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Strongly nonlinear parabolic inequalities with L^1-data in Musielak-spaces

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Abstract: In this study, we prove an entropy solutions to some nonlinear parabolic inequalities with L^1-data. The proof is based on the penalization methods.

1 Introduction and essential assumptions

In this note, we consider as a model, the following problem parabolic inequalities:

$$\begin{cases} w \geq \Lambda \quad \text{a.e. in } \Omega \times (0,T), \\ \frac{\partial b(w)}{\partial t} - \text{div}\left(\varrho(x,t,w,\nabla w)\right) + \text{div}(\mathbb{F}(x,t,w)) + \mathbb{H}(x,t,w,\nabla w) = f \quad \text{in } Q, \\ w = 0 \quad \text{in } \partial\Omega \times (0,T), \\ w(x,0) = w_0 \quad \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where, Ω be a bounded open set of $\mathbb{R}^N$ with the segment property and Q be the cylinder $\Omega \times (0,T)$, $T > 0$. let Ψ and Φ two complementary Musielak-Orlicz functions.

Let $M : D(M) \subset W_0^{1,x} L_\Psi(Q) \longrightarrow W^{-1,x} L_\Phi(Q)$ be a mapping such that

$$\mathbb{M}(w) = -\text{div}(\varrho(x,t,w,\nabla w)),$$

where $\varrho : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that

$$\varrho(x,t,s,\xi).\xi \geq \alpha\Psi(x,|\xi|) + \Psi(x,|s|), \tag{1.2}$$

$$[\varrho(x,t,s,\xi) - \varrho(x,t,s,\xi^*)][\xi - \xi^*] > 0, \tag{1.3}$$

for all ξ and ξ^* in $\mathbb{R}^N$, $\xi \neq \xi^*$.

There exist two Musielak Orlicz functions Ψ and Φ such that $\Phi \prec\prec \Psi$ such that for a.e. $(x,t) \in Q$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$

$$|\varrho(x,t,s,\xi)| \leq \beta\left(a_0(x,t) + \Phi_x^{-1}\gamma\left(x, k_1|s|\right) + \Phi_x^{-1}\Psi\left(x, k_1|\xi|\right)\right), \tag{1.4}$$

with $a_0(.) \in E_\Phi(Q)$, $k_1 \in \mathbb{R}^+$ and $\alpha, \beta > 0$. $b : \mathbb{R} \to \mathbb{R}$ is a strictly increasing $\mathcal{C}^1(\mathbb{R})$-function, $b(0) = 0$

$$b_0 < b'(s) < b_1, \quad \forall s \in \mathbb{R} \quad \text{such that} \quad b_1 < \frac{1}{\alpha_0} \tag{1.5}$$

where α_0 is the constant appearing in (1.7).

Let $\mathbb{H} : \Omega \times [0,t] \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a Caratheodory function satisfying for a.e. $(x,t) \in \Omega \times [0,t]$ and $\forall s \in \mathbb{R}, \xi \in \mathbb{R}^N$:

$$|\mathbb{H}(x,t,s,\xi)| \leq \rho(s)\Psi(x,|\xi|); \tag{1.6}$$

where $\rho : \mathbb{R} \to \mathbb{R}^+$ is continuous positive function belongs to $L^1(\mathbb{R})$.

Furthermore $\mathbb{F} : Q \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function satisfying the following natural growth condition

$$|\mathbb{F}(x,t,s)| \leq c(x,t)\Phi_x^{-1}\Psi\left(x,\alpha_0|s|\right) \tag{1.7}$$

where $\|c(.,.)\|_{L^\infty(Q)} \leq \min\left(\frac{\alpha}{\alpha_0+1}; \frac{\alpha}{2(\alpha_0 b_1+1)}\right)$ and $0<\alpha_0<1$.

$$f \in L^1\left(Q\right)), \tag{1.8}$$

$$w_0 \in L^1(\Omega) \text{ such that } b\left(w_0\right) \in L^1(\Omega). \tag{1.9}$$

A large of papers was devoted to the study the similar problem (1). As an example ([9, 20]) where the authors considered the problem under study in order to prove the existence solution in the classical Sobolev spaces when $b(w)=w, f\in L^1(Q)$ and $\mathbb{H}$ is the non-linearity term satisfying the following conditions

$$|\mathbb{H}(x,t,s,\xi)| \leq b(s)\left(|\xi|^p + c(x,t)\right), \tag{1.10}$$

$$\mathbb{H}(x,t,s,\xi)s \geq 0. \tag{1.11}$$

This result was extended to the Orlicz-Sobolev-spaces (see[1]) when Aberqi et al have been proved the existence and uniqueness solution for some nonlinear parabolic parobleme like

$$\begin{cases} \frac{\partial b(w)}{\partial t} - \Delta_M w - \operatorname{div}\left(\bar{c}(x,t)\bar{M}^{-1}M\left(\frac{\alpha_0}{\lambda}|b(w)|\right)\right) = f \text{ in } Q_T, \\ u(x,t)=0 & \text{on } \partial\Omega\times(0,T), \\ b(w)(t=0) = b\left(w_0\right) & \text{in } \Omega. \end{cases} \tag{1.12}$$

where $-\Delta_M w = -\operatorname{div}\left((1+|w|)^2 Dw\frac{\log(e+Dw)}{|Dw|}\right), \bar{c} \in \left(L^\infty\left(Q_T\right)\right)^N$, $f\in L^1\left(Q_T\right)$, $b\left(w_0\right) \in L^1(\Omega)$. and $M(t)=t\log(e+t)$ is an N -function.

In generalized-Orlicz spaces, the existence and uniqueness of weak solutions for some nonlinear parabolic equation with non standard anisotropic growth hypothesise in the variable exponent Lebesgue spaces have been shown by Antontsev and Shmarev ([3]) when some equations generalize the evolution $p(x,t)$ -Laplacian looks like

$$\begin{cases} \frac{\partial w}{\partial t} - \sum_i \frac{\partial}{\partial x_i}\left[m_i(x,t,w)\left|D_i w\right|^{p_i(x,t)-2} D_i w + b_i(x,t,w)\right] = 0 & \text{in } Q_T \\ u(x,t)=0 & \text{on } \partial\Omega\times(0,T) \\ u(x,0)=w_0(x) & \text{in } \Omega. \end{cases} \cdot \tag{1.13}$$

Several studies of certain elliptical and parabolic problems which are interested in the results of existence and uniqueness have been carried out by many researchers (see [6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 18, 17]).

Our goal in this paper, is to prove the existence of entropy solution for the problem in generalized Sobolev spaces without the sign condition (1.11) and no coercivity condition will be assumed, then we assume that the growth of $\mu(x,t,w,\nabla w)$ is not controlled with respect to w in order to prove the existence results in generalized sobolev spaces.

The outline of this paper is as follows : After giving some preliminaries and background concerning the musielak-Orlicz space,we present in Section 3 some technical lemmas which will be needed later, and the section 4, will be devoted to states the main results and giving te steps of the proof of an existence theorem. The final section 5, we finish with a conclusion.

