{T}}} \left|D^{\alpha} f(t)\right| < +\infty$.

Corollary 3.2.3
In the case of $\mathbb{T} = h\mathbb{Z},\ h > 0$ in Theorem 3.2.1 we have

$$\left| \left\{ (1-\lambda)f(t) + \lambda \left[\frac{\sum_{k=\frac{a}{h}}^{\frac{t}{h}-1} k^{\alpha-1} g(hk)}{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1} g(hk)} f(a) + \frac{\sum_{k=\frac{t}{h}}^{\frac{b}{h}-1} k^{\alpha-1} g(hk)}{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1} g(hk)} f(b) \right] \right\} \left(\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1} g(hk) \right) \right.$$
$$\left. - \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1} g(hk) f\left(h(k+1)\right) \right| \le K \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1} |Q(t,hk)|$$

Where $g(t) = t^{1-\alpha}.\frac{\varphi(t+h)-\varphi(t)}{h}$ *on* $[a,b]_{h\mathbb{Z}}$ and $K = \sup_{t\in[a,b]_{h\mathbb{Z}}} \left| \frac{t^{1-\alpha}\left(f(t+h) - f(t)\right)}{h} \right| < +\infty$, and

$$Q(t,s) = \begin{cases} \varphi(s) - [(1-\lambda)\varphi(a) + \lambda\varphi(t)] \ ; \ \ s \in \{a, a+h, t-h\}, \\ \varphi(s) - [\lambda\varphi(t) + (1-\lambda)\varphi(b)] \ ; \ \ s \in \{t, t+h, ...b\}, \end{cases}$$

Corollary 3.2.4
In the case of $\mathbb{T} = q^{\mathbb{N}}\ (q > 1), a = q^m\ and\ b = q^n\ with\ m < n$ in Theorem 3.2.1 we have

$$\left| \left\{ (1-\lambda)f(q^k) + \lambda \left[\frac{\sum_{j=m}^{k-1} q^{\alpha j} g(q^j)}{\sum_{j=m}^{n-1} q^{\alpha j} g(q^j)} f(q^m) + \frac{\sum_{j=k}^{n-1} q^{\alpha j} g(q^j)}{\sum_{j=m}^{n-1} q^{\alpha j} g(q^j)} f(q^n) \right] \right\} \left(\sum_{j=m}^{n-1} q^{\alpha j} g(q^j) \right) \right.$$
$$\left. - \sum_{j=m}^{n-1} q^{\alpha j} g(q^j) f(q^{j+1}) \right| \le K. \sum_{j=m}^{n-1} q^{\alpha j} |Q(q^k, q^j)|$$

Where $K = \sup_{m\le k\le n} \left| \frac{f(q^{1+k}) - f(q^k)}{(q-1)q^{\alpha k}} \right| < \infty$. and

$$Q(q^k, q^{k'}) = \begin{cases} \varphi(q^{k'}) - \left[(1-\lambda)\varphi(q^m) + \lambda\varphi(q^k)\right] \ ; \ \ m \le k' < k, \\ \varphi(q^{k'}) - \left[\lambda\varphi(q^k) + (1-\lambda)\varphi(q^n)\right] \ ; \ \ k \le k' \le n, \end{cases}$$

If $\varphi(t) = h_{\alpha,1}(t,a)$ then,

$$Q(t,s) = \begin{cases} h_{\alpha,1}(s,a) - \lambda h_{\alpha,1}(t,a) \ ; \ a \le s < t, \\ h_{\alpha,1}(s,a) - \left[\lambda h_{\alpha,1}(t,a) + (1-\lambda)h_{\alpha,1}(b,a)\right] \ ; \ t \le s \le b. \end{cases}$$

and

$$\begin{aligned} \int_a^t \left|Q(t,s)\right| \Delta^{\alpha} s &\le \int_a^t \left(h_{\alpha,1}(s,a) + \lambda h_{\alpha,1}(t,a)\right) \Delta^{\alpha} s \\ &= h_{\alpha,2}(t,a) + \lambda h_{\alpha,1}(t,a) \int_a^t 1\Delta^{\alpha} s \\ &= h_{\alpha,2}(t,a) + \lambda \left(h_{\alpha,1}(t,a)\right)^2 \end{aligned} \tag{3.3}$$

$$\begin{aligned} \int_t^b \left|Q(t,s)\right| \Delta^{\alpha} s &\le \int_t^b \left(h_{\alpha,1}(s,a) + \left(\lambda h_{\alpha,1}(t,a) + (1-\lambda)h_{\alpha,1}(b,a)\right) \Delta^{\alpha} s\right) \\ &= \int_t^b h_{\alpha,1}(s,a)\Delta^{\alpha} s + \left(\lambda h_{\alpha,1}(t,a) + (1-\lambda)h_{\alpha,1}(b,a)\right) h_{\alpha,1}(b,t) \\ &= \int_a^b h_{\alpha,1}(s,a)\Delta^{\alpha} s - \int_a^t h_{\alpha,1}(s,a)\Delta^{\alpha} s + \left(\lambda h_{\alpha,1}(t,a) + (1-\lambda)h_{\alpha,1}(b,a)\right) h_{\alpha,1}(b,t) \\ &= h_{\alpha,2}(b,a) - h_{\alpha,2}(t,a) + \left(\lambda h_{\alpha,1}(t,a) + (1-\lambda)h_{\alpha,1}(b,a)\right) h_{\alpha,1}(b,t) \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$\begin{aligned}\int_a^b \left|Q(t,s)\right|\Delta^\alpha s &= \int_a^t \left|Q(t,s)\right|\Delta^\alpha s + \int_t^b \left|Q(t,s)\right|\Delta^\alpha s \\ &\le h_{\alpha,2}(b,a) + \lambda(h_{\alpha,1}(t,a))^2 + \lambda h_{\alpha,1}(t,a)h_{\alpha,1}(b,t) + (1-\lambda)h_{\alpha,1}(b,a)h_{\alpha,1}(b,t) \\ &\le h_{\alpha,2}(b,a) + \lambda(h_{\alpha,1}(b,a))^2 + \lambda h_{\alpha,1}(b,a)h_{\alpha,1}(b,t) + (1-\lambda)h_{\alpha,1}(b,a)h_{\alpha,1}(b,t) \\ &= h_{\alpha,2}(b,a) + \lambda(h_{\alpha,1}(b,a))^2 + h_{\alpha,1}(b,a)h_{\alpha,1}(b,t) \end{aligned} \tag{3.5}$$

Using the inequality (3.5), we have the following Corollary
Corollary 3.2.5
In the case of $\varphi(t) = h_{\alpha,1}(t,a)$ in theorem, we have

$$\left|(1-\lambda)f(t) + \lambda\left[\frac{h_{\alpha,1}(t,a)}{h_{\alpha,1}(b,a)}f(a) + \frac{h_{\alpha,1}(b,t)}{h_{\alpha,1}(b,a)}f(b)\right] - \frac{1}{h_{\alpha,1}(b,a)}\int_a^b f^\sigma(s)\Delta^\alpha s\right| \le K.\left[h_{\alpha,2}(b,a) + \lambda\left(h_{\alpha,1}(b,a)\right)^2 + h_{\alpha,1}(b,a)h_{\alpha,1}(b,t)\right].$$

Corollary 3.2.6
In the case of $\mathbb{T} = \mathbb{R}$ in Corollary 3.2.5 we have

$$\left|(1-\lambda)f(t) + \lambda\left[\frac{t^\alpha - a^\alpha}{b^\alpha - a^\alpha}f(a) + \frac{b^\alpha - t^\alpha}{b^\alpha - a^\alpha}f(b)\right] - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(s)d_\alpha s\right| \le \frac{K(b^\alpha - a^\alpha)}{\alpha^2}.\left[\left(\frac{1}{2}+\lambda\right)(b^\alpha - a^\alpha) + (b^\alpha - t^\alpha)\right]$$

