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ON THE EXISTENCE OF POSITIVE RADIAL SOLUTIONS OF A NONLINEAR ELLIPTIC EQUATION WITH A CRITICAL POTENTIAL

Mouhssin Bakhadda and Arij Bouzelmate

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Abstract The paper deals with the following elliptic equation with a critical potential

$$div\left(|\nabla u|^{p-2}\nabla u\right) + \gamma\, |x|^{-p}\, |u|^{p-2}\, u = 0, \quad x \in \mathbb{R}^N - \{0\},$$

where $N > p > 2$ and $\gamma > 0$.

We prove the existence of positive solutions in the radial case. The study relies heavily on comparing γ with $((N-p)/p)^p$.

1 Introduction

This paper is concerned with the following singular radial equation

$$\left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}|u'|^{p-2}u' + \gamma r^{-p}|u|^{p-2}u = 0, \quad r > 0, \tag{1.1}$$

where $N > p > 2$ and $\gamma > 0$.

To carefully analyze the radial solutions of (1.1), we study the following problem

$$(Q)\begin{cases} \left(|u'|^{p-2}u'\right)' + \dfrac{N-1}{r}|u'|^{p-2}u' + \gamma r^{-p}|u|^{p-2}u = 0, \quad r > 0, \\ u(0) = \xi, \end{cases}$$

where $N > p > 2$, $\gamma > 0$ and $\xi \in \mathbb{R}^*$.

It's easy to see that $u(r,\xi) = -u(r,-\xi)$, then we restrict to $\xi > 0$.

Note that the problem (Q) is a second-order problem with a single initial data. We focus on the study of the behavior near the origin of solutions of problem (Q). In fact, we will prove that if u is a solution of problem (Q), then $\lim_{r\to 0} r^{(N-1)/(p-1)}u'(r) = 0$. So we consider this following problem

$$(P)\begin{cases} \left(|u'|^{p-2}u'\right)' + \dfrac{N-1}{r}|u'|^{p-2}u' + \gamma r^{-p}|u|^{p-2}u = 0, \quad r > 0, \\ u(0) = \xi, \quad \lim\limits_{r\to 0} r^{(N-1)/(p-1)}u'(r) = 0, \end{cases}$$

where $N > p > 2$, $\gamma > 0$ and $\xi > 0$.

We mean by a solution of problem (P), a function u defined on $[0,+\infty[$ such that $u \in C^0([0,+\infty[) \cap C^1(]0,+\infty[)$, $|u'|^{p-2}u' \in C^1(]0,+\infty[)$ and satisfying (1.1) in $]0,+\infty[$ with $u(0) = \xi$.

The difficulty lies in the fact that the problem (P) is singular near the origin. More precisely, we will show that the problem (P) has a unique solution u which satisfies

$$\lim_{r\to 0} ru'(r) = -\left(\frac{\gamma}{N-p}\right)^{1/(p-1)}\xi$$

and

$$\lim_{r\to 0} r^2 u''(r) = \left(\frac{\gamma}{N-p}\right)^{1/(p-1)} \xi.$$

We will then show that this solution is strictly positive on $[0, +\infty[$.

The equation (1.1) has been the subject of study by a number of authors, the main one being the survey in [1, 2, 3, 4, 8, 10, 7, 11, 12, 14, 15] and the references therein. If $p = 2$, the equation (1.1) was studied by [10] and [12]. The non-radial case was studied by [11]. In the papers [10, 12], there was a large discussion about the sign of the real $((N-p)/p)^p - \gamma$ which has a direct relation with the characteristic equation of (1.1). Specifically, if $u = r^{-\alpha}$ is a simple solution of the equation (1.1), then we obtain the characteristic equation

$$\chi(\alpha) \equiv |\alpha|^{p-2} \alpha(N - p - \alpha(p-1)) - \gamma = 0, \quad \alpha \in \mathbb{R}.$$

It has been proven in [12] that

$$\max_{\alpha\in\mathbb{R}} \chi(\alpha) = \chi(\frac{N-p}{p}) = \left(\frac{N-p}{p}\right)^p - \gamma.$$

According to the study carried out in [10, 12], the sign of $((N-p)/p)^p - \gamma$ influences the variation of the function χ and consequently the behavior of the solutions of equation (1.1). In this paper, our main aim is to add value to previous work by proving the existence of positive solutions of the singular equation (1.1) using the energy method introduced by [13]. We will also prove that these positive solutions are decreasing on $(0, +\infty)$. Throughout the work we will use some ideas introduced in the papers [2, 3, 4, 5, 6, 9].