2 Background

Here we give some definitions and notations concerning Musielak-Orlicz spaces ([21]).

2.1 Musielak-Orlicz functions

Let Ω be an open subset of $\mathbb{R}^n$.

A Musielak-Orlicz function Ψ is a real-valued function defined in $\Omega \times \mathbb{R}_+$ such that
a) $\Psi(x,t)$ is an N-function i.e. convex, nondecreasing, continuous, $\Psi(x,0) = 0, \Psi(x,t) > 0$ for all $t > 0$ and

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\Psi(x,t)}{t} = 0, \qquad \lim_{t\to\infty} \inf_{x\in\Omega} \frac{\Psi(x,t)}{t} = 0.$$

b) $\Psi(\cdot,t)$ is a Lebesgue measurable function.

Put $\Psi_x(t) = \Psi(x,t)$ and let Ψ_x^{-1} be the non-negative reciprocal function with respect to t, i.e

$$\Psi_x^{-1}(\Psi(x,t)) = \Psi\left(x, \Psi_x^{-1}(t)\right) = t.$$

We said that Ψ satisfy the Δ_2 -condition if for some $k > 0$, and a non negative function h, integrable in Ω, we have

$$\Psi(x,2t) \leq k\Psi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \geq 0. \tag{2.1}$$

Ψ is said to satisfy the Δ_2 -condition near infinity When 2.1 holds only for $t \geq t_0 > 0$.

Let Ψ and γ be two Musielak-orlicz functions, we say that Ψ dominate γ and we write $\gamma \prec \Psi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x \in \Omega$

$$\gamma(x,t) \leq \Psi(x,ct) \text{ for all } t \geq t_0, \quad (\text{ resp. for all } t \geq 0 \text{ i.e. } t_0 = 0).$$

We say that γ grows essentially less rapidly than Ψ at 0 (resp. near infinity) and we write $\gamma \prec\prec \Psi$ if for every constant $c > 0$ one has

$$\lim_{t\to 0}\left(\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\Psi(x,t)}\right) = 0, \qquad \left(\text{resp. } \lim_{t\to\infty}\left(\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\Psi(x,t)}\right) = 0\right).$$

2.2 Musielak-Orlicz-Sobolev spaces

For a Musielak-Orlicz function Ψ and a measurable function $w : \Omega \longrightarrow \mathbb{R}$, we put

$$\rho_{\Psi,\Omega}(w) = \int_\Omega \Psi(x, |w(x)|)dx.$$

The set $K_\Psi(\Omega) = \{w : \Omega \longrightarrow \mathbb{R} \text{ measurable } / \rho_{\Psi,\Omega}(w) < \infty\}$ is named the Musielak-Orlicz class. The Musielak-Orlicz space $L_\Psi(\Omega)$ is the vector space generated by $K_\Psi(\Omega)$, that is, $L_\Psi(\Omega)$ is the smallest linear space containing the set $K_\Psi(\Omega)$. That's to say

$$L_\Psi(\Omega) = \left\{w : \Omega \longrightarrow \mathbb{R} \text{ measurable } / \rho_{\Psi,\Omega}\left(\frac{w}{\lambda}\right) < \infty, \text{ for some } \lambda > 0\right\}.$$

For a Musielak-Orlicz function Ψ we put: $\Phi(x,s) = \sup_{t>0}\{st - \Psi(x,t)\}, \Phi$ is the conjugate Musielak-Orlicz function of Ψ in the sens of Young with respect to the variable s in the space $L_\Psi(\Omega)$

we give the following norms:

$$\|w\|_{\Psi,\Omega} = \inf\left\{\lambda > 0 / \int_\Omega \Psi\left(x, \frac{|w(x)|}{\lambda}\right) dx \leq 1\right\}, \text{ (the Luxemburg norm)}$$

$$|||w|||_{\Psi,\Omega} = \sup_{\|v\|_\Phi \leq 1} \int_\Omega |w(x)v(x)|dx, \text{ (so-called Orlicz norm)}$$

where Φ is the Musielak Orlicz function complementary to Ψ. These two norms are equivalent ([21])

We will need the space $E_\Psi(\Omega)$ given by

$$E_\Psi(\Omega) = \left\{w : \Omega \longrightarrow \mathbb{R} \text{ measurable } / \rho_{\Psi,\Omega}\left(\frac{w}{\lambda}\right) < \infty, \text{ for all } \lambda > 0\right\}.$$

A Musielak function Ψ is locally integrable on Ω if $\rho_{\Psi}(t\chi_D) < \infty$ for all $t > 0$ and all measurable $D \subset \Omega$ with meas $(D) < \infty$.

We say that sequence of functions $w_n \in L_{\Psi}(\Omega)$ is modular convergent to $w \in L_{\Psi}(\Omega)$ if there exists $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\Psi,\Omega}\left(\frac{w_n - w}{\lambda}\right) = 0.$$

$$ts \leq \Psi(x,t) + \Phi(x,s), \quad \forall t, s \geq 0, x \in \Omega, \qquad \text{Young inequality ([21])} \tag{2.2}$$

this implies that

$$\|w\|_{\Psi,\Omega} \leq \rho_{\Psi,\Omega}(w) + 1. \tag{2.3}$$

$$\|w\|_{\Psi,\Omega} \leq \rho_{\Psi,\Omega}(w) \text{ if } \|w\|_{\Psi,\Omega} > 1, \tag{2.4}$$

$$\|w\|_{\Psi,\Omega} \geq \rho_{\Psi,\Omega}(w) \text{ if } \|w\|_{\Psi,\Omega} \leq 1. \tag{2.5}$$

For a Musielak Orlicz functions Ψ and her conjugate Φ, let $w \in L_{\Psi}(\Omega)$ and $v \in L_{\Phi}(\Omega)$, then we have

$$\left|\int_{\Omega} w(x)v(x)dx\right| \leq \|w\|_{\Psi,\Omega}\||v|\|_{\Phi,\Omega}. \text{ Holder inequality (see[21])} \tag{2.6}$$

2.3 Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let Ω a bounded open subset of $\mathbb{R}^N$ and let $Q = \Omega\times]0,T[$ with $T > 0$. Let Ψ and Φ be two conjugate Musielak-Orlicz functions. For each $\alpha \in \mathbb{N}^N$ denote by D_x^{α} the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Generalized sobolev spaces (Musielak-Orlicz-Sobolev) of order 1 are defined as follows.

$$W^{1,x}L_{\Psi}(Q) = \{w \in L_{\Psi}(Q) : \forall|\alpha| \leq 1 D_x^{\alpha}w \in L_{\Psi}(Q)\}$$

et

$$W^{1,x}E_{\Psi}(Q) = \{w \in E_{\Psi}(Q) : \forall|\alpha| \leq 1 D_x^{\alpha}w \in E_{\Psi}(Q)\}.$$

This second space is a subspace of the first one, and both are Banach spaces under the norm

$$\|w\| = \sum_{|\alpha|\leq 1} \|D_x^{\alpha}w\|_{\Psi,Q}$$

Now we may consider the weak topologies $\sigma(\Pi L_{\Psi}, \Pi E_{\Phi})$ and $\sigma(\Pi L_{\Psi}, \Pi L_{\Phi})$ If $w \in W^{1,x}L_{\Psi}(Q)$ then the function $t \to w(t) = w(\cdot,t)$ is defined on $[0,T]$ with values in $W^1L_{\Psi}(\Omega)$. If $w \in W^{1,x}E_{\Psi}(Q)$, then $w \in W^1E_{\Psi}(\Omega)$ and it is strongly measurable. Furthermore, the imbedding $W^{1,x}E_{\Psi}(Q) \subset L^1(0,T,W^1E_{\Psi}(\Omega))$ holds.