Where $K = \sup\limits_{t\in[a,b]}\left|D^\alpha f(t)\right| < \infty$.
Corollary 3.2.7
In the case of $\mathbb{T} = h\mathbb{Z}, (h>0)$ in Corollary 3.2.5 we have

$$\left|(1-\lambda)f(t) + \lambda\left[\frac{\sum_{k=\frac{a}{h}}^{\frac{t}{h}-1}k^{\alpha-1}}{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}}f(a) + \frac{\sum_{k=\frac{t}{h}}^{\frac{b}{h}-1}k^{\alpha-1}}{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}}f(b)\right] - \frac{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}f(h(k+1))k^{\alpha-1}}{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}}\right| \le K.h^{2\alpha}\left[\sum_{i=\frac{a}{h}}^{\frac{b}{h}-1}\sum_{j=\frac{a}{h}}^{i-1}(ij)^{\alpha-1} + \lambda\left(\sum_{j=\frac{a}{h}}^{\frac{b}{h}-1}j^{\alpha-1}\right)^2 + \left(\sum_{j=\frac{a}{h}}^{\frac{b}{h}-1}j^{\alpha-1}\right).\left(\sum_{j=\frac{t}{h}}^{\frac{b}{h}-1}j^{\alpha-1}\right)\right]$$

Corollary 3.2.8
If we let $\alpha = 1$ in Corollary 3.2.7, we have

$$\left|(1-\lambda)f(t) + \lambda\left[\frac{t-a}{b-a}f(a) + \frac{b-t}{b-a}f(b)\right] - \frac{h.\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}f(h(k+1))}{b-a}\right| \le K.(b-a)\left[(b-a)\left(\lambda+\frac{1}{2}\right) + b - t - \frac{h}{2}\right]$$

Corollary 3.2.9
In the case of $\mathbb{T} = q^{\mathbb{N}}\ (q>1), a = q^m\ and\ b = q^n\ with\ m<n$ in Corollary 3.2.5 we have

$$\left|(1-\lambda)f(q^k) + \lambda\left[\frac{q^{\alpha k} - q^{\alpha m}}{q^{\alpha n} - q^{\alpha m}}f(q^m) + \frac{q^{\alpha n} - q^{\alpha k}}{q^{\alpha n} - q^{\alpha m}}f(q^n)\right] - \frac{(q^\alpha - 1)\sum_{j=m}^{n-1}q^{\alpha j}f(q^{j+1})}{q^{\alpha n} - q^{\alpha m}}\right| \le K\frac{(q-1)^2}{(q^\alpha - 1)^2}.(q^{\alpha n} - q^{\alpha m})\left[\frac{q^{\alpha n} - q^{\alpha(m+1)}}{q^\alpha + 1} + \lambda(q^{\alpha n} - q^{\alpha m}) + \left(q^{\alpha n} - q^{\alpha k}\right)\right].$$

3.3 Weighted fractional Trapezoid type inequality on time scales

Theorem 3.3.1
Let $0 \leq \lambda \leq 1$,$g : [a,b]_{\mathbb{T}} \to [0,+\infty)$ be rd-continous and positive and $\varphi : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be conformable differentiable of order $\alpha \in (0,1]$ such that $T_\alpha \varphi(t) = g(t)$ *on* $[a,b]_{\mathbb{T}}$. If $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is conformable differentiable of order $\alpha \in (0,1]$.Then for all $t \in [a,b]_{\mathbb{T}}$, we have

$$\Bigg|(1-\lambda)\left(f^2(b)-f^2(a)\right)+\left[(\lambda-1)\frac{f(b)-f(a)}{\int_a^b g(s)\Delta^\alpha s}+\lambda.\frac{f(\sigma(b))-f(\sigma(a))}{\int_a^b g(s)\Delta^\alpha s}\right].\int_a^b g(s)f^\sigma(s)\Delta^\alpha s$$
$$+\lambda\left(f(\sigma(a))f(b)-f(\sigma(b)f(a))\right)-\frac{f(b)-f(a)}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b g(s)f(\sigma^2(s))\Delta^\alpha s\Bigg|$$
$$\leq \frac{K(K+K^\sigma)}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b\left(\int_a^b |Q(t,s)|\Delta^\alpha s\right)\Delta^\alpha t. \tag{3.6}$$

Where $K = \sup_{t\in[a,b]_{\mathbb{T}}}\left|T_\alpha f(t)\right| < \infty$, and $K^\sigma = \sup_{t\in[a,b]_{\mathbb{T}}}\left|T_\alpha f(\sigma(t))\right| < \infty$. And

$$Q(t,s) = \begin{cases}\varphi(s)-[(1-\lambda)\varphi(a)+\lambda\varphi(t)]\ ;\ a\leq s<t,\\ \varphi(s)-[\lambda\varphi(t)+(1-\lambda)\varphi(b)]\ ;\ t\leq s\leq b.\end{cases}$$

Proof
From **Lemma 3.3.1**, we have

$$(1-\lambda)f(t) = -\lambda\left[\frac{\int_a^t g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}f(a)+\frac{\int_t^b g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}f(b)\right]+\frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b g(s)f^\sigma(s)\Delta^\alpha s$$
$$+\frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b Q(t;s)T_\alpha f(s)\Delta^\alpha s. \tag{3.7}$$

$$(1-\lambda)f(\sigma(t)) = -\lambda\left[\frac{\int_a^t g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}f(\sigma(a))+\frac{\int_t^b g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}f(\sigma(b))\right]+\frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b g(s)f\left(\sigma^2(s)\right)\Delta^\alpha s$$
$$+\frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b Q(t;s)T_\alpha f(\sigma(s))\Delta^\alpha s. \tag{3.8}$$

Adding (3.7) and (3.8), we get

$$(1-\lambda)\left(f(t)+f(\sigma(t))\right) = -\lambda\left[\frac{\int_a^t g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}\left(f(a)+f(\sigma(a))\right)+\frac{\int_t^b g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}\left(f(b)+f(\sigma(b))\right)\right]$$
$$+\frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b g(s)\left(f(\sigma(s))+f\left(\sigma^2(s)\right)\right)\Delta^\alpha s+\frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b Q(t;s)\left(T_\alpha f(s)+T_\alpha f(\sigma(s))\right)\Delta^\alpha s. \tag{3.9}$$

Multiplying (3.9) by $T_\alpha f(t)$, using Theorem 2.3 and fractional integrating the result identity on $[a,b]_{\mathbb{T}}$, we have

$$(1-\lambda)\int_a^b\left(f(t)+f(\sigma(t))\right)T_\alpha f(t)\Delta^\alpha t = (1-\lambda)\int_a^b T_\alpha(f(t))^2\Delta^\alpha t$$
$$= -\lambda\Bigg[\frac{\left(f(a)+f(\sigma(a))\right)}{\int_a^b g(s)\Delta^\alpha s}\int_a^b T_\alpha f(t)\left(\int_a^t g(s)\Delta^\alpha s\right)\Delta^\alpha t$$
$$+\frac{\left(f(b)+f(\sigma(b))\right)}{\int_a^b g(s)\Delta^\alpha s}\int_a^b T_\alpha f(t)\left(\int_t^b g(s)\Delta^\alpha s\right)\Delta^\alpha t\,]$$
$$+\frac{1}{\int_a^b g(s)\Delta^\alpha s}.\left(\int_a^b T_\alpha f(t)\Delta^\alpha t\right)\left(\int_a^b g(s)\left(f(\sigma(s))+f\left(\sigma^2(s)\right)\right)\Delta^\alpha s\right)$$
$$+\frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b T_\alpha f(t)\left(\int_a^b Q(t,s)\left(T_\alpha f(s)+T_\alpha f(\sigma(s))\right)\Delta^\alpha s\right)\Delta^\alpha t.$$

