Palestine Journal of Mathematics

Vol 13(Special Issue I)(2024) , 1–8

Notes on complex symmetric Toeplitz operators on Hardy space and truncated Toeplitz operators

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MSC 2010 Classifications: 46C99,47C99.

Keywords and phrases: Complex symmetric operators, Hardy space, Toeplitz Operator,truncated Toeplitz operators, Model spaces .

Abstract In this paper, we characterize the complex symmetric Toeplitz operator on the Hardy space via a kind of canonical conjugation on H^2 introduced by M.S. Ferreira in [1]. In model space equipped with a conjugation, we prove some results for truncated toeplitz operators in relationship with complex symmetry according to [2].

1 Introduction

The concept of Toeplitz operators is related to the matrix defined by O.Toeplitz whose its name, Toeplitz matrices have infinitely many rows and columns, indexed by non-negative integers, and the coefficients of the matrix are complex numbers. Thus a Toeplitz matrix is determined by a sequence $(a_n)_{n\in\mathbb{Z}}$ of complex numbers, with the coefficients in row i, column j for $(i; j = ..., -1, 0, 1, ...)$ of the Toeplitz matrix equal to a_{ij}. We can think of the Toeplitz matrix above as acting on the usual Hilbert space l^2 of summable sequences into squares of complex numbers, with its standard orthonormal basis. The question then arises of characterizing the sequences (a_n) of complex numbers such that the corresponding Toeplitz matrix is the matrix of an operator bounded on l^2. The answer to this question highlights the fascinating link between Toeplitz operators and complex analysis and theory of functions.

Complex symmetric operators on Hilbert spaces are natural generalizations of complex symmetric matrices, and their general study was initiated by Putinar, Wogen and Garcia in [2],[3]. The class of complex symmetric operators includes a large examples including all Hankel operators, normal operators, the Volterra intetrgal operator and truncated Toeplitz operators.

In the field of applied mathematics, complex symmetric matrices appear in the study of the dynamics of quantum reactions, the modeling of electrical energy, digital simulation of high voltage insulators, propagation of thermoelastic waves, the maximum clique graph theory problem, inverse spectral problems for semi-simple damped vibrating systems, study of decay phenomena, the diffusion matrices in the theory of atomic collisions and the numerical solution of the time-harmonic Maxwell equation in an axi-symmetric cavity. Over the years, complex symmetric matrices have also been the subject of spor-adic digital work.

This paper is organized as following. In Section 2, we collect some preliminaries concerning the notion of complex semmytry in relation to the Toeplitz operators, especially we recall the notion of canonical conjugation introduced by [1], and we close with a result on the symmetry of the matrix associated with such operator. In Section 3, we caracterized truncared toeplitz operators with complex symmetric. We start by citing some basic properties and we finished by demonstrating a statement concerning the complex symmetry of the Truncated operator of Toeplitz and a result of unitarity equivalence with complex symmetry was proved in $\mathcal{H}_u$.

2 Complex symmetric Toeplitz operator

A bounded operator T on a Hilbert space H is complex symmetric if there exists an orthonormal basis for H with respect to which T has a self-transpose matrix representation. An equivalent definition also exists. A conjugation is a conjugate-linear operator $C : H \to H$ that satisfes the conditions:

(a) C is isometric: $< Cf; Cg >=< g; f > \forall\, f\, g \in H$

(b) C is involutive: $C^2 = I$.

We say that T is C-symmetric if $CT = T^*C$, and complex symmetric if there exists a conjugation C with respect to which T is C-symmetric.This is equivalent to the symmetry of T with respect to the bilinear form $[f; g] =< f; Cg >$.It was showing in [7, Lemma 1] that there exists an orthonormal basis $(e_i)_{i\in I}$ of H which is left invariant by $C : Ce_i = e_i$.

Lemma 2.1. *If C is a conjugation on H, then there exists an orthonormal basis(e_i) of H such that $Ce_i = e_i \;\; \forall i$. In particular,*

$$C(\sum_i \alpha_i\, e_i) = \sum_{i=0} \overline{\alpha_i} C(e_i)$$

where (α_i) is in l^2.