The rest of the work is as follows. In the section 2, we present fundamental properties concerning the asymptotic behavior near the origin of solutions of problem (P). We give a main result which shows that $\lim_{r\to 0} u'(r) = -\infty$ and $\lim_{r\to 0} u''(r) = +\infty$. In the section 3, we prove the existence of positive solutions of problem (P). The study strongly depends on the sign of $((N-p)/p)^p - \gamma$. The final section 4 concludes with an overview of the main results and their prospects.

2 Fundamental properties

This section concerns the study of the asymptotic behavior near 0 of solutions of problem (Q). We start with this theorem.

Theorem 2.1. *Let u be a solution of problem (Q). Then*

$$\lim_{r\to 0} r^{(N-1)/(p-1)} u'(r) = 0, \tag{2.1}$$

$$\lim_{r\to 0} r u'(r) = -\left(\frac{\gamma}{N-p}\right)^{1/(p-1)} \xi \tag{2.2}$$

and

$$\lim_{r\to 0} r^2 u''(r) = \left(\frac{\gamma}{N-p}\right)^{1/(p-1)} \xi. \tag{2.3}$$

Before proving the previous theorem, we will need this following proposition.

Proposition 2.2. *Let u be a solution of problem (Q). then*

(i) *u is strictly monotone near* 0.

(ii) $\lim_{r\to 0} r^{(N-1)/(p-1)} u'(r) = 0.$

(iii) $\lim_{r\to 0} r u'(r) = -\left(\frac{\gamma}{N-p}\right)^{1/(p-1)} \xi.$

In particular, $u'(r) < 0$ for small r.

Proof. First, we show that $u'(r) \neq 0$ near 0.
Since $u(0) = \xi > 0$, then by continuity of u, there exists $\rho > 0$ small enough such that $u(r) > 0$ for $r \in]0, \rho[$. If there exists $r_0 \in]0, \rho[$ the first zero of u', then by equation (1.1)

$$\left(|u'|^{p-2}u'\right)'(r_0) = -\gamma r_0^{-p}|u|^{p-2}(r_0)u(r_0) < 0. \tag{2.4}$$

So u is strictly monotone near 0.

To prove (ii), let

$$V(r) = r^{N-1}|u'|^{p-2}u'(r), \qquad r > 0. \tag{2.5}$$

Then by (1.1), we have

$$V'(r) = -\gamma r^{N-1-p}|u|^{p-2}u(r), \qquad \text{for any } r > 0. \tag{2.6}$$

Therefore $V'(r) < 0$ for small r. It follows that $\lim_{r\to 0} V(r) \in]-\infty, +\infty]$ and so by (2.5), $\lim_{r\to 0} r^{(N-1)/(p-1)}u'(r) \in]-\infty, +\infty]$. Suppose that $r^{(N-1)/(p-1)}u'(r)$ does not converge to 0 when $r \to 0$, then there exists a small $\rho > 0$ and a constant $C > 0$ such that

$$|u'(r)| > Cr^{(1-N)/(p-1)}, \qquad \text{for any } r \in (0, \rho).$$

But $r^{(1-N)/(p-1)} \notin L^1(0, \rho)$. This contradicts $u' \in L^1(0, \rho)$. Consequently, $r^{(N-1)/(p-1)}u'(r)$ converges to 0 when $r \to 0$.