However, the scalar function $t \to \|u(t)\|_{\Psi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_{\Psi}(Q)$ is defined as the norm closure of $\mathcal{D}(Q)$ in $W^{1,x}E_{\Psi}(Q)$.

Theorem 2.1.
If $w \in W^{1,x}L_{\Psi}(Q) \cap L^1(Q)$ and $\frac{\partial w}{\partial t} \in W^{-1,x}L_{\Phi}(Q) + L^1(Q)$, then there exists (v_j) in $\mathcal{D}(\bar{Q})/v_j \to w$ in $W^{1,x}L_{\Psi}(Q)$ and

$$\frac{\partial v_j}{\partial t} \to \frac{\partial w}{\partial t} \text{ in } W^{-1,x}L_{\Phi}(Q) + L^1(Q)$$

for the modular convergence.

3 Auxiliary lemma

The truncation function will be given by $T_k(r) = \max(-k, \min(k,r)), k > 0$.

Definition 3.1. If there exists a constant $A > 0$ such that

$$\frac{\Psi(x,t)}{\Psi(y,t)} \leq t\left(\frac{A}{\log\left(\frac{1}{|x-y|}\right)}\right)$$

for all $t \geq 1$ and for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$

we said that the Musielak function Ψ verify the log-Hölder continuity condition on Ω

Lemma 3.2. *[2] Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N (N \geq 2)$ and let Ψ be a Musielak function satisfying the log-Hölder continuity such that*

$$\bar{\Psi}(x,1) \leq c_1 \quad \text{a.e in } \Omega \text{ for some } c_1 > 0 \tag{3.1}$$

Then $\mathfrak{D}(\Omega)$ is dense in $L_\Psi(\Omega)$ and in $W_0^1 L_\Psi(\Omega)$ for the modular convergence.

Remark 3.3. Note that if $\lim_{t\to\infty} \inf_{x\in\Omega} \frac{\Psi(x,t)}{t} = \infty$, then (3.1) holds (see [2]).

Lemma 3.4. *[2] (Poincare's inequality: Integral form) Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N (N \geq 2)$ and consider Ψ a Musielak function which verify the log-Hölder continuity. Then there exists a constants $\beta, \eta > 0$ and λ depending only on Ω and Ψ such that*

$$\int_\Omega \Psi(x,|v|)dx \leq \beta + \eta \int_\Omega \Psi(x,\lambda|\nabla v|)dx \text{ for all } v \in W_0^1 L_\Psi(\Omega). \tag{3.2}$$

Lemma 3.5. *[2] (Poincare's inequality) Let Ω be a bounded Lipchitz domain of $\mathbb{R}^N (N \geq 2)$ and let us consider Ψ be a Musielak function satisfying the log-Hölder continuity. Then there exists a constant $C > 0$ such that*

$$\|v\|_\Psi \leq C\|\nabla v\|_\Psi \quad \forall v \in W_0^1 L_\Psi(\Omega).$$

Lemma 3.6. *[?]. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let Ψ be a Musielak- Orlicz function and let $w \in W_0^1 L_\Psi(\Omega)$. Then $F(w) \in W_0^1 L_\Psi(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i} F(w) = \begin{cases} F'(w)\frac{\partial w}{\partial x_i} & \text{a.e in } \{x \in \Omega : w(x) \in D\}. \\ 0 & \text{a.e in } \{x \in \Omega : w(x) \notin D\}. \end{cases}$$

Lemma 3.7. *[22] Let $w_n, w \in L_\Psi(\Omega)$. If $w_n \to w$ with respect to the modular convergence, then $w_n \to w$ for $\sigma(L_\Psi(\Omega), L_\Phi(\Omega))$.*

4 Existence results

Let Λ a measurable function with values in $\mathbb{R}$ such that

$$\Lambda \in W_0^1 E_\Psi(Q) \cap L^\infty(Q), \quad \frac{\partial \Lambda}{\partial t} \in L^1(Q) \quad \text{such that} \quad w_0 \geq \Lambda$$

and let

$$K_\Lambda = \left\{ w \in W_0^{1,x} L_\Psi(Q) : w \geq \Lambda \text{ a.e. in } Q \right\}.$$

The existence theorem can be stated as follows.

Theorem 4.1. *Under the assumptions (1.2)-(1.9). Then the problem (1) admit at least one solution defined as follows:*

$$w \in T_0^{1,\Psi}(Q) \text{ and } w \geq \Lambda \quad \text{a.e. in } \Omega \times (0,T), \tag{4.1}$$

and for all $v \in W_0^{1,x} L_\Psi(Q) \cap L^\infty(Q)$, $\frac{\partial v}{\partial t} \in W_0^{-1,x} L_\Phi(Q)$ *such that* $v \geq \Lambda$ *a.e. in* Q *and* $\forall k > 0$, $\tau \in (0,T)$

$$\begin{aligned}
&\int_\Omega S_k(b(w(\tau)) - v(\tau)\, dx + \int_0^\tau \langle \frac{\partial v}{\partial t}, T_k(b(w) - v) \rangle \, dt \\
&\quad + \int_Q \varrho(x,t,w,\nabla w) \nabla T_k(w - v)\, dx\, dt + \int_Q \mathbb{H}(x,t,u,\nabla w) T_k(w - v)\, dx\, dt \\
&\quad + \int_Q \mathbb{F}(x,t,w) \nabla T_k(w - v)\, dx\, dt \leq \int_Q f T_k(w - v)\, dx\, dt + \int_\Omega S_k(b(w_0) - v(0)\, dx,
\end{aligned} \tag{4.2}$$

where $S_k(s) = \displaystyle\int_0^s T_k(r)\, dr$.

Step 1: Approximate problems

For each $n > 0, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N$, let us define the approximations:

$$b_n(s) = b\left(T_n(s)\right), \forall s \in \mathbb{R}, \tag{4.3}$$

$$\varrho_n(x,t,s,\xi) = \varrho\left(x,t,T_n(s),\xi\right) \quad \text{a.e. } (x,t) \in Q, \tag{4.4}$$

$$\mathbb{F}_n(x,t,s) = \mathbb{F}\left(x,t,T_n(s)\right) \quad \text{a.e. } (x,t) \in Q, \tag{4.5}$$

$$\mathbb{H}_n(x,t,s,\xi) = \frac{\mathbb{H}(x,t,s,\xi)}{1 + \frac{1}{n}|\mathbb{H}(x,t,s,\xi)|}, \tag{4.6}$$

$$w_{0n} \in \mathcal{C}_0^\infty(\Omega) \text{ such that } b_n\left(w_{0n}\right) \to b\left(w_0\right) \text{ strongly in } L^1(\Omega), \tag{4.7}$$

f_n a sequence of smooth functions which converges strongly to f in $L^1(Q)$, with $\|f_n\|_{L^1(Q)} \leq \|f\|_{L^1(Q)}$.