Proof. Consider the $\mathbb{R}$-linear subspace, $\mathcal{R} = (I + C)H$.of H and note $\forall f \in \mathcal{R}$ we have $Cf = f$.Consequently $\mathcal{R}$ is a real Hilbert space under the inner product $< \cdot; \cdot >$ since,

$$< f; g >=< Cg; Cf >=< g; f >= \overline{< f; g >},\ \forall f\,, g \in \mathcal{R}$$

Let (e_i) be an orthonormal basis for $\mathcal{R}$. Since $H = \mathcal{R} + i\mathcal{R}$, it follows easily that (e_i) is an orthonormal basis for space H as well.$\square$

Definition 2.2. A $f \in H$ that satisfies $Cf = f$ is called a C-real vector. We refer to a basis having the properties described in Lemma 1.1 as a C-real orthonormal basis.

With respect to the basis $(e_i)_{i\in I}$, C-symmetry is simply complex symmetry of the associated matrix.

Some authors prefer to use the term conjugate linear instead of anti-linear. So with this perspective, a function that satisfies the first and last conditions listed previously is called anti-unitary operator. A conjugation is simply an involution operator that is unitary conjugate-linear. In light of the polarization identity:

$$4 < f, g >= ||f + g||^2 - ||f - g||^2 + i||f + ig||^2 - i||f - ig||^2,$$

the isometric condition is equivalent to asserting that $||Cf|| = ||f||\ \forall f \in H$. Let us consider a few standard examples of conjugations.

Example 2.3. One of the simplest, and perhaps most important, families of C-symmetric operators are the finite Jordan blocks. Let λ be a complex number and consider the Jordan block $J_n(\lambda)$ of order n corresponding to λ: $J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & \lambda \end{pmatrix}$

If C_n denotes the isometric antilinear operator(conjugation):

$$C_n(z_1, z_2, ..., z_n) := (\overline{z_n}, ..., \overline{z_2}, \overline{z_1})$$

on $\mathbb{C}^n$, then that $J_n(\lambda)$ is a C_n-symmetric for any λ.

Example 2.4. The operator and its adjoint

$$\mathcal{F}fv(a) = \int_0^a v(x)\, dx \quad \mathcal{F}^*v(a) = \int_a^1 v(x)\, dx$$

on $L^2([0, 1])$ satisfy $\mathcal{F} = C\mathcal{F}^*C$ where $Cv(x) = \overline{v(1 - x)}$ denotes the conjugation, and the orthonormal basis $e_n = e^{[\pi in(2x-1)]}, \forall n \in \mathbb{Z}$ is C-real

Let $L^2 = \{f\colon \int |f|^2 dm < \infty\}$ L^∞ the space of essentially bounded functions, and C($\mathbb{T}$) the space of continuous functions on the unit circle $\mathbb{T}$. The Hardy space, denoted by H^2, consists of all analytic functions $f(z) = \sum_{n\geq 0} a_n z^n$ on the unit disk D such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. It is clear that $\{z^n : n = 0, 1, 2, ...\}$ is an orthonormal basis for H^2.Analogously, For an analytic function f on $\mathbb{D}$ the map :

$$r \mapsto \frac{1}{2\pi} \int_{\mathbb{T}} |f(r\theta))|^2 \, d\theta,$$

is increasing on $]0, 1[$.So if $f(z) = \sum_{n\geq 0} a_n z^n$ then

$$\frac{1}{2\pi} \int_{\mathbb{T}} |f(r\theta))|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

This fact leads to define the Hardy space H^2: the function f such that:

$$||f|| = \lim_{r\to 1^-} \left(\frac{1}{2\pi} \int_{\mathbf{T}} |f(r\theta)|^2 \, d\theta \right)^{\frac{1}{2}} < \infty \tag{2.1}$$