Then

$$(1-\lambda)\left(f^2(b)-f^2(a)\right) = -\lambda\Bigg[\frac{\left(f(a)+f(\sigma(a))\right)}{\int_a^b g(s)\Delta^\alpha s}\int_a^b T_\alpha f(t)\left(\int_a^t g(s)\Delta^\alpha s\right)\Delta^\alpha t$$
$$+\frac{\left(f(b)+f(\sigma(b))\right)}{\int_a^b g(s)\Delta^\alpha s}\int_a^b T_\alpha f(t)\left(\int_t^b g(s)\Delta^\alpha s\right)\Delta^\alpha t\,]$$
$$+\frac{\left(f(b)-f(a)\right)}{\int_a^t g(s)\Delta^\alpha s}.\left(\int_a^b g(s)\left(f(\sigma(s))+f\left(\sigma^2(s)\right)\right)\Delta^\alpha s\right)$$
$$+\frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b T_\alpha f(t)\left(\int_a^b Q(t,s)\left(T_\alpha f(s)+T_\alpha f(\sigma(s))\right)\Delta^\alpha s\right)\Delta^\alpha t. \tag{3.10}$$

Using the fractional integrating by parts(2.1) we have

$$\int_a^b T_\alpha f(t)\left(\int_a^t g(s)\Delta^\alpha s\right)\Delta^\alpha t=\left[f(t)\int_a^t g(s)\Delta^\alpha s\right]_a^b-\int_a^b T_\alpha\left(\int_a^t g(s)\Delta^\alpha s\right).f^\sigma(t)\Delta^\alpha t$$
$$=f(b)\int_a^b g(s)\Delta^\alpha s-\int_a^b g(t)f(\sigma(t))\Delta^\alpha t \tag{3.11}$$

and

$$\int_a^b T_\alpha f(t)\left(\int_t^b g(s)\Delta^\alpha s\right)\Delta^\alpha t=-f(a)\int_a^b g(s)\Delta^\alpha s+\int_a^b g(t)f(\sigma(t))\Delta^\alpha t \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10),

$$\begin{aligned}(1-\lambda)\left(f^2(b)-f^2(a)\right)&=-\lambda\left[\frac{(f(a)+f(\sigma(a)))}{\int_a^b g(s)\Delta^\alpha s}\left(f(b)\int_a^b g(s)\Delta^\alpha s-\int_a^b g(t)f(\sigma(t))\Delta^\alpha t\right)\right.\\&+\frac{(f(b)+f(\sigma(b)))}{\int_a^b g(s)\Delta^\alpha s}\left(-f(a)\int_a^b g(s)\Delta^\alpha s+\int_a^b g(t)f(\sigma(t))\Delta^\alpha t\right]\\&+\frac{(f(b)-f(a))}{\int_a^b g(s)\Delta^\alpha s}.\left(\int_a^b g(s)\left(f(\sigma(s))+f\left(\sigma^2(s)\right)\right)\Delta^\alpha s\right.\\&+\frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b T_\alpha f(t)\left(\int_a^b Q(t,s)\left(T_\alpha f(s)+T_\alpha f(\sigma(s))\right)\Delta^\alpha s\right)\Delta^\alpha t.\end{aligned}$$

Then

$$\begin{aligned}&(1-\lambda)\left(f^2(b)-f^2(a)\right)+\left[(\lambda-1)\frac{(f(b)-f(a))}{\int_a^b g(s)\Delta^\alpha s}+\lambda.\frac{(f(\sigma(b))-f(\sigma(a)))}{\int_a^b g(s)\Delta^\alpha s}\right].\int_a^b g(s)f^\sigma(s)\Delta^\alpha s\\&+\lambda.\left(f(\sigma(a))f(b)-f(\sigma(b))f(a)\right)-\frac{(f(b)-f(a))}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b g(s)f(\sigma^2(s))\Delta^\alpha s\\&=\frac{1}{\int_a^b g(s)\Delta^\alpha s}.\left(\int_a^b T_\alpha f(t)\left(\int_a^b Q(t,s)\left(T_\alpha f(s)+T_\alpha f(\sigma(s))\right)\Delta^\alpha s\right)\Delta^\alpha t.\end{aligned}$$

This implies that

$$\begin{aligned}&\left|(1-\lambda)\left(f^2(b)-f^2(a)\right)+\left[(\lambda-1)\frac{(f(b)-f(a))}{\int_a^b g(s)\Delta^\alpha s}+\lambda.\frac{(f(\sigma(b))-f(\sigma(a)))}{\int_a^b g(s)\Delta^\alpha s}\right].\int_a^b g(s)f^\sigma(s)\Delta^\alpha s\right.\\&\left.+\lambda.\left(f(\sigma(a))f(b)-f(\sigma(b))f(a)\right)-\frac{(f(b)-f(a))}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b g(s)f(\sigma^2(s))\Delta^\alpha s\right|\\&\le\frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b\left|T_\alpha f(t)\right|\left(\int_a^b\left|Q(t,s)\right|\left(\left|T_\alpha f(s)\right|+\left|T_\alpha f(\sigma(s))\right|\right)\Delta^\alpha s\right)\Delta^\alpha t\\&\le\frac{K(K+K^\sigma)}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b\left(\int_a^b\left|Q(t,s)\right|\Delta^\alpha s\right)\Delta^\alpha t.\end{aligned}$$

Which is the desired inequality (3.6) the proof is complete.

Corollary 3.3.2

In the case of $\mathbb{T}=\mathbb{R}$ in Theorem 3.3.1, we have

$$\left|(1-\lambda)\frac{\left(f^2(b)-f^2(a)\right)}{2}+(\lambda-1)\frac{(f(b)-f(a))}{\int_a^b g(s)d_\alpha s}\int_a^b g(s)f(s)d_\alpha s\right|\le\frac{K^2}{\int_a^b g(s)d_\alpha s}\int_a^b\left(\int_a^b\left|Q(t,s)\right|d_\alpha s\right)d_\alpha t,$$

Where $K=\sup_{a\le t\le b}\left|D^\alpha f(t)\right|<\infty$, and

$$Q(t,s)=\begin{cases}\varphi(s)-[(1-\lambda)\varphi(a)+\lambda\varphi(t)]\ ;\ a\le s<t,\\ \varphi(s)-[\lambda\varphi(t)+(1-\lambda)\varphi(b)]\ ;\ t\le s\le b.\end{cases}$$

Corollary 3.3.3

In the case of $\mathbb{T}=h\mathbb{Z}, h>0$ in Theorem 3.3.1, we have

$$\left|(1-\lambda)\left(f^2(b)-f^2(a)\right)+\left[(\lambda-1)\frac{f(b)-f(a)}{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}g(hk)}+\lambda\frac{f(b+h)-f(a+h)}{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}g(hk)}\right]\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}g(hk)f(h(k+1))\right.$$

$$\left.+\lambda\left(f(a+h)f(b)-f(b+h)f(a)\right)-\frac{f(b)-f(a)}{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}g(hk)}\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}g(hk)f(h(k+2))\right|$$

$$\leq\frac{K(K+K^{\sigma})h^{\sigma}}{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}g(hk)}.\sum_{i=\frac{a}{h}}^{\frac{b}{h}-1}\sum_{j=\frac{a}{h}}^{\frac{b}{h}-1}(ij)^{\alpha-1}\left|Q(h.i;h.j)\right|$$