Let us define the approximate problems

$$\begin{cases}
\frac{\partial b(w_n)}{\partial t} - \operatorname{div}\left(\varrho_n(x,t,w_n,\nabla w_n)\right) + \mathbb{H}_n\left(x,t,w_n,\nabla w_n\right) & \\
+ n T_n\left(w_n - \Lambda\right)^- = f_n + \operatorname{div}\left(\mathbb{F}_n\left(x,t,w_n\right)\right) & \text{in } Q, \\
w_n(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\
w_n(x,0) = w_{0n} & \text{in } \Omega.
\end{cases} \tag{4.8}$$

Since $\mathbb{H}_n$ is bounded for any $n > 0$, the problem (4.8) admit one solution $w_n \in W_0^{1,x} L_\Psi(Q)$ (see [19]).

Step 2: A priori estimates.

By fixing $k > 0$ Let $\tau \in (0,T)$ and taking $\exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+ \chi_{(0,\tau)}$ as a test in problem (4.8) where $G(s) = \displaystyle\int_0^s \frac{\rho(r)}{\alpha'} dr$, we get

$$\begin{aligned}
&\int_{Q_\tau} \frac{\partial b_n(w_n)}{\partial t} \exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+ dxdt \\
&+ \int_{Q_\tau} \varrho_n(x,t,w_n,\nabla w_n) \nabla \left(\exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+\right) dxdt \\
&+ \int_{Q_\tau} \mathbb{F}_n\left(x,t,w_n\right) \nabla \left(\exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+\right) dxdt \\
&+ \int_{Q_\tau} \mathbb{H}\left(x,t,w_n,\nabla w_n\right) \exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+ dxdt \\
&+ \int_{Q_\tau} n T_n\left(w_n - \Lambda\right)^- \exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+ dxdt \\
&\leq k \exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right) \|f_n\|_{L^1(Q)}.
\end{aligned} \tag{4.9}$$

Put

$$\widetilde{T}_k(r) = \int_0^r \exp(G(s))T_k(s)^+ ds$$

then

$$\int_{Q_\tau} \frac{\partial b_n(w_n)}{\partial t} \exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+ dxdt = \int_\Omega \widetilde{T}_k\left(b_n(w_n(\tau)\right) dx - \int_\Omega \widetilde{T}_k\left(b_n(w_n(0))\right) dx \tag{4.10}$$

By definition we may write

$$\int_\Omega \widetilde{T}_k\left(b_n(w_n(\tau)\right) dx \geq 0, \tag{4.11}$$

and

$$\int_\Omega \widetilde{T}_k\left(b_n(w_n(0))\right) dx \leq k \exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right) \|b(w_0)\|_{L^1(\Omega)}. \tag{4.12}$$

By using (1.6) one has

$$\begin{aligned} &\int_{Q_\tau} \mathbb{H}_n\left(x,t,w_n,\nabla w_n\right) \exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+ dxdt \\ &\leq \int_{Q_\tau} \rho\left(w_n\right) \exp\left(G\left(w_n\right)\right) \Psi\left(x,\nabla w_n\right) T_k\left(w_n\right)^+ dxdt \end{aligned} \tag{4.13}$$

By (1.7) and Young inequality we have

$$\begin{aligned} &\int_{Q_\tau} \mathbb{F}_n\left(x,t,w_n\right) \nabla\left(\exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+\right) dxdt \\ &\quad\leq \frac{\|c(.,.)\|_{L^\infty(Q)}}{\alpha'}\left[\alpha_0 \int_{Q_\tau} \Psi\left(x,w_n\right)\rho\left(w_n\right)\exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+ dxdt\right. \\ &\quad\quad \left. + \int_{Q_r} \Psi\left(x,\nabla w_n\right)\rho\left(w_n\right)\exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+ dxdt\right] \\ &\quad\quad + \|c(\dots,)\|_{L^\infty(Q)}\alpha_0 \int_{Q_r} \Psi\left(x,w_n\right)\exp\left(G\left(w_n\right)\right) dxdt \\ &\quad\quad + \|c(.,.)\|_{L^\infty(Q)} \int_{Q_r} \Psi\left(x,\left|\nabla T_k\left(w_n\right)^+\right|\right)\exp\left(G\left(w_n\right)\right) dxdt \end{aligned} \tag{4.14}$$

According to (4.14) and (1.2) we get

$$\begin{aligned} &\frac{1}{\alpha'}\int_{Q_\tau} \Psi\left(x,w_n\right)\rho\left(w_n\right)\exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+ dxdt \\ &\frac{\alpha}{\alpha'}\int_{Q} \Psi\left(x,\nabla w_n\right)\rho\left(w_n\right)\exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+ dxdt \\ &+ \int_{Q_\tau} \varrho\left(x,t,w_n,\nabla w_n\right)\exp\left(G\left(w_n\right)\right) \nabla T_k\left(w_n\right)^+ dxdt \\ &+ \int_{Q_t} nT_n\left(w_n - \Lambda\right)^- \exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+ dxdt \\ &\leq \frac{\|c(.,.)\|_{L^\infty(Q)}}{\alpha'}\left[\alpha_0 \int_{Q_\tau} \Psi\left(x,w_n\right)\rho\left(w_n\right)\exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+ dxdt\right. \\ &\left. + \int_{Q_r} \Psi\left(x,\nabla w_n\right)\rho\left(w_n\right)\exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+ dxdt\right] \\ &+\|c(\dots,)\|_{L^\infty(Q)}\alpha_0 \int_{Q_r} \Psi\left(x,w_n\right)\exp\left(G\left(w_n\right)\right) dxdt \\ &+\|c(.,.)\|_{L^\infty(Q)} \int_{Q_r} \Psi\left(x,\left|\nabla T_k\left(w_n\right)^+\right|\right)\exp\left(G\left(w_n\right)\right) dxdt \\ &+ \int_{Q_\tau} \Psi\left(x,\nabla w_n\right)\rho\left(w_n\right)\exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^+ dxdt \\ &+k\exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right)\left[\|f\|_{L^1(Q)} + \|b(w_0)\|_{L^1(\Omega)} + \int_Q |P(x,t)| dxdt\right]. \end{aligned} \tag{4.15}$$