Remark 2.5. Fatou's and Riesz's lemmas show that the limit:

$$f(\theta) = \lim_{r\to 1^-} f(r\theta)$$

exist *a.e* in $\mathbb{T}$.
So that

$$||f||_{H^2} \equiv ||f||_{L^2(\mathbb{T})}$$

Indeed, The bilinear form on the Hardy space H^2 defined by:

$$< f; g >= \frac{1}{2\pi} \int_{\mathbb{T}} f(\theta) \, \overline{g(\theta)} \, d\theta = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

is a inner product where $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$ denote the elements of H^2.
Recall that for $z_0 \in \mathbb{D}$, the Function defined by:

$$k_{z_0}(z) = \frac{1}{1 - \overline{z_0} z}$$

is called the reproducing kernel. For fixed $z_0 \in \mathbb{D}$ functional $f \mapsto f(z_0)$ are bounded on H^2 and, by the Riesz Representation Theorem, we have:

$$f(z_0) =< f; k_{z_0} >$$

or, with Cauchy integral form:

$$f(z_0) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{f(\theta)}{1 - z_0 \overline{\theta}} \, d\theta$$

For each $\psi \in L^\infty$, the Toeplitz operator $T_\psi\colon H^2 \to H^2$ is defined by:

$$T_\psi f = P(\psi f)$$

with P is the projection of L^2 onto H^2 and ψ is called the symbol of T_ψ.
The concept of Toeplitz operators was introcuced in [4] by Halomos and its a generalization of Toeplitz matrices concept. The question of characterizing complex symmetric Toeplitz operators on H^2 in the unit disk is lifting by Guo and Zhu in [5]. Ko and Lee [6] introduced the family of conjugations $S_\mu : H^2 \to H^2$, given by:

$$S_\mu f(z) = \overline{f(\mu \overline{z})}$$

with $\mu \in \mathbb{T}$ and proved the following result:

Theorem 2.6. *[6, Theorem2.4] Let* $\psi(z) = \sum_{n=-\infty}^{+\infty} \hat{\psi}(n)z^n \in L^\infty$ *then* T_ψ *is* S_μ*-symmetric if, and only if,* $\hat{\psi}(-n) = \mu^n \hat{\psi}(n), \quad \forall n \in \mathbb{Z}$

M.S. Ferreira define the conjugation $\mathcal{J}$ by:$\mathcal{J}f(z) = \overline{f(\overline{z})}$ for each $f \in H^2$, the conjugation is a kind of canonical conjugation on H^2, and he proved in [1, theorem 2.1] this relationship with conjugations of H^2

Theorem 2.7. *If* C *is an conjugation on* H^2*, then exists an unitary operator* T *on* H^2 *such that* $TC = \mathcal{J}T$*.*

For the proof we need this Lemma:

Lemma 2.8. *[7] If* C *and* J *are conjugations on a Hilbert space H, then* $U = CJ$ *is a unitary operator. Moreover,* U *is both* C*-symmetric and* J*-symmetric.*

Proof[Theorem1.7] according to the assumpation we just take $U = C\mathcal{J}$ and we have the result.

Remark 2.9. M.S.Ferreirra used the caracterization of conjugation C with respect to an C-real orthonormal basis to prove the result.

Note that the converse of Theorem 1.7 is a simple corollary of The Godiˇc-Lucenko Theorem[7, theorem 3.1]. summerized

Corollary 2.10. *If* T *an isomorphism linear on* H^2 *and* $C = T^{-1}\mathcal{J}T$*, then T is unitary if,and only if, C is a conjugation.*

In the next result we determine the matrix of operators $\mathcal{J}$-symmetic on H^2

Proposition 2.11. *Let* T *an Bounded operator on* H^2*.Then* T *is* $\mathcal{J}$*-symmetric if, and only if, the matrix of* T *with respect the canonical basis of* H^2 *is symmetric.*

Proof. suppose that $T = \mathcal{J}T^*\mathcal{J}$ et let $\{z^n : n = 0, 1, 2, ...\}$ the standard basis of H^2 and $M = m_{ij}$ the matrix of T with respect of standard basis on H^2. it's clear that $\{z^n : n = 0, 1, 2, ...\}$ is $\mathcal{J}$-real ($\mathcal{J}z^n = z^n$).

$$\begin{aligned} m_{ij} &=< Tz^j, z^i >=< \mathcal{J}T^*\mathcal{J}z^j, z^i > \\ &=< \mathcal{J}z^i, T^*\mathcal{J}z^j > \\ &=< z^i, T^*z^j > \\ &=< Tz^i, z^j > \\ &= m_{ji} \end{aligned}$$

Then M is symmetric. The converse sense is showing by similar computation.□

3 Truncated Toeplitz operators on model spaces

3.1 Basics properties

H^2 is a linear space, it is its multiplicative structure that reveals its true function-theoretic depth. We recall here somes important facts on H^2 of H^2 functions.