Where $K=\sup_{t\in[a,b]_{h\mathbb{Z}}}\left|\frac{t^{1-\alpha}(f(t+h)-f(t))}{h}\right|<\infty$, and $K^{\sigma}=\sup_{t\in[a,b]_{h\mathbb{Z}}}\left|\frac{t^{1-\alpha}(f(t+2h)-f(t+h))}{h}\right|<\infty$, and

$$Q(t,s)=\begin{cases}\varphi(s)-[(1-\lambda)\varphi(a)+\lambda\varphi(t)]\ ;\ s\in\{a,a+h,...t-h\},\\ \varphi(s)-[\lambda\varphi(t)+(1-\lambda)\varphi(b)]\ ;\ s\in\{t,t+h,...b\}.\end{cases}$$

Corollary 3.3.4
In the case of $\mathbb{T}=q^{\mathbb{N}},\ q>1\ a=q^m,\ and\, b=q^n$ with $m<n$ in Theorem 3.3.1, we have

$$\left|(1-\lambda)\left(f^2(q^n)-f^2(q^m)\right)+\left[(\lambda-1)\frac{f(q^n)-f(q^m)}{\sum_{j=m}^{n-1}q^{\alpha j}g(q^j)}+\lambda\frac{f(q^{n+1})-f(q^{m+1})}{\sum_{j=m}^{n-1}q^{\alpha j}g(q^j)}\right]\sum_{j=m}^{n-1}q^{\alpha j}g(q^j)f(q^{j+1})\right.$$

$$\left.+\lambda\left(f(q^{m+1})f(q^n)-f(q^{n+1})f(q^m)\right)-\frac{f(q^n)-f(q^m)}{\sum_{j=m}^{n-1}q^{\alpha j}g(q^j)}.\sum_{j=m}^{n-1}q^{\alpha j}g(q^j)f(q^{j+2})\right|$$

$$\leq\frac{K(K+K^{\sigma})(q-1)}{\sum_{j=m}^{n-1}q^{\alpha j}g(q^j)}.\sum_{i=m}^{n-1}\sum_{j=m}^{n-1}q^{\alpha(i+j)}\left|Q(q^i;q^j)\right|$$

Where $K=\sup_{m\leq k\leq n}\left|\frac{f(q^{1+k})-f(q^k)}{(q-1)q^{\alpha k}}\right|<\infty$, and $K^{\sigma}=\sup_{m\leq k\leq n}\left|\frac{f(q^{k+2})-f(q^{k+1})}{(q-1)q^{\alpha k}}\right|<\infty$, and

$$Q(q^i,q^j)=\begin{cases}\varphi(q^j)-\left[(1-\lambda)\varphi(q^m)+\lambda\varphi(q^i)\right]\ ;\ m\leq j<i,\\ \varphi(q^j)-\left[\lambda\varphi(q^i)+(1-\lambda)\varphi(q^n)\right]\ ;\ i\leq j\leq n.\end{cases}$$

Using (3.5) we have the following Corollary

Corollary 3.3.5
In the case of $\varphi(t)=h_{\alpha,1}(t;a)$ in Theorem 3.3.1, we have

$$\left|(1-\lambda)\left(f^2(b)-f^2(a)\right)+\left[(\lambda-1)\frac{f(b)-f(a)}{h_{\alpha,1}(b,a)}+\lambda\frac{f(\sigma(b))-f(\sigma(a))}{h_{\alpha,1}(b,a)}\right]\int_a^b f^{\sigma}(s)\Delta^{\alpha}s\right.$$

$$\left.+\lambda\left(f(\sigma(a))f(b)-f(\sigma(b))f(a)\right)-\frac{f(b)-f(a)}{h_{\alpha,1}(b,a)}\int_a^b f^{\sigma 2}(s)\Delta^{\alpha}s\right|$$

$$\leq K(K+K^{\sigma})\left[h_{\alpha,2}(b,a)+\lambda\left(h_{\alpha,1}(b,a)\right)^2+h_{\alpha,2}(a,b)\right]$$

Where $K=\sup_{t\in[a,b]_{\mathbb{T}}}\left|T_{\alpha}f(t)\right|<\infty$, and $K^{\sigma}=\sup_{t\in[a,b]_{\mathbb{T}}}\left|T_{\alpha}f(\sigma(t))\right|<\infty$.

Corollary 3.3.6
In the case of $\mathbb{T}=\mathbb{R}$ in Corollary 3.3.5, we have

$$\left|\frac{1}{2}(1-\lambda)\left(f^2(b)-f^2(a)\right)+\frac{\alpha(\lambda-1)}{b^{\alpha}-a^{\alpha}}\left(f(b)-f(a)\right)\int_a^b f(s)d_{\alpha}s\right|\leq K^2\left(\frac{b^{\alpha}-a^{\alpha}}{\alpha}\right)^2(1+\lambda)$$

Where $K = \sup_{t\in[a,b]_{\mathbb{T}}} \left|D^\alpha f(t)\right| < \infty$.

Corollary 3.3.7
In the case of $\mathbb{T} = q^{\mathbb{N}},\ q > 1\ a = q^m, and\ b = q^n$ with $m < n$ in Corollary 3.3.5, we have

$$\Bigg|(1-\lambda)\left(f^2(q^n) - f^2(q^m)\right) + \left[(\lambda-1)\frac{f(q^n) - f(q^m)}{q^{\alpha n} - q^{\alpha m}} + \lambda\frac{f(q^{n+1}) - f(q^{m+1})}{q^{\alpha n} - q^{\alpha m}}\right](q^\alpha - 1)\sum_{k=m}^{n-1} q^{\alpha k} f(q^{k+1})$$
$$+\lambda\left(f(q^{m+1})f(q^n) - f(q^{n+1})f(q^m)\right) - (q^\alpha - 1)\frac{\sum_{k=m}^{n-1} q^{\alpha k} f(q^{k+2})\left(f(q^n) - f(q^m)\right)}{q^{\alpha n} - q^{\alpha m}}\Bigg|$$
$$\leq K(K + K^\sigma)\frac{(q-1)^2}{(q^\alpha - 1)^2}\left(q^{\alpha n} - q^{\alpha m}\right)^2(1+\lambda)$$

Where $K = \sup_{m\leq k\leq n}\left|\frac{f(q^{k+1}) - f(q^k)}{(q-1)q^{\alpha k}}\right| < \infty$, and $K^\sigma = \sup_{m\leq k\leq n}\left|\frac{f(q^{k+2}) - f(q^{k+1})}{(q-1)q^{\alpha k}}\right| < \infty$.

Corollary 3.3.8
In the case of $\mathbb{T} = h\mathbb{Z}, h > 0$ in Corollary 3.3.5, we have

$$\Bigg|(1-\lambda)\left(f^2(b) - f^2(a)\right) + \left[(\lambda-1)\frac{f(b) - f(a)}{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1}} + \lambda\frac{f(b+h) - f(a+h)}{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1}}\right]\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1} f(h(k+1))$$
$$+\lambda\left(f(a+h)f(b) - f(b+h)f(a)\right) - \frac{f(b) - f(a)}{\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1}}\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} k^{\alpha-1} f(h(k+2))\Bigg|$$
$$\leq h^{2\alpha}K(K + K^\sigma)\left[\sum_{i=\frac{a}{h}}^{\frac{b}{h}-1}\sum_{j=\frac{a}{h}}^{i-1}(ij)^{\alpha-1} + \lambda\left(\sum_{i=\frac{a}{h}}^{\frac{b}{h}-1} i^{\alpha-1}\right)^2 + \sum_{i=\frac{a}{h}-1}^{\frac{b}{h}}\sum_{j=i-1}^{\frac{b}{h}}(ij)^{\alpha-1}\right]$$

Where $K = \sup_{t\in[a,b]_{h\mathbb{Z}}}\left|\frac{t^{1-\alpha}(f(t+h) - f(t))}{h}\right| < +\infty$, and $K^\sigma = \sup_{t\in[a,b]_{h\mathbb{Z}}}\left|\frac{t^{1-\alpha}(f(t+2h) - f(t+h))}{h}\right| < \infty$.