Now, we show (iii). We have by (ii) and (2.5), $\lim_{r\to 0} V(r) = 0$. Therefore, by (2.6) and Hopital's rule, we have

$$\lim_{r\to 0} r^{p-N}V(r) = \frac{-\gamma}{N-p}\xi^{p-1}. \tag{2.7}$$

This gives

$$\lim_{r\to 0} r^{p-1}|u'|^{p-2}u'(r) = \frac{-\gamma}{N-p}\xi^{p-1}. \tag{2.8}$$

Which leads to the desired result. □

We now give the proof of the Theorem 2.1.

Proof. According to the previous proposition, it remains to show (2.3). Note that since $u' \neq 0$ near 0, then u'' exists near 0. Multiply equation (1.1) by r^p and use Proposition 2.2, we get

$$\lim_{r\to 0} r^p\left(|u'|^{p-2}u'\right)'(r) = \lim_{r\to 0}(p-1)r^p|u'|^{p-2}u''(r) = \frac{\gamma(p-1)}{N-p}\xi^{p-1} \tag{2.9}$$

and so

$$\lim_{r\to 0} r^2u''(r) = \left(\frac{\gamma}{N-p}\right)^{1/(p-1)}\xi.$$

This completes the proof. □

3 Existence results

The aim of this section is to give existence results for positive solutions of the problem (P). We begin with local and global existence results.

Theorem 3.1. *For each* $\xi > 0$*, the problem* (P) *has a unique solution* $u = u(., \xi)$ *on* $[0, +\infty[$.

Proof. . Three stages will be taken in the proving process.
Step 1: Existence of a local solution.
Integrating equation (1.1) twice from 0 to r, we see that the problem (P) is equivalent to the following equation.

$$u(r) = \xi - \int_0^r g(f[u](\tau))d\tau, \tag{3.1}$$

where $g(s) = |s|^{(2-p)/(p-1)}s$ and f is a nonlinear function given by

$$f(u)(\tau) = \gamma\tau^{1-N}\int_0^\tau \rho^{N-p-1}(|u|^{p-2}.u)d\rho. \tag{3.2}$$

For $R > 0$, the Banach space of real continuous functions on $[0, R]$ with uniform norm $\|\ .\|_0$ is denoted by $C([0, R])$. Given $\xi,\ M > 0$, we consider the complete metric space

$$E_{\xi,M,R} = \{\varphi \in C([0, R]); \|\varphi - \xi\|_0 \leq M\}. \tag{3.3}$$

The operator Γ on $E_{\xi,M,R}$ is also defined by

$$\Gamma[\varphi](r) = \xi - \int_0^r g(f[\varphi](\tau))d\tau. \tag{3.4}$$

Claim 1: Γ transforms $E_{\xi,M,R}$ into itself when M is small and $R > 0$.
Obviously, $\Gamma[\varphi] \in C([0, R]$. According to the definition of space $E_{\xi,M,R}$, we have $\varphi \in [\xi - M, \xi+M]$ for all $r \in [0, R]$. we show that $f[\varphi]$ has a constant sign on $[0, R]$ for every $\varphi \in E_{\xi,M,R}$. Additionally, there is a constant $K > 0$ such that

$$f[\varphi](\tau) \geq K\tau^{1-p} \text{ for all } \tau \in [0, R], \tag{3.5}$$

where $K = \dfrac{\gamma}{N-p}\xi^{p-1}$.
Considering that the function $r \to \dfrac{g(r)}{r^{p-1}}$ decreases on $(0, +\infty)$, we obtain

$$|\Gamma[\varphi](r) - \xi| \leq \int_0^r \frac{g(f[\varphi](\tau))}{f[\varphi](\tau)}|f[\varphi](\tau)|d\tau \leq \int_0^r \frac{g\left(K.\tau^{1-p}\right)}{K\tau^{1-p}}|f[\varphi](\tau)|d\tau$$

for $r \in [0, R]$. On the other side,

$$f[\varphi](\tau) \leq C \cdot \xi^{1-p}, \quad C = \frac{\gamma}{N-p}(\xi + M)^{p-1}.$$

We thus get

$$|\Gamma[\varphi](r) - \xi| \leq C\frac{K^{2(1-p)+1}}{p-1} \cdot r^{p-1}$$

for every $r \in [0, R]$. Choose R small enough such that

$$|\Gamma[\varphi](r) - \xi| \leq M \quad \text{ for } \varphi \in E_{\xi,M,R}. \tag{3.6}$$