Definition 3.1. An inner function is a bounded analytic function u on $\mathbb{D}$ such that :

$$|u(e^{i\theta})| = 1, \quad \forall\, e^{i\theta} \in \mathbb{T} \ , 0 \leq \theta \leq 2\pi$$

Example 3.2. The Möbius transformation is giving by :

$$e^{i\theta}\frac{\lambda - z}{1-\overline{\lambda}z}$$

with $\lambda \in \mathbb{D}$ and, $0 \leq \theta \leq 2\pi$, it's easy to see that an automorphism mapping $\mathbb{T}$ to $\mathbb{T}$.

Recall that Shift operators play a major role in functions theory. the forward shift $F\colon H^2 \to H^2$ defined by:

$$F\,g(z) = z\,g(z),$$

or, by Taylor coefficients:

$$F(a_0, a_1, \cdots) = (0, a_0, a_1, \cdots)$$

It's so easy to verifie that F is isometrie. The adjoint F^* is the backward shift

$$F^*\,g(z) = \frac{g(z) - g(0)}{z},$$

or, by Taylor coefficients:

$$F^*(a_0, a_1, \cdots) = (a_1, a_2, \cdots)$$

the operator $f \mapsto u\,f$ is an isometry for u is a inner function ; so uH^2 is a closed subspace of H^2 (i.e., a closed linear manifold). Moreover, we have that $F(uH^2) \subset uH^2$ (Beurling's).

3.2 Model spaces

We are now ready to introduce the model spaces.

Definition 3.3. If u is an inner function, then the corresponding model space is:

$$\mathcal{H}_u = (uH^2)^{\perp} = H^2 \ominus uH^2 \tag{3.1}$$

Let consider the Functions,

$$K_w(z) = \frac{1-\overline{u(w)}u(z)}{1-\overline{w}\,z}, \quad w\,, z \in \mathbb{D} \tag{3.2}$$

It's clair that $K_w(z) \in \mathcal{H}_u$ and

$$f(w) = < f,\, K_w >, \quad w \in \mathbb{D}\,, f \in \mathcal{H}_u \tag{3.3}$$

Remark 3.4. For $w \in \mathbb{T}$ formula remains valid such that u has an angular derivative at sense of Caratheodory.

For fixed u the function K_w is called the reproducing kernel for $\mathcal{H}_u$,it follows that if (e_n) is un othogonal basis for $\mathcal{H}_u$,then

$$K_w(z) = \sum_{n\geq 1} \overline{e_n(w)}\, e_n(z) \tag{3.4}$$

The orthogonal projection:

$$P_u : L^2 \mapsto \mathcal{H}_u \tag{3.5}$$

plays an axial role in the studies of certain types of operators such as of truncated Toeplitz operators, Hankel operators and model Toeplitz operators. this importance is expressed via the use of the reproducing kernels

Proposition 3.5. *Let* $f \in L^2$, $w \in \mathbb{D}$

$$P_u f(w) = < f, K_w > \tag{3.6}$$

Proof. P_u is self-adjoint then we have :

$$< f, K_w >=< f, P_u K_w >=< P_u f, K_w >= P_u f(w) \qquad \square$$

An alternative definition of $\mathcal{H}_u$,via the functions defined on $\mathbb{T}$, is giving by :

Proposition 3.6. *Let u an inner function,Then*

$$\mathcal{H}_u = H^2 \cap u\overline{zH^2} \tag{3.7}$$

Proof. Let $f \in H^2$, $< f, ug >= 0$, $\forall g \in H^2$ if,and only if,$< \overline{u}f, g >= 0$, $\forall g \in H^2$ if,and only if $\overline{u}f \in \overline{zH^2}$. Since $|u(w)|^2 = 1$ For(a.e.) $w \in \mathbb{T}$, we have $f \in (uH^2)^\perp \iff f \in u\overline{zH^2}$. $\square$

we state the following:

Corollary 3.7. *Let u an inner function, Then*

$$F^*\mathcal{H}_u \subset \mathcal{H}_u.$$

where F^* the operator shift defined previously.