Corollary 3.3.9
If we let $\alpha = 1$ in Corollary 3.3.8, we have

$$\Bigg|(1-\lambda)\left(f^2(b) - f^2(a)\right) + \left[h(\lambda-1)\frac{(f(b) - f(a))}{b-a} + h\lambda\frac{(f(b+h) - f(a+h))}{b-a}\right]\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(h(k+1))$$
$$+\lambda\left(f(a+h)f(b) - f(b+h)f(a)\right) - \frac{h(f(b) - f(a))}{b-a}\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(h(k+2))\Bigg|$$
$$\leq h^{2\alpha}K(K + K^\sigma)\left[(1+\lambda)\left(\frac{b-a}{h}\right)^2 + \frac{3(b-a)}{h} + 5\right]$$

Where $K = \sup_{t\in[a,b]_{h\mathbb{Z}}}\left|\frac{(f(t+h) - f(t))}{h}\right| < \infty$, and $K^\sigma = \sup_{t\in[a,b]_{h\mathbb{Z}}}\left|\frac{(f(t+h) - f(t+h))}{h}\right| < \infty$.

3.4 Weighted fractional Grss type inequality on time scales

Theorem 3.4.1
Let $0 \leq \lambda \leq 1$, $g : [a,b]_{\mathbb{T}} \to [0, +\infty)$ be rd-continous and positive and $\varphi : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be conformable differentiable of order $\alpha \in (0,1]$ such that $T_\alpha\varphi(t) = g(t)\ on\ [a,b]_{\mathbb{T}}$.

Let $\Psi; \Phi\ : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be conformable differentiable of order $\alpha \in (0,1]$, then we have

$$\Bigg| 2(1-\lambda).\left(\int_a^b g(t)\Delta^\alpha t\right).\left(\int_a^b \Psi(t).\Phi(t)\Delta^\alpha t\right) + \lambda.\int_a^b \Bigg\{\Phi(t)\left[\left(\int_a^t g(s)\Delta^\alpha s\right)\Psi(a) + \left(\int_t^a g(s)\Delta^\alpha s\right)\Psi(b)\right] + \Psi(t)\left[\left(\int_a^t g(s)\Delta^\alpha s\right)\Phi(a) + \left(\int_t^b g(s)\Delta^\alpha s\right)\Phi(b)\right]\Bigg\}\Delta^\alpha t - \left[\left(\int_a^b \Phi(t)\Delta^\alpha t\right)\left(\int_a^b g(t)\Psi(\sigma(t))\Delta^\alpha t\right) + \left(\int_a^b \Psi(t)\Delta^\alpha t\right)\left(\int_a^b g(t)\Phi(\sigma(t))\Delta^\alpha t\right)\right]\Bigg| \le \int_a^b (M.|\Phi(t)| + N|\Psi(t)|).\left(\int_a^b |Q(t,s)|\Delta^\alpha s\right)\Delta^\alpha t \tag{3.13}$$

Where

$$Q(t,s) = \begin{cases} \varphi(s) - [(1-\lambda)\varphi(a) + \lambda\varphi(t)]\ ;\ a \le s < t, \\ \varphi(s) - [\lambda\varphi(t) + (1-\lambda)\varphi(b)]\ ;\ t \le s \le b. \end{cases}$$

and $M = \sup_{t\in[a,b]_{\mathbb{T}}} \left|T_\alpha \Psi(t)\right| < \infty$, and $N = \sup_{t\in[a,b]_{\mathbb{T}}} \left|T_\alpha \Phi(t)\right| < \infty$.

Proof
From Lemma 3.1.1, we have

$$(1-\lambda)\Psi(t) = -\lambda\left[\frac{\int_a^t g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}\Psi(a) + \frac{\int_t^b g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}\Psi(b)\right] + \frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b g(s)\Psi(\sigma(s))\Delta^\alpha s + \frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b Q(t;s)T_\alpha \Psi(s)\Delta^\alpha s. \tag{3.14}$$

and

$$(1-\lambda)\Phi(t) = -\lambda\left[\frac{\int_a^t g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}\Phi(a) + \frac{\int_t^b g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}\Phi(b)\right] + \frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b g(s)\Phi(\sigma(s))\Delta^\alpha s + \frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b Q(t;s)T_\alpha \Phi(s)\Delta^\alpha s. \tag{3.15}$$

Multiplying (3.14) by $\Phi(t)$ and (3.15) by $\Psi(t)$, adding and then fractional integrating the result from a to b, we have

$$\begin{aligned} 2(1-\lambda)\int_a^b \Psi(t).\Phi(t)\Delta^\alpha t &= -\lambda.\int_a^b \Bigg\{\Phi(t)\left[\frac{\int_a^t g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}\Psi(a) + \frac{\int_t^b g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}\Psi(b)\right] \\ &+ \Psi(t)\left[\frac{\int_a^t g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}\Phi(a) + \frac{\int_t^b g(s)\Delta^\alpha s}{\int_a^b g(s)\Delta^\alpha s}\Phi(b)\right]\Bigg\}\Delta^\alpha t \\ &+ \frac{1}{\int_a^b g(s)\Delta^\alpha s}.\left(\int_a^b \Phi(t)\Delta^\alpha s\right).\left(\int_a^b g(s)\Psi(\sigma(s))\Delta^\alpha s\right) \\ &+ \frac{1}{\int_a^b g(s)\Delta^\alpha s}.\left(\int_a^b \Psi(t)\Delta^\alpha s\right).\left(\int_a^b g(s)\Phi(\sigma(s))\Delta^\alpha s\right) \\ &+ \frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b \Phi(t)\left(\int_a^b Q(t;s)T_\alpha \Psi(s)\Delta^\alpha s\right)\Delta^\alpha t \\ &+ \frac{1}{\int_a^b g(s)\Delta^\alpha s}.\int_a^b \Psi(t)\left(\int_a^b Q(t;s)T_\alpha \Phi(s)\Delta^\alpha s\right)\Delta^\alpha t \end{aligned} \tag{3.16}$$

From (3.16), we get

$$\Bigg|2(1-\lambda)\left(\int_a^b g(t)\Delta^\alpha t\right)\left(\int_a^b \Psi(t).\Phi(t)\Delta^\alpha t\right)$$
$$+\lambda.\int_a^b\Bigg\{\Phi(t)\left[\left(\int_a^t g(s)\Delta^\alpha s\right)\Psi(a)+\left(\int_t^b g(s)\Delta^\alpha s\right)\Psi(b)\right]$$
$$+\Psi(t)\left[\left(\int_a^t g(s)\Delta^\alpha s\right)\Phi(a)+\left(\int_t^b g(s)\Delta^\alpha s\right)\Phi(b)\right]\Bigg\}\Delta^\alpha t$$
$$-\left[\left(\int_a^b \Phi(t)\Delta^\alpha s\right).\left(\int_a^b g(s)\Psi(\sigma(s))\Delta^\alpha s\right)+\left(\int_a^b \Psi(t)\Delta^\alpha s\right).\left(\int_a^b g(s)\Phi(\sigma(s))\Delta^\alpha s\right)\right]\Bigg|$$
$$\leq \int_a^b|\Phi(t)|\left(\int_a^b|Q(t;s)||T_\alpha\Psi(s)|\Delta^\alpha s\right)\Delta^\alpha t+\int_a^b|\Psi(t)|\left(\int_a^b|Q(t;s)||T_\alpha\Phi(s)|\Delta^\alpha s\right)\Delta^\alpha t$$
$$\leq \int_a^b M|\Phi(t)|\left(\int_a^b|Q(t;s)|\Delta^\alpha s\right)\Delta^\alpha t+\int_a^b N|\Psi(t)|\left(\int_a^b|Q(t;s)|\Delta^\alpha s\right)\Delta^\alpha t$$
$$=\int_a^b M|\Phi(t)|+N|\Psi(t)|\left(\int_a^b|Q(t;s)|\Delta^\alpha s\right)\Delta^\alpha t.$$

This completes the proof of the inequality (3.13).