This means that $\Gamma[\varphi] \in E_{\xi,M,R}$. The Claim 1 is proved.
Claim 2: Γ is a contraction of the interval $[0, r_\xi]$.
According to Claim 1, if r_ξ is small enough, the space E_{ξ,M,r_ξ} applies into itself. For such r_ξ and any $\varphi, \psi \in E_{\xi,M,r_\xi}$ we have

$$|\Gamma[\varphi](r) - \Gamma[\psi](r)| \leq \int_0^r |g(f[\varphi](\tau)) - g(f[\psi](\tau))|d\tau,$$

where $f[\varphi]$ is given by (3.2). Next, let

$$\phi(\tau) = \min(f[\varphi](\tau), f[\psi](\tau)).$$

As a result of estimation (3.5), we have

$$\phi(\tau) \geq K\tau^{1-p} \quad \text{ for } 0 \leq \tau \leq r < r_\xi \tag{3.7}$$

and then

$$\begin{aligned} |g(f[\varphi](\tau)) - g(f[\psi](\tau))| &\leq \frac{g(\phi(\tau))}{\phi(\tau)}|f[\varphi](\tau) - f[\psi](\tau)| \\ &\leq \frac{g\left(K\tau^{1-p}\right)}{K \cdot \tau^{1-p}}|f[\varphi](\tau) - f[\psi](\tau)|. \end{aligned} \tag{3.8}$$

Moreover,

$$|f[\varphi](\tau) - f[\psi](\tau)| \leq C' \cdot \|\varphi - \psi\|_0 \cdot \tau^{1-p}, \quad C' = \frac{\gamma}{N-p}(\xi + M)^p. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we

$$|\Gamma[\varphi](r) - \Gamma[\psi](r)| \leq C' \cdot \frac{K^{2(1-p)+1}}{p-1} r^{p-1} \cdot \|\varphi - \psi\|_0, \quad \text{for all } r \in [0, r_\xi]. \tag{3.10}$$

Finally, we choose r_ξ small enough so that

$$C' \cdot \frac{K^{2(1-p)+1}}{p-1} r_\xi^{p-1} < 1.$$

This means that Γ is a contraction. Consequently, the existence of a unique fixed point of Γ in E_{ξ,M,r_ξ} is implied by the Banach fixed point Theorem. This fixed point is a solution of problem (P) which can be stretched to a maximum interval $[0, r_{\max}[$ with $0 < r_{\max} \leq +\infty$.
Step 2: Existence of a global solution.
We introduce the energy function

$$E(r) = \frac{p-1}{P} |u'|^p + \frac{\gamma}{p} r^{-p} |u|^p \tag{3.11}$$

From equation (1.1), we obtain

$$E'(r) = -\left(\frac{N-1}{r} |u'|^p + \gamma r^{-p-1} |u|^p\right). \tag{3.12}$$

Since $\gamma > 0$, then E is positive and decreasing. Therefore E is bounded. Hence u and u' are also bounded and $u(r)$ exists for all $r \geq 0$. □

The existence of positive solutions of problem (P) is given by this main theorem.

Theorem 3.2. *Assume that* $0 < \gamma < ((N-p)/p)^p$. *Let* u *be a solution of problem* (P). *Then* u *is strictly positive on* $[0, +\infty[$.

The proof requires this following preliminary results.

Proposition 3.3. *Let* u *be a solution of problem* (P). *If* $r_0 > 0$ *is the first zero of* u, *then* $u'(r_0) < 0$.

Proof. Let $r_0 > 0$ the first zero of u. Then $u'(r_0) \leq 0$ and by continuity of u, there exists a left neighborhood $(r_0 - \varepsilon, r_0)$ (for some $\varepsilon > 0$) where $u > 0$ and $u' < 0$.