Thus,

$$\begin{aligned}&\left[\frac{1-\alpha_0\|c(.,.)\|_{L^\infty(Q)}}{\alpha'}\right]\int_{Q_r}\Psi\left(x,w_n\right)\rho\left(w_n\right)\exp\left(G\left(w_n\right)\right)T_k\left(w_n\right)^+dxdt\\&\quad+\left[\frac{\alpha-\|c(...)\|_{L^\infty(Q)-\alpha'}}{\alpha'}\right]\int_{Q_\tau}\Psi\left(x,\nabla w_n\right)\rho\left(w_n\right)\exp\left(G\left(w_n\right)\right)T_k\left(w_n\right)^+dxdt\\&+\int_{Q_\tau}\varrho\left(x,t,w_n,\nabla w_n\right)\exp\left(G\left(w_n\right)\right)\nabla T_k\left(w_n\right)^+dxdt\\&+\int_{Q_t}nT_n\left(w_n-\varLambda\right)^-\exp\left(G\left(w_n\right)\right)T_k\left(w_n\right)^+dxdt\\&\le\frac{\|c(.,.)\|_{L^\infty(Q)}}{\alpha}\left[\alpha_0\alpha\int_{\{0\le w_n\le k\}}\Psi\left(x,w_n\right)\exp\left(G\left(w_n\right)\right)dxdt+\alpha\Psi\left(x,\nabla T_k\left(w_n\right)^+\right)\exp\left(G\left(w_n\right)\right)dxdt\right]\\&\qquad+kc_1.\end{aligned}\tag{4.16}$$

We can take α' such that $\alpha'<\alpha-\|c(.,.)\|_{L^\infty(Q)}$ and thanks to (1.2) we obtain

$$\begin{aligned}&\left[1-\frac{\|c(.,.)\|_{L^\infty(Q)}}{\alpha}\right]\int_{Q_r}\varrho\left(x,t,w_n,\nabla w_n\right)\exp\left(G\left(w_n\right)\right)\nabla T_k\left(w_n\right)^+dxdt\\&+\int_{Q_t}nT_n\left(w_n-\varLambda\right)^-\exp\left(G\left(w_n\right)\right)T_k\left(w_n\right)^+dxdt\le kc_1.\end{aligned}\tag{4.17}$$

Taking $\frac{1}{c_2}=\left[1-\frac{\|c(.,.)\|_{L^\infty(Q)}}{\alpha}\right]$
Thus,

$$\begin{aligned}&\int_{Q_\tau}\varrho\left(x,t,w_n,\nabla w_n\right)\exp\left(G\left(w_n\right)\right)\nabla T_k\left(w_n\right)^+dxdt\\&\quad+c_2\int_{Q_\tau}nT_n\left(w_n-\varLambda\right)^-\exp\left(G\left(w_n\right)\right)T_k\left(w_n\right)^+dxdt\le kc_1c_2.\end{aligned}$$

It follow that

$$0\le\int_{Q_\tau}nT_n(w-\varLambda)^-\exp\left(G\left(w_n\right)\right)\frac{T_k\left(w_n\right)^+}{k}dxdt\le c_1,$$

as $k\to 0$ Fatou's lemma implies that

$$0\le\int_{\{u_n\ge 0\}}nT_n\left(w_n-\varLambda\right)^-\exp\left(G\left(w_n\right)\right)dxdt\le c_1.$$

Thanking to (4.17) we can have

$$\int_{\{0\le w_n\le k\}}\varrho\left(x,t,w_n,\nabla w_n\right)\exp\left(G\left(w_n\right)\right)\nabla T_k\left(w_n\right)dxdt\le kc_1c_2,$$

since $\exp\left(G\left(w_n\right)\right)\ge 1$ for $0\le w_n\le k$, then

$$\int_{\{0\le w_n\le k\}}\varrho\left(x,t,w_n,\nabla w_n\right)\nabla T_k\left(w_n\right)dxdt\le kc_1c_2\tag{4.18}$$

by (1.2)

$$\int_{Q_\tau}\Psi\left(x,\left|\nabla T_k\left(w_n\right)^+\right|\right)dxdt\le\frac{kc_1c_2}{\alpha},\tag{4.19}$$

and

$$0\le\int_{\{u_n\ge 0\}}nT_n\left(w_n-\varLambda\right)^-dxdt\le c_1.\tag{4.20}$$

By the similar idea, we choose $\exp\left(-G\left(w_n\right)\right)T_k\left(w_n\right)^-\chi_{(0,\tau)}$ as a test function in (4.8) we obtain

$$\begin{aligned}
&\int_{Q_\tau} \frac{\partial b_n(w_n)}{\partial t} \exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^- \,dxdt \\
&+\int_{Q_\tau} \varrho_n(x,(x,t,w_n,\nabla w_n)\,\nabla\left(\exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^-\right) dxdt \\
&+\int_{Q_\tau} \mathbb{F}_n\left(x,t,w_n\right) \nabla\left(\exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^-\right) dxdt \\
&+\int_{Q_\tau} \mathbb{H}\left(x,t,w_n,\nabla w_n\right) \exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^- \,dxdt \\
&+\int_{Q_\tau} nT_n\left(w_n-\Lambda\right)^- \exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^- \,dxdt \\
&\geq \int_{Q_\tau} f_n \exp\left(-G\left(w_n\right)\right) T_k\left(w_n\right)^- \,dxdt
\end{aligned} \tag{4.21}$$

by choosing

$$\widetilde{T}_k(r)=\int_0^r \exp(G(s))T_k(s)^- ds$$

we get

$$\int_{Q_\tau} \frac{\partial b_n(w_n)}{\partial t} \exp\left(G\left(w_n\right)\right) T_k\left(w_n\right)^- \,dxdt = \int_\Omega \widetilde{T}_k\left(b_n(w_n(\tau)\right) dx - \int_\Omega \widetilde{T}_k\left(b_n(w_n(0))\right) dx \tag{4.22}$$

By definition we have

$$\int_\Omega \widetilde{T}_k\left(b_n(w_n(\tau)\right) dx \geq 0, \tag{4.23}$$

and

$$\int_\Omega \widetilde{T}_k\left(b_n(w_n(0))\right) dx \leq k \exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right) \|b(w_0)\|_{L^1(\Omega)}\,. \tag{4.24}$$

and using the similar techniques, we get

$$\begin{aligned}
&\int_{Q_\tau} \varrho\left(x,t,w_n,\nabla w_n\right) \exp\left(-G\left(w_n\right)\right) \nabla T_k\left(w_n\right) dxdt \\
&+c_2\int_{Q_\tau} nT_n(w-\Lambda)^- \exp\left(-G\left(w_n\right)\right) T_k\left(w_n\right)^- \,dxdt \leq kc_1c_2.
\end{aligned} \tag{4.25}$$