Corollary 3.4.2

In the case of $\mathbb{T}=\mathbb{R}$ in Theorem 3.4.1, we have

$$\Bigg|2(1-\lambda)\left(\int_a^b g(t)d_\alpha t\right)\left(\int_a^b \Psi(t).\Phi(t)d_\alpha t\right)$$
$$+\lambda.\int_a^b\Bigg\{\Phi(t)\left[\left(\int_a^t g(s)d_\alpha s\right)\Psi(a)+\left(\int_t^b g(s)d_\alpha s\right)\Psi(b)\right]$$
$$+\Psi(t)\left[\left(\int_a^t g(s)d_\alpha s\right)\Phi(a)+\left(\int_t^b g(s)d_\alpha s\right)\Phi(b)\right]\Bigg\}d_\alpha t$$
$$-\left[\left(\int_a^b \Phi(t)d_\alpha t\right).\left(\int_a^b g(t)\Psi((t)d_\alpha t\right)+\left(\int_a^b \Psi(t)d_\alpha t\right).\left(\int_a^b g(t)\Phi(t)d_\alpha s\right)\right]\Bigg|$$
$$\leq \int_a^b (M|\Phi(t)|+N|\Psi(t)|)\left(\int_a^b|Q(t;s)|d_\alpha t\right)d_\alpha t,$$

Where

$$Q(t,s)=\begin{cases}\varphi(s)-[(1-\lambda)\varphi(a)+\lambda\varphi(t)]\ ;\ a\leq s<t,\\ \varphi(s)-[\lambda\varphi(t)+(1-\lambda)\varphi(b)]\ ;\ t\leq s\leq b.\end{cases}$$

and

$$M=\sup_{t\in[a,b]}\left|D^\alpha\Psi(t)\right|<\infty.\ and\ N=\sup_{t\in[a,b]}\left|D^\alpha\Phi(t)\right|<\infty.$$

Corollary 3.4.3

In the case of $\mathbb{T}=h\mathbb{Z}; h>0$ in Theorem 3.4.1, we have

$$\Bigg|2(1\text{-}\lambda)\left(\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}g(hk)\right)\left(\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}\Psi(hk).\Phi(hk)\right)$$
$$+\lambda.\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}\Bigg\{k^{\alpha-1}\Phi(hk)\left[\left(\sum_{j=\frac{a}{h}}^{k-1}j^{\alpha-1}g(hj)\right)\Psi(a)+\left(\sum_{j=k}^{\frac{b}{h}-1}j^{\alpha-1}g(hj)\right)\Psi(b)\right]$$
$$+k^{\alpha-1}\Psi(hk)\left[\left(\sum_{j=\frac{a}{h}}^{k-1}j^{\alpha-1}g(hj)\right)\Phi(a)+\left(\sum_{j=k}^{\frac{b}{h}-1}j^{\alpha-1}g(hj)\right)\Phi(b)\right]\Bigg\}$$
$$-\left[\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}\Phi(hk).\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}g(hk)\Psi(h(k+1))+\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}\Psi(hk).\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}g(hk)\Phi(h(k+1))\right]\Bigg|$$
$$\leq\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}\left(M|\Phi(hk)|+N|\Psi(hk)|\right)\left(\sum_{j=\frac{a}{h}}^{\frac{b}{h}-1}j^{\alpha-1}|Q(hk;hj)|\right),$$

Where

$$Q(t,s)=\begin{cases}\varphi(s)-[(1-\lambda)\varphi(a)+\lambda\varphi(t)]\ ;\ \ s\in\{a,a+h,...t-h\},\\ \varphi(s)-[\lambda\varphi(t)+(1-\lambda)\varphi(b)]\ ;\ \ s\in\{t,t+h,...b\}.\end{cases}$$

and $M = \sup_{t\in[a,b]_{h\mathbb{Z}}} \left|\frac{t^{1-\alpha}(\Psi(t+h)-\Psi(t))}{h}\right| < \infty$, and $N = \sup_{t\in[a,b]_{h\mathbb{Z}}} \left|\frac{t^{1-\alpha}(\Phi(t+h)-\Phi(t))}{h}\right| < \infty$.

Corollary 3.4.4

In the case of $\mathbb{T} = q^{\mathbb{N}}; q > 1$, $a = q^m$ and $b = q^n$ with $m < n$ in Theorem 3.4.1, we have

$$\begin{aligned}
&\Bigg|2(1-\lambda)\left(\sum_{k=m}^{n-1} q^{\alpha k} g(q^k)\right)\left(\sum_{k=m}^{n-1} q^{\alpha k}\Psi(q^k).\Phi(q^k)\right)\\
&+\lambda.\sum_{k=m}^{n-1} q^{\alpha k}\Phi(q^k)\left[\left(\sum_{j=m}^{k-1} q^{\alpha j} g(q^j)\right)\Psi(a) + \left(\sum_{j=k}^{n-1} q^{\alpha j} g(q^j)\right)\Psi(b)\right]\\
&+q^{\alpha k}\Psi(q^k)\left[\left(\sum_{j=m}^{k-1} q^{\alpha j} g(q^j)\right)\Phi(a) + \left(\sum_{j=k}^{n-1} q^{\alpha j} g(q^j)\right)\Phi(b)\right]\Bigg\}\\
&-\left[\left(\sum_{k=m}^{n-1} q^{\alpha k}\Phi(q^k)\right).\left(\sum_{k=m}^{n-1} q^{\alpha k} g(q^k)\Psi(q^{k+1})\right) + \left(\sum_{k=m}^{n-1} q^{\alpha k}\Psi(q^k)\right).\left(\sum_{k=m}^{n-1} q^{\alpha k} g(q^k)\Phi(q^{k+1}))\right)\right]\Bigg|\\
&\le \sum_{k=m}^{n-1} q^{\alpha k}\left(M|\Phi(q^k)| + N|\Psi(q^k)|\right)\left(\sum_{j=m}^{n-1} q^{\alpha j}|Q(q^k;q^j)|\right),
\end{aligned}$$

Where

$$Q(q^i,q^j) = \begin{cases}\varphi(q^j) - \left[(1-\lambda)\varphi(q^m) + \lambda\varphi(q^i)\right] \;;\; m \le j < i,\\ \varphi(q^j) - \left[\lambda\varphi(q^i) + (1-\lambda)\varphi(q^n)\right] \;;\; i \le j \le n.\end{cases}$$

and $M = \sup_{m\le k\le n}\left|\frac{\Psi(q^{k+1})-\Psi(q^k))}{(q-1)q^{\alpha k}}\right| < \infty$, and $N = \sup_{m\le k\le n}\left|\frac{\Phi(q^{k+1})-\Phi(q^k))}{(q-1)q^{\alpha k}}\right| < \infty$.