On $\mathcal{H}_u$ define ,for functions on $\mathbb{T}$, the operator by:

$$Cf = \overline{fz}u \tag{3.8}$$

Proposition 3.8. *C is a anti-linear involution isometrie on $\mathcal{H}_u$*

Proof. Since $u\overline{u} = 1$ on $\mathbb{T}$ (a.e), Then C is anti-linear, isometric and involution. verify that C is on $\mathcal{H}_u$. Let $f \in (uH^2)^\perp$,by computing we have :

$$\begin{aligned} < Cf; \overline{zg} > &=< \overline{fz}u; \overline{zg} > \\ &= \frac{1}{2\pi}\int_{\mathbb{T}} \overline{f(\lambda)\lambda}u(\lambda)\,\lambda g(\lambda)\,d\lambda. \\ &=< ug; f > \\ &= 0. \end{aligned}$$

for each $g \in H^2$, then $Cf \in H^2$. by similar argument we get

$$< Cf; ug >=< \overline{fz}u; ug >=< \overline{fz}, g >$$

(i.e) $Cf \in \mathcal{H}_u$. $\square$

Example 3.9. For the reproducing kernel K_w we have

$$\begin{aligned} CK_w(z) &= \overline{\left(\frac{1-\overline{u(w)}u(z)}{1-\overline{w}\,z}\right)z}\,u(z) \\ &= \frac{1-u(w)\overline{u(z)}}{1-w\overline{z}}\;\frac{u(z)}{z} \\ &= \frac{u(z)-u(w)}{z-w}. \end{aligned}$$

We note $Q_\lambda = CK_w$ is called the conjugate kernel.

3.3 Truncated Toeplitz operators

For each symbol $\phi \in L^2$ the corresponding truncated Toeplitz operator T_ϕ^u is the densely defined operator on $\mathcal{H}_u$ given by:

$$\mathrm{T}_\phi^u f = P_u(\phi f) \quad \forall f \in L^\infty \tag{3.9}$$

or

$$\mathrm{T}_\phi^u = P_u T_\phi P_u.$$

where T_ϕ is Toerplitz operator defined on H^2, and $P_u : L^2 \mapsto \mathcal{H}_u$ othogonal projection.

Recall that operator $Cf = \overline{fzu}$ is a conjugation on $\mathcal{H}_u$. The following result is from [9]

Proposition 3.10. *Each Truncated Toeplitz operator* T_ϕ^u *is* C*-symmetric*

Proof. For $f, g \in \mathcal{H}_u$

$$\begin{aligned}
< CT_\phi^u f; g > &=< Cg; T_\phi^u f >=< Cg; P_u T_\phi P_u f > \\
&=< P_u Cg; T_\phi f >=< Cg; P(\phi f) > \\
&=< PCg; \phi f >=< Cg; \phi f > \\
&=< \overline{gz}u; \phi f >=< \overline{fz}u; \phi g > \\
&=< Cf; P(\phi g) > \\
&=< Cf; P_u T_\phi P_u g > \\
&=< (T_\phi^u)^* Cf; g >
\end{aligned}$$

Then $C\mathrm{T}_\phi^u = (\mathrm{T}_\phi^u)^* C$ □

According to [1, Remark2.2] we have the following result:

Corollary 3.11. *For each truncated Toeplitz operator* T_ϕ^u*, the operator* $\mathbf{T}^u = U\mathrm{T}_\phi^u U^*$ *is* UCU^**-symmetric where* U *unitary operator on* $\mathcal{H}_u$*.*

Proof. Via [9, Lemma 1] and we have the result by [7, p.1291].□

Recall the canonical conjugation introduced previously $f \mapsto \mathcal{J}f$ on H^2 by:

$$\mathcal{J}f(z) := \overline{f(\overline{z})}$$

Now let $\mathcal{J}C$ denote the corresponding conjugation on the model space $\mathcal{H}_{\mathcal{J}u}$.