It follow that

$$0 \leq \int_{Q_r} nT_n\left(w_n-\Lambda\right)^- \exp\left(-G\left(w_n\right)\right) \frac{T_k\left(w_n\right)^-}{k} dxdt \leq c_1,$$

as $k \to 0$ Fatou's lemma implies that

$$0 \leq \int_{\{w_n \leq 0\}} nT_n\left(w_n-\Lambda\right)^- \exp\left(-G\left(w_n\right)\right) dxdt \leq c_1,$$

since $\exp\left(-G\left(w_n\right)\right) \geq 1$ and as $-k \leq w_n \leq 0$, thus

$$\int_{\{-k \leq w_n \leq 0\}} \varrho\left(x,t,w_n,\nabla w_n\right) \nabla T_k\left(w_n\right) dxdt \leq kc_1c_2, \tag{4.26}$$

$$\int_{Q_\tau} \Psi\left(x,\left|\nabla T_k\left(w_n\right)^-\right|\right) dxdt \leq \frac{kc_1c_2}{\alpha} \tag{4.27}$$

and

$$0 \leq \int_{\{w_n \leq 0\}} nT_n\left(w_n-\Lambda\right)^- dxdt \leq c_1 \tag{4.28}$$

Combining now (4.20) and (4.26) we get,

$$\int_Q \varrho\left(x, t, w_n, \nabla w_n\right) \nabla T_k\left(w_n\right) dxdt \leq kC_1. \tag{4.29}$$

Of the same with (4.19) and (4.27) we get,

$$\int_Q \Psi\left(x, \left|\nabla T_k\left(w_n\right)\right|\right) dxdt \leq kC_2. \tag{4.30}$$

Then $T_k\left(\left(w_n\right)\right)$ is bounded in $W_0^{1,x} L_\Psi(Q)$ independently of n and for any $k>0$, consequently there exists a subsequence still denoted by w_n such that

$$T_k\left(w_n\right) \rightharpoonup \xi_k \quad \text{weakly in} \quad W_0^{1,x} L_\Psi(Q) \tag{4.31}$$

Now, according to (4.30), we obtain

$$\begin{aligned}\inf_{x\in\Omega} \Psi\left(x, \frac{k}{\delta}\right) \operatorname{meas}\left\{\left|u_n\right|>k\right\} &\leq \int_{\left|w_n\right|>k} \Psi\left(x, \frac{\left|T_k\left(w_n\right)\right|}{\delta}\right) dxdt \\ &\leq \int_{Q_T} \Psi\left(x, \left|\nabla T_k\left(w_n\right)\right|\right) dxdt \leq kC\end{aligned}$$

Then

$$\operatorname{meas}\left\{\left|w_n\right|>k\right\} \leq \frac{kC}{\inf_{x\in\Omega} \Psi\left(x, \frac{k}{\delta}\right)}$$

Thanks to (**??**) , we get

$$\lim_{k\rightarrow\infty} \operatorname{meas}\left\{\left|w_n\right|>k\right\}=0. \tag{4.32}$$

Step 3:Almost everywhere convergence of w_n and of $b_n\left(w_n\right)$

Let $\lambda>0$ then

$$\begin{gathered}\operatorname{meas}\{\ \{w_m-w_n\right|>\lambda\} \leq \operatorname{meas}\left\{\left|w_m\right|>k\right\} \\ +\operatorname{meas}\{\ \{w_n\right|>k\}+\operatorname{meas}\left\{\left|T_k\left(w_m\right)-T_k\left(w_n\right)\right|>\lambda\right\}\end{gathered}$$

By (4.31) we suppose that $T_k\left(w_n\right)$ is a Cauchy sequence in measure in Q and thanks to (4.32) we conclude that for any $\epsilon>0$ there exists $k(\epsilon)>0$ such that

$$\operatorname{meas}\left\{\left|w_m-w_n\right|>\lambda\right\} \leq \epsilon \quad \text{for all} \quad n, m>N_{k(\epsilon), \lambda}.$$

Consequently w_n is a Cauchy sequence in measure in Q, thus converge almost every where to w

For $k<n$, let $\mathbb{H}_k \in W^{2,\infty}(\mathbb{R})$, such that $\mathbb{H}_k^{\prime}$, has a compact support $\operatorname{supp}\left(\mathbb{H}_k^{\prime}\right) \subset[-k, k]$. We multiply (4.8) by $\mathbb{H}_k^{\prime}\left(w_n\right)$, to obtain in $\mathcal{D}^{\prime}(Q)$

$$\begin{aligned}\frac{\partial B_{\mathbb{H}k}^n\left(w_n\right)}{\partial t}= & \operatorname{div}\left(\mathbb{H}_k^{\prime}\left(w_n\right)\left(\varrho_n(x,\left(w_n, \nabla w_n\right)+\mathbb{F}_n\left(w_n\right)\right)\right. \\ & -\mathbb{H}_k^{\prime\prime}\left(w_n\right)\left(\varrho_n(x,\left(w_n, \nabla w_n\right)+\mathbb{F}_n\left(w_n\right)\right) \nabla w_n+f_n \mathbb{H}_k^{\prime}\left(w_n\right)\end{aligned} \tag{4.33}$$

where $B_{\mathbb{H}k}^n(r)=\displaystyle\int_0^r \mathbb{H}_k^{\prime}(s) \frac{\partial b_n(s)}{\partial s} ds$ Then, we show that

$$\left(B_{\mathbb{H}k}^n\left(w_n\right)\right) \text{ is bounded in } W_0^{1,x} L_\Psi(Q), \tag{4.34}$$

and

$$\left(\frac{\partial B_{\mathbb{H}_k}^n\left(w_n\right)}{\partial t}\right) \text{ is bounded in } L^1(Q)+W^{-1,x} L_\Phi(Q). \tag{4.35}$$

Indeed, we obtain

$$\left|\nabla B_{g_t}^n\left(w_n\right)\right| \leq b_1\left|\nabla T_k\left(w_n\right)\right|\|\mathbb{H}_k'\|_{L^\infty(R)} \text{ a.e. in } Q,$$

and according to (4.29) we obtain (4.34). In the other hand since supp($\mathbb{H}_k'$) and supp ($\mathbb{H}_k''$) are both included in $[-k,k]$, w_n can be changed by $T_k\left(w_n\right)$ in each of these terms. As a consequence, each term in the right hand side of (4.33) is bounded either in $W^{-1,x}L_\Phi(Q)$ or in $L^1\left(Q\right)$ which implies that (4.35) holds true. As in (1.3) estimates (4.34)and (4.35) leads, for a subsequence, still indexed by n

$$b_n\left(w_n\right) \rightarrow b(w) \text{ a.e in } Q, \quad b(w) \in L^\infty\left(0,T,L^1(\Omega)\right) \tag{4.36}$$

Step 4: Convergence of $\varrho_n(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right))$