Using the inequality (3.5) we have the following Corollary

Corollary 3.4.5

In the case of $\varphi(t) = h_{\alpha,1}(t,a)$ in Theorem 3.4.1, we have

$$\begin{aligned}
&\Bigg|2(\lambda-1)h_{\alpha,1}(b,a)\int_a^b \Psi(t).\Phi(t)d_\alpha t\\
&+\lambda.\int_a^b \left\{\Phi(t)\left[h_{\alpha,1}(t,a)\Psi(a) + h_{\alpha,1}(b,t)\Psi(b)\right] + \Psi(t)\left[h_{\alpha,1}(t,a)\Phi(a) + h_{\alpha,1}(b,t)\Phi(b)\right]\right\}\Delta^\alpha t\\
&-\left[\left(\int_a^b \Phi(t)\Delta^\alpha t\right).\left(\int_a^b \Psi(\sigma(t))\Delta^\alpha t\right) + \left(\int_a^b \Psi(t)\Delta^\alpha t\right).\left(\int_a^b \Phi(\sigma(t))\Delta^\alpha s\right)\right]\Bigg|\\
&\le \int_a^b (M|\Phi(t)| + N|\Psi(t)|)\left[h_{\alpha,2}(b,a) + \lambda(h_{\alpha,1}(b,a))^2 + h_{\alpha,1}(b,a)h_{\alpha,1}(b,t)\right]\Delta^\alpha t,
\end{aligned}$$

Where $M = \sup_{t\in[a,b]_{\mathbb{T}}}\left|T_\alpha\Psi(t)\right| < \infty$, and $N = \sup_{t\in[a,b]_{\mathbb{T}}}\left|T_\alpha\Phi(t)\right| < \infty$.

Corollary 3.4.6

In the case of $\mathbb{T} = \mathbb{R}$ in Corollary 3.4.5, we have

$$\begin{aligned}
&\Bigg|2(1-\lambda)(b^\alpha - a^\alpha)\int_a^b \Psi(t).\Phi(t)d_\alpha t) + \lambda.\int_a^b \Big\{\Phi(t)\left[(t^\alpha - a^\alpha)\Psi(a) + (b^\alpha - t^\alpha)\Psi(b)\right]\\
&+\Psi(t)\left[(t^\alpha - a^\alpha)\Phi(a) + (b^\alpha - t^\alpha)\Phi(b)\right]\Big\}d_\alpha t - 2\alpha\left(\int_a^b \Phi(t)d_\alpha t\right)\left(\int_a^b \Psi(t)d_\alpha t\right)\Bigg|\\
&\le \left(\frac{b^\alpha - t^\alpha}{\alpha}\right)\int_a^b (M|\Phi(t)| + N|\Psi(t)|)\left[\left(\lambda + \frac{1}{2})\right)(b^\alpha - a^\alpha) + (b^\alpha - t^\alpha)\right]d_\alpha t,
\end{aligned}$$

Where $M = \sup_{t\in[a,b]}\left|D^\alpha\Psi(t)\right| < \infty$, and $N = \sup_{t\in[a,b]}\left|D^\alpha\Phi(t)\right| < \infty$.

Corollary 3.4.7

In the case of $\lambda = 0$ in Corollary 3.4.6, we have

$$\begin{aligned}
&\left|\int_a^b \Psi(t).\Phi(t)d_\alpha t) - \frac{\alpha}{b^\alpha - a^\alpha}\left(\int_a^b \Psi(t)d_\alpha t\right)\left(\int_a^b \Phi(t)d_\alpha t\right)\right|\\
&\le \frac{1}{2\alpha}\int_a^b (M|\Phi(t)| + N|\Psi(t)|)\left[\frac{1}{2}(b^\alpha - a^\alpha) + (b^\alpha - t^\alpha)\right]d_\alpha t,
\end{aligned}$$

Where $M = \sup_{t\in[a,b]}\left|D^\alpha\Psi(t)\right| < \infty$, and $N = \sup_{t\in[a,b]}\left|D^\alpha\Phi(t)\right| < \infty$

Corollary 3.4.8

In the case of $\mathbb{T}=h\mathbb{Z}; h>0$ in Theorem 3.4.5, we have

$$\Bigg|2(1-\lambda)\left(\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}\right)\left(\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}\Psi(hk).\Phi(hk)\right)$$
$$+\lambda.\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}\Bigg\{\Phi(hk)\left[\left(\sum_{j=\frac{a}{h}}^{k-1}j^{\alpha-1}\right)\Psi(a)+\left(\sum_{j=k}^{\frac{b}{h}-1}j^{\alpha-1}\right)\Psi(b)\right]$$
$$+\Psi(hk)\left[\left(\sum_{j=\frac{a}{h}}^{k-1}j^{\alpha-1}\right)\Phi(a)+\left(\sum_{j=k}^{\frac{b}{h}-1}j^{\alpha-1}\right)\Phi(b)\right]\Bigg\}$$
$$-\left[\left(\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}\Phi(hk)\right).\left(\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}\Psi(h(k+1))\right)+\left(\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}\Psi(hk)\right).\left(\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}\Phi(h(k+1)))\right)\right]\Bigg|$$
$$\leq h^{\alpha}\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}k^{\alpha-1}\left(M|\Phi(hk)|+N|\Psi(hk)|\right)\left[\sum_{i=\frac{a}{h}}^{\frac{b}{h}-1}\sum_{j=\frac{a}{h}}^{i-1}(ij)^{\alpha-1}+\lambda\left(\sum_{j=\frac{a}{h}}^{\frac{b}{h}-1}j^{\alpha-1}\right)^2+\left(\sum_{j=\frac{a}{h}}^{\frac{b}{h}-1}j^{\alpha-1}\right)\left(\sum_{j=k}^{\frac{b}{h}-1}j^{\alpha-1}\right)\right],$$

Where $M=\sup_{t\in[a,b]_{h\mathbb{Z}}}\left|\frac{t^{1-\alpha}(\Psi(t+h)-\Psi(t))}{h}\right|<\infty$, and $N=\sup_{t\in[a,b]_{h\mathbb{Z}}}\left|\frac{t^{1-\alpha}(\Phi(t+h)-\Phi(t))}{h}\right|<\infty.$

Corollary 3.4.9
If we let $\alpha=1$ in Corollary 3.4.8, we have

$$\Bigg|2(1-\lambda)\left(\frac{b-a}{h}\right)\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}\Psi(hk).\Phi(hk))+\lambda.\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}\Bigg\{\Phi(hk)\left[\Psi(a)\left(k-\frac{a}{h}\right)+\Psi(b)\left(\frac{b}{h}-k\right)\right]$$
$$+\Psi(hk)\left[\Phi(a)\left(k-\frac{a}{h}\right)+\Phi(b)\left(\frac{b}{h}-k\right)\right]\Bigg\}$$
$$-\left[\left(\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}\Phi(hk)\right).\left(\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}\Psi(h(k+1))\right)+\left(\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}\Psi(hk)\right).\left(\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}\Phi(h(k+1))\right)\right]\Bigg|$$
$$\leq(b-a)\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}\left(M|\Phi(hk)|+N|\Psi(hk)|\right)\left[\left(\frac{b-a}{h}\right)\left(\lambda+\frac{1}{2}\right)+\frac{b}{h}-k-\frac{1}{2}\right],$$

Where $M=\sup_{t\in[a,b]_{h\mathbb{Z}}}\left|\frac{t^{1-\alpha}(\Psi(t+h)-\Psi(t))}{h}\right|<\infty$, and $N=\sup_{t\in[a,b]_{h\mathbb{Z}}}\left|\frac{t^{1-\alpha}(\Phi(t+h)-\Phi(t))}{h}\right|<\infty.$