Remark 3.12. u is an inner function $\iff$ $\mathcal{J}u$ is inner

Proposition 3.13. *The operator* $\mathcal{J}C : \mathcal{H}_u \to \mathcal{H}_{\mathcal{J}u}$ *is unitary.*

Proof. Since $\mathcal{J}$ et C is a conjugations then by [9, Lemma 1] we have the unitary.□

4 Conclusion

In summary, the operator T_ψ on H^2 symbol $\psi \in L^\infty$ is defined by $P(\psi f)$, where P is the orthogonal projection from L^2 onto H^2.One can show that T_ψ is a bounded operator that satisfies $T_\psi^* = T_{\overline{\psi}}$ and $||T_\psi|| = ||\psi||_\infty$. The coefficients $(t_{i,j})$ of the matrix representation of T_ψ in a standard orthonormal basis $\{1, z^1, z^2, ...\}$of H^2 is $\hat{\psi}(j-i)$(Fourier coefficients) , which an infinite Toeplitz matrix (constant in a diagonals). It's was a algebriac caracterization of Toeplitz operators giving by the forward shift F on H^2 due to Halmos [4], a bounded operator T on H^2 is a Toeplitz operator if and only:

$$T = FTF^*$$

There are some similarities between truncated Toeplitz operators and Toeplitz operators. For instance, one has the adjoint formula, and an analogue of Halmos's caracterization.However the However, the differences are greater. for examples , the inequality:

$$||T_\psi|| \leq ||\psi||_\infty$$

is large for Toepltiz operators and it's frequently strict for truncatred Toeplitz operators; and there are somes of bounded truncated Toeplitz operator cannot be represented with a bounded symbol. Finally, The class of complex symmetric operators is surprisingly large and many of them can be shown to be unitarily equivalent to truncated Toeplitz operators.

References

[1] M. C. Ferreira, Some notes on complex symmetric operators, *e-J of Anal and Appl. Math.* **1**, 90–96 (2021).

[2] S. R. Garcia and M. Putinar, Complex symmetric operators and applications, *Trans. Amer. Math. Soc.* **358(3)**, 1285–1315 (2006).

[3] S. R. Garcia and Wogen, Some new classes of complex symmetric operators, *Trans. Amer. Math. Soc.* **362**, 6065–6077 (2010).

[4] A. Brown and P.R. Halmos, Algebraic properties of Toeplitz operators, *J.Reine Angew. Math.* **213**, 89–102 (1963–1964).

[5] K. Guo and S. Zhu, A canonical decomposition of complex symmetric operators, *J.Operator Theory* **72**, 529-547 (2014).

[6] E. Ko and J. Lee, On complex symmetric Toeplitz operators, *J. Math. Anal. Appl.* **434**, 20–34 (2016).

[7] S. R. Garcia, E. Prodan, M. Putinar, Mathematical and physics aspects of complex symmetric operators *J. Phys. A: Math. Theor.* **47**, 353001 (2014).

[8] M. Fatehi, Complex symmetric weighted composition operators, *Complex Variables and Elliptic Equations* **64**, 710–720 (2019).

[9] S. R. Garcia and M. Putinar, symmetric operators and applications. II, *Trans. Amer. Math. Soc.* **359(8)**, 3913–3931 (2007).

[10] S. R. Garcia, Conjugation and Clark operators, Recent advances in operator-related function theory, *Contemp. Math.* **393**, 67–111 (2006).

[11] R. T. W. Martin, Unitary perturbations of compressed n-dimensional shifts, *Complex Anal. Oper. Theory* **7(4)**, 765–799 (2013).

[12] R. G. Douglas, Banach Algebra Techniques in Operator Theory, *second ed., Graduate Texts in Mathematics* **179**, Springer-Verlag, New York, (1998).

[13] S. Zhu and C. G. Li, Complex symmetric weighted shifts, *Trans. Amer. Math. Soc.* **365(1)**, 511–530 (2013).

[14] S. R. Garcia, D. E. Poore, and W.T. Ross, Unitary equivalence to a truncated Toeplitz operator: analytic symbols, *Proc. Amer. Math. Soc.* **140(4)**, 1281–1295 (2012).

[15] J. B. Conway, A course in operator theory,*Graduate Studies in Mathematics*, **21**, American Mathematical Society, Providence, RI (2000).

[16] J. A. Cima, W. T. Ross, and W. R. Wogen, Truncated Toeplitz operators on finite dimensional spaces, *Oper. Matrices* **2(3)**, 357–369, (2008).

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