Let $w \in \left(E_\Psi(\Omega)\right)^N$. By (1.3) we get,

$$\left(\varrho\left(x,t,w_n,\nabla w_n\right)-\varrho\left(x,t,w_n,w\right)\right)\left(\nabla w_n-w\right)>0$$

then

$$\begin{aligned}\int_{\{|w_n|\leq k\}}\varrho\left(x,t,w_n,\nabla w_n\right)wdxdt \leq& \int_{\{|w_n|\leq k\}}\varrho\left(x,t,w_n,\nabla w_n\right)\nabla w_n dxdt\\ &+\int_{\{|w_n|\leq k\}}\varrho\left(x,t,w_n,w\right)\left(w-\nabla w_n\right)dxdt\end{aligned}$$

by (1.4) we have for $\nu>\beta$

$$\begin{aligned}\int_{\{|w_n|\leq k\}}\varPhi_x\left(x,\frac{\varrho\left(x,t,w_n,\frac{w}{k_2}\right)}{3\nu}\right)dxdt \leq& \frac{\beta}{3\nu}\int_{Q_T}\left[\varPhi\left(x,a_0(x,t)\right)+\gamma\left(x,k_1|T_k\left(w_n\right)\right)\right]dxdt\\ &+\frac{\beta}{3\nu}\int_{Q_T}[\varPsi(x,|w|)]dxdt\\ \leq&\frac{\beta}{3\nu}\left[\int_{Q_T}\varPhi\left(x,a_0(x,t)\right)+\gamma\left(x,k_1k\right)dxdt\right]\\ &+\frac{\beta}{3\nu}\left[\int_Q\varPsi(x,|w|)dxdt\right]\end{aligned} \tag{4.37}$$

Thus $\left\{\varrho\left(x,t,T_k\left(w_n\right),\frac{w}{k_2}\right)\right\}$ is bounded in $\left(L_\Phi(\Omega)\right)^N$. By (4.29), (4.37) and by the theorem of Banach-Steinhaus, the sequence $\left\{\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right)\right)\right\}$ is bounded in $\left(L_\Phi(\Omega)\right)^N$ and we deduce

$$\varrho_n(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right) \rightharpoonup \varpi_k \text{ in } \left(L_\Phi(Q)\right)^N, \text{ for } \sigma\left(\Pi L_\Phi,\Pi E_\Psi\right) \text{ for some } \varpi_k \in \left(L_\Phi(Q)\right)^N. \tag{4.38}$$

Then,

$$T_k\left(w_n\right) \rightharpoonup \text{ weakly } T_k(w) \text{ in } W_0^{1,x}L_\Psi(Q) \text{ for } \sigma\left(\prod L_\Psi,\prod E_\Phi\right). \tag{4.39}$$

Step 5: Almost everywhere convergence of the gradients.

Choosing $Z_m\left(w_n\right)=T_1\left(w_n-T_m\left(w_n\right)\right)$ as a test in (4.8) leads

$$\int_{\{m\leq|w_n|\leq m+1\}}\varrho_n(x,t,w_n,\nabla w_n)\nabla w_n dxdt \leq C\Big(\int_Q f_n Z_m\left(w_n\right)dxdt+\int_{\{|w_{0n}|>m\}}\left|b_n\left(w_{0n}\right)\right|dxdt\Big)$$

where $\frac{1}{C}=\left[1-\frac{(\alpha_0 b_1+1)}{\alpha}\|c(.,.)\|_{L^\infty(Q)}\right]>0.$

Passing to the limit as $n\rightarrow+\infty$, using the pointwise convergence of w_n and strongly convergence in $L^1\left(Q\right)$ of f_n we get

$$\lim_{n\to+\infty}\int_{\{m\le|w_n|\le m+1\}}\varrho_n(x,t,w_n,\nabla w_n)\nabla w_n dxdt\le C\Big(\int_Q fZ_m(w)dxdt+\int_{\{|w_0|>m\}}|b\,(w_0)|\,dxdt\Big).$$

By applying Lebesgue's theorem and as $m\to+\infty$, in the all terms of the right-hand side, we get

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|w_n|\le m+1\}}\varrho\,(x,t,w_n,\nabla w_n)\,\nabla w_n dxdt=0 \tag{4.40}$$

From (1.2), we deduce also

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|w_n|\le m+1\}}\varPsi\,(x,|\nabla Z_m\,(w_n)|)\,dxdt=0 \tag{4.41}$$

Now, one has

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T}\mathbb{F}_n\,(x,t,w_n)\,\nabla Z_m\,(w_n)\,dxdt\le\lim_{m\to+\infty}\lim_{n\to+\infty}\int_Q\varPsi\,(x,|\nabla Z_m\,(w_n)|)\,dxdt$$
$$+\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|w_n|\le m+1\}}\varPhi\,(x,|\mathbb{F}_n\,(x,t,w_n)|)\,dxdt.$$

By applying Lebegue's theorem and using the pointwise convergence of w_n in the second term of the right side of this last expression, we get

$$\lim_{n\to+\infty}\int_{\{m\le|w_n|\le m+1\}}\varPhi\,(x,|\mathbb{F}_n\,(x,t,w_n)|)\,dxdt=\int_{\{m\le|w|\le m+1\}}\varPhi(x,|\mathbb{F}(x,t,w)|)dxdt$$

Lebesgue's theorem gives us

$$\lim_{m\to+\infty}\int_{\{m\le|w|\le m+1\}}\varPhi(x,|\mathbb{F}(x,t,w)|)dxdt=0. \tag{4.42}$$

Thus with (4.41) and (4.42), we get

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|w_n|\le m+1\}}\mathbb{F}\,(x,t,w_n)\,\nabla w_n dxdt=0. \tag{4.43}$$

We need the following lemma

Lemma 4.2. *Under the Assumptions (1.2)-(1.9), let (z_n) be a sequence in $W_0^{1,x}L_\varPsi\,(Q)$ such that:*

$$z_n\to z \text{ for } \sigma\,(\Pi L_\varPsi,\Pi E_\varPhi) \tag{4.44}$$

$$(\varrho\,(x,t,z_n,\nabla z_n))\ \text{ is bounded in }\ (L_\varPhi\,(Q))^N \tag{4.45}$$

$$\int_{Q_T}[\varrho\,(x,t,z_n,\nabla z_n)-\varrho\,(x,t,z_n,\nabla z\chi_s)]\,[\nabla z_n-\nabla z\chi_s]\,dxdt\to 0 \tag{4.46}$$

as $n,s\longrightarrow+\infty$, and where χ_s is the characteristic function of $Q^s=\{x\in Q;|\nabla z|\le s\}$.
Then,

$$\nabla z_n\to\nabla z \text{ a.e. in } Q, \tag{4.47}$$

$$\lim_{n\to+\infty}\int_{Q_T}\varrho\,(x,t,z_n,\nabla z_n)\,\nabla z_n dxdt=\int_Q\varrho(x,t,z,\nabla z)\nabla z dxdt, \tag{4.48}$$

$$\varPsi\,(x,|\nabla z_n|)\to\varPsi(x,|\nabla z|) \text{ in } L^1\,(Q)\,. \tag{4.49}$$

Proof: (see [4]).