Suppose by contradiction that $u'(r_0) = 0$. Then, since $u > 0$ on $(0, r_0)$, we have by (2.5) and (2.6), $V'(r) < 0$ on $(0, r_0)$. Therefore

$$V(r) > V(r_0) = 0 \quad \text{for any } r \in (r_0 - \varepsilon, r_0).$$

This means that

$$u'(r) > 0 \quad \text{for any } r \in (r_0 - \varepsilon, r_0).$$

This is a contradiction with the fact that $u'(r) \leq 0$ in $(r_0 - \varepsilon, r_0)$. Consequently $u'(r_0) < 0$. □

Proposition 3.4. *Let* u *be a solution of problem* (P) *and let* $D_u := \{r > 0 : u(r) > 0\}$. *Then* $u'(r) < 0$ *for any* $r \in D_u$.

Proof. Suppose by contradiction that there exists $r_1 > 0$ the first zero of u'. Since by Proposition 2.2, $u'(r) < 0$ for $r \sim 0$, then u' is strictly increasing and strictly negative on a left neighborhood $]r_1 - \varepsilon, r_1[$ (for some $\varepsilon > 0$). This means that $(|u'|^{p-2}u')'(r_1) \geq 0$. But by equation (1.1), we have $(|u'|^{p-2}u')'(r_1) = -\gamma r_1^{-p} |u|^{p-2} u(r_1) < 0$ because $u(r_1) > 0$, $u'(r_1) = 0$ and $\gamma > 0$. This contradiction completes the proof. □

Now, we give the proof of Theorem 3.2.

Proof. Suppose by contradiction that there exists $r_0 > 0$ the first zero of u. Then by Proposition 3.4, $u'(r) < 0$ on $(0, r_0)$.

Define the energy function

$$F(r) = \frac{p-1}{p}|u'(r)|^p - \frac{\gamma(p-1)}{p} r^{-p}|u(r)|^p. \tag{3.13}$$

So by equation (1.1), we have

$$F'(r) = -\frac{N-1}{r}|u'|^p - \gamma p r^{-p}|u|^{p-2}uu' + \gamma(p-1)r^{-p-1}|u|^p. \tag{3.14}$$

We show that there exists $\theta \in (0, r_0)$ such that $F(\theta) = 0$ and $F'(\theta) \geq 0$.
Since $u'(r_0) < 0$ by Proposition 3.3, then $F(r_0) > 0$.
On the other side, the function F can be written as follows:

$$F(r) = \frac{p-1}{p} r^{-p}|u(r)|^p \left[\frac{r^p|u'|^p}{|u|^p} - \gamma\right]. \tag{3.15}$$

According to Proposition 2.2, we see that

$$\lim_{r\to 0} r^p F(r) = \frac{p-1}{p}\xi^p \left[\left(\frac{\gamma}{N-p}\right)^{p/(p-1)} - \gamma\right].$$

Since $0 < \gamma < ((N-p)/p)^p$, then $\lim_{r\to 0} r^p F(r) < 0$. This means that $F(r) < 0$ for small r. Combining this with $F(r_0) > 0$, we deduce that there exists $\theta \in (0, r_0)$ the first zero of F, that is $F(\theta) = 0$, $F'(\theta) \geq 0$ and by (3.14),

$$F'(\theta) = -\left[N - p - \gamma^{1/p}p\right]\frac{|u'(\theta)|^p}{\theta}. \tag{3.16}$$

Again, since $0 < \gamma < ((N-p)/p)^p$ and $u'(\theta) < 0$, then $F'(\theta) < 0$. This contradicts $F'(\theta) \geq 0$. Consequently $u(r) > 0$ on $[0, +\infty[$. □

4 Conclusion

In this work we have used the energy method to study the existence of positive radial solutions of a nonlinear elliptic equation with a critical potential in the case $0 < \gamma < ((N-p)/p)^p$. These solutions are strictly decreasing. The question of the existence of positive solutions in the case $\gamma \geq ((N-p)/p)^p$ remains open and will be studied in the future.

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Author information

Mouhssin Bakhadda and Arij Bouzelmate, LaR2A Laboratory, Faculty of Sciences, Abdelmalek Essaadi University, Tetouan, Morocco.
E-mail: mouhssin017@gmail.com, abouzelmate@uae.ac.ma