Corollary 3.4.10
In the case of $\mathbb{T}=q^{\mathbb{N}},\ q>1$,$a=q^m$ and $b=q^n$ with $m<n$ in Corollary 3.4.5, we have

$$\Bigg|2(1-\lambda)\left(\frac{q^{\alpha n}-q^{\alpha m}}{q^{\alpha}-1}\right)\sum_{k=m}^{n-1}q^{\alpha k}\Psi(q^k).\Phi(q^k)$$
$$+\lambda.\sum_{k=m}^{n-1}\Bigg\{q^{\alpha k}\Phi(q^k)\left[\Psi(q^m)\left(\frac{q^{\alpha k}-q^{\alpha m}}{q^{\alpha}-1}\right)+\Psi(q^n)\left(\frac{q^{\alpha n}-q^{\alpha k}}{q^{\alpha}-1}\right)\right]$$
$$+q^{\alpha k}\Psi(q^k)\left[\Phi(q^m)\left(\frac{q^{\alpha k}-q^{\alpha m}}{q^{\alpha}-1}\right)+\Phi(q^n)\left(\frac{q^{\alpha n}-q^{\alpha k}}{q^{\alpha}-1}\right)\right]\Bigg\}$$
$$-\left[\left(\sum_{k=m}^{n-1}q^{\alpha k}\Phi(q^k)\right).\left(\sum_{k=m}^{n-1}q^{\alpha k}\Psi(q^{k+1})\right)+\left(\sum_{k=m}^{n-1}q^{\alpha k}\Psi(q^k)\right).\left(\sum_{k=m}^{n-1}q^{\alpha k}\Phi(q^{k+1}))\right)\right]\Bigg|$$
$$\leq\frac{(q-1)(q^{\alpha n}-q^{\alpha m})}{(q^{\alpha}-1)^2}\sum_{k=m}^{n-1}q^{\alpha k}\left(M|\Phi(q^k)|+N|\Psi(q^k)|\right)\left[\frac{q^{\alpha n}-q^{\alpha(m+1)}}{q^{\alpha}+1}+\lambda(q^{\alpha n}-q^{\alpha m})+(q^{\alpha n}-q^{\alpha k})\right].$$

Where $M=\sup_{m\leq k\leq n}\left|\frac{\Psi(q^{k+1})-\Psi(q^k))}{(q-1)q^{\alpha k}}\right|<\infty$, and $N=\sup_{m\leq k\leq n}\left|\frac{\Phi(q^{k+1})-\Phi(q^k))}{(q-1)q^{\alpha k}}\right|<\infty.$

References

[1] S. S. Dragomir and S. Wang, An inequality of Ostrowski-Grss' type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Computers & Mathematics with Applications* **33 (11)**, 15–20 (1997).

[2] Q. A. Ng W. J. Liu, A sharp Grss type inequality on time scales and application to the sharp Ostrowski-Grss inequality, *Commun. Math. Anal.* **6 (2)**, 33–41 (2009).

[3] M. Bohner and T. Matthews, The Grss inequality on time scales, *Communications in Mathematical Analysis* **3 (1)**, 1–8 (2007).

[4] M. Bohner and T. Matthews, Ostrowski inequalities on time scales, *J.Inequal.Pure Appl.Math* **9 (1)**, 1–8 (2008).

[5] B. Karpuz and U. M. Ozkan, Generalised Ostrowski's Inequality on time scales, *J. Inequal. Pure and Appl. Math* **9 (4)**, 1–15 (2008).

[6] S. Kermausuor, E. R. Nwaeze and D.F.M. Torres, Generalized weighted Ostrowski and Ostrowski-Grss type inequalities on time scales via a parameter function, *J.Math.inequal* **11 (4)**, 1185–1199 (2017).

[7] W. J. LIU and A. NGO, A new generalization of Ostrozski type ineauality on time scales, *An. st. univ. Ovidius constanta* **17**, 101–114 (2009).

[8] W. J. Liu, A. Tuna and Y. Jiang, On weighted Ostrowski type, Tropezoid type Grss type and Ostrowski-Grss like inequalities on time scales, *Appl. anal.* **93 (3)**, 551–571 (2014).

[9] W. J. Liu, A. Tuna, and Y. Jiang, New weighted Ostrowski and Ostrowski-Grss type inequalities on time scales, *An stiint. univ. Al. I. Cusa iasi. Mat. (N.S)* **60 (1)**, 57–76 (2014).

[10] E. R. Nwaeze, A new weighted Ostrowski type inequality on arbitrary time scale, *Journal of King Saud University-Science* **29 (2)**, 230–234 (2017).

[11] E. R. Nwaeze, Generalized Weighted Trapezoid and Grss type inequalities on time scales, *Aust. J. Math. anal. appl.* **14 (4)**, 1–13 (2017).

[12] W. S. Feng, Q. L. Xue and W. J. Liu, Further generalization of Ostrowski-Grss type inequalities, *Adv. Appl. Math. Anal.* **3 (1)**, 17–20 (2008).

[13] A. Tuna and D. Durmus, Generalization of Ostrowski and Ostrowski, Grss type inequalities on time scales, *Computers & Mathematics with Applications* **60 (3)**, 803–811 (2010).

[14] W. J. Liu, Some weighted integral inequalities with a parameter and applications, *Acta applicandae mathematicae* **109 (2)**, 389–400 (2010).

[15] K. Bogolan and B. Dyda, The best constant in a fractional hardy inequality, *Math. Nachr* **284**, 629–638 (2011).

[16] M. Jleli and B. Samet, Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions, *Math.Inequal.Appl.* **18 (2)**, 443–451 (2015).

[17] C. Yildiz, M. E. Ozdemir and H. K. Onelan, Fractional integral inequalities for different functions, *New trends Math. sci* **3 (2)**, 110–117 (2015).

[18] A. Thabet, On conformable fractional calculus, *Journal of computational and Applied Mathematics* **279**, 57–66 (2015).

[19] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, A new definition of fractional derivative, *Journal of computational and applied mathematics* **264**, 65–70 (2014).

[20] M. Z. Sarikaya and H. Budak, New inequalities of Opial type for conformable fractional integrals, *Turkish Journal of Mathematics* **41 (5)**, 1164–1173 (2017).

[21] M. Z. Sarikaya and H. Budak, Opial-type inequalities for conformable fractional integrals, *Journal of Applied Analysis* **25**, 155–163 (2019).

[22] Y. M. Chu, M. A. Khan, S. S. Dragomir and M. Z. Sarikaya, Hermite-Hadamard type inequalities for fractional integral, *Rev. R. Acod: cienc. exactas Fis. Nat. Ser. A Mat RASCAM* **112**, 1033–1048 (2018).

[23] S. Saker, Oscillation Theory of Dynamic Equations on Time Scales: Second and Third Orders, *LAP Lambert Academic Publishing* (2010).

[24] N. Benkhettou, S. Hassani and D. F. Torres, A conformable fractional calculus on arbitrary time scales, *Journal of King Saud University-Science* **28 (1)**, 93–98 (2016).

[25] M. Bohner and A. Peterson, Dynamic equations on time scales: An introduction with applications, *Birkhauser Boston* (2001).

[26] M. Bohner and A. Peterson, Advances in dynamic equations on time scales, *Birkhauser Boston* (2003).

[27] E.R. Nwaeze and D.F. M. Torres, Chain rules and inequalities for the BHT fractional calculus on arbitrary time scales, *Arabian Journal of Mathematics* **6 (1)**, 13–20 (2017).

Author information

Mohamed Lahrouz and Mohamed El Amri, Department of Mathematics, Faculty of science, University Abdelmalek Essaadi, Tetouan, Morocco.
E-mail: m.lahrouz@uae.ac.ma
med.elamri.2021@gmail.com