Let $D(Q)\ni v_j\to w\in W_0^{1,x}L_\varPsi(Q)$ for the modular convergence.
Let $(\alpha_0^\mu)_\mu$ be a sequence of functions defined on Ω as follows

$$\alpha_0^\mu\in L^\infty(\Omega)\cap W_0^1L_\varPsi(\Omega)\text{ for all }\mu>0 \tag{4.50}$$

$$\left\|\alpha_0^\mu\right\|_{L^\infty(\Omega)} \leq k \quad \forall \mu > 0$$

$$\alpha_0^\mu \to T_k\left(w_0\right) \text{ a.c. in } \Omega \text{ and } \frac{1}{\mu}\left\|\alpha_0^\mu\right\|_{\Psi,\Omega} \to 0, \text{ as } \mu \to +\infty$$

For fixed $k, \mu > 0$, let $T_k\left(v_j\right)_\mu \in L^\infty(Q) \cap W_0^{1,x} L_\Psi(Q)$ be the unique solution of the problem like:

$$\begin{aligned} \frac{\partial T_k\left(v_j\right)_\mu}{\partial t} + \mu\left(T_k\left(v_j\right)_\mu - T_k\left(v_j\right)\right) = 0 \text{ in } D'(Q), \\ T_K\left(v_j\right)_\mu(t=0) = \alpha_0^\mu \text{ in } \Omega. \end{aligned} \tag{4.51}$$

Remark that due to (4.51), we have for $\mu > 0, j > 0$ and $k \geq 0$

$$\frac{\partial T_k\left(v_j\right)_\mu}{\partial t} \in W_0^{1,x} L_\Psi(Q)$$

Recalling that,

$$\left(T_k\left(v_j\right)\right)_\mu \to T_k(w) \text{ a.e. in } Q, \text{ weakly-* in } L^\infty(Q)$$

$\left(T_k\left(v_j\right)\right)_\mu \to \left(T_k(w)\right)_\mu$ in $W_0^{1,x} L_\Psi(Q)$ for the modular convergence as $j \to +\infty$,
$\left(T_k(w)\right)_\mu \to T_k(w)$ in $W_0^{1,x} L_\Psi(Q)$ for the modular convergence as $\mu \to +\infty$.
$\left\|\left(T_k\left(v_j\right)\right)_\mu\right\|_{L^\infty(Q)} \leq \max\left(\left\|\left(T_k(w)\right)\right\|_{L^\infty(Q)}, \quad \left\|\alpha_0^\mu\right\|_{L^\infty(\Omega)}\right) \leq k$ for all $\mu > 0$, and for all $k > 0$. We introduce a sequence of increasing $\mathbf{C}^1(\mathbb{R})-$ functions S_m such that

$$S_m(r) = 1 \text{ for } |r| \leq m, S_m(r) = m + 1 - |r|, \text{ for } m \leq |r| \leq m+1, S_m(r) = 0$$

for $|r| \geq m+1$ for any $m \geq 1$, and $\epsilon(n, \mu, \eta, j, m)$ is the quantities such that

$$\lim_{m \to +\infty} \lim_{j \to +\infty} \lim_{\eta \to +\infty} \lim_{\mu \to +\infty} \lim_{n \to +\infty} \epsilon(n, \mu, \eta, j, m) = 0.$$

The main estimate is

Lemma 4.3. *We have*

$$\int_0^T \left\langle \frac{\partial b_n\left(w_n\right)}{\partial t}, T_\eta\left(w_n - \left(T_k\left(v_j\right)\right)_\mu\right)^+ \exp\left(G\left(w_n\right)\right) S_m'\left(w_n\right) \right\rangle \geq w(n, \mu, \eta, j), \quad \forall m \geq 1$$

Proof :
For fixed $k \geq 0$, let $W_{\nu,\eta}^{n,j} = T_\eta\left(T_k\left(w_n\right) - T_k\left(v_j\right)_\mu\right)^+$ and $W_{\nu,\eta}^j = T_\eta(T_k(w) - T_k\left(v_j\right)_\mu)^+$
By choosing $\exp\left(G\left(w_n\right)\right)) W_{\nu,\eta}^{n,j} S_m\left(w_n\right)$ as a function test in (4.8) and by the similar idea used in step 2 we obtain:

$$\left\langle \frac{\partial b_n\left(w_n\right)}{\partial t}, \exp\left(G\left(w_n\right)\right) W_{\nu,\eta}^{n,j} S_m\left(w_n\right) \right\rangle \tag{4.52}$$

$$+ \int_Q \varrho_n(x,t,w_n,\nabla w_n) \exp\left(G\left(w_n\right)\right) \nabla\left(W_{\nu,\eta}^{n,j}\right) S_m\left(w_n\right) dxdt \tag{4.53}$$

$$+ \int_Q \varrho_n(x,t,w_n,\nabla w_n) \nabla w_n \exp\left(G\left(w_n\right)\right) W_{\nu,\eta}^{n,j} S_m'\left(w_n\right) dxdt \tag{4.54}$$

$$- \int_Q \mathbb{F}_n\left(x,t,w_n\right) \exp\left(G\left(w_n\right)\right) \nabla\left(W_{\nu,\eta}^{n,j}\right) S_m\left(w_n\right) dxdt \tag{4.55}$$

$$- \int_Q \mathbb{F}_n\left(x,t,w_n\right) \nabla w_n \exp\left(G\left(w_n\right)\right) W_{\nu,\eta}^{n,j} S_m'\left(w_n\right) dxdt \tag{4.56}$$

$$\leq \int_Q f_n \exp\left(G\left(w_n\right)\right) W_{\nu,\eta}^{n,j} S_m\left(w_n\right) dxdt \tag{4.57}$$

Now we pass to the limit in (4.53),(4.54),(4.55),(4.56)and in (4.57) for k real number fixed.

By lemma 4.3 we have for any fixed $k \geq 0$

$$\int_Q \frac{\partial b_n(w_n)}{\partial t} \exp\left(G\left(w_n\right)\right) W_{\nu,\eta}^{n,j} S_m\left(w_n\right) dxdt \geq \epsilon(n,\mu,\eta,j) \quad \text{for any } m \geq 1 \tag{4.58}$$

About (4.55):
If we take $n > m+1$, we get

$$\begin{aligned}\mathbb{F}_n\left(x,t,w_n\right) \exp\left(G\left(w_n\right)\right) S_m\left(w_n\right) =&\mathbb{F}\left(x,t,T_{m+1}\left(w_n\right)\right) \exp\left(G\left(T_{m+1}\left(w_n\right)\right)\right) \\ &\times S_m\left(T_{m+1}\left(w_n\right)\right)\end{aligned}$$

then $\mathbb{F}_n\left(x,t,w_n\right) \exp\left(G\left(w_n\right)\right) S_m\left(w_n\right)$ is bounded in $L_\Phi(Q)$, thus, by using the pointwise convergence of w_n and Lebesgue's theorem we obtain

$$\mathbb{F}_n\left(x,t,w_n\right) \exp\left(G\left(w_n\right)\right) S_m\left(w_n\right) \rightarrow \mathbb{F}(x,t,w) \exp(G(w)) S_m(w),$$

with the modular convergence as $n \rightarrow +\infty$ then

$$\mathbb{F}_n\left(x,t,w_n\right) \exp\left(G\left(w_n\right)\right) S_m\left(w_n\right) \rightarrow \mathbb{F}(x,t,w) \exp(G(w)) S_m(w)$$

for $\sigma\left(\prod L_\Phi, \prod L_\Psi\right)$.
In the other hand $\nabla W_{\nu,\eta}^{n,j} = \nabla T_k\left(w_n\right) - \nabla\left(T_k\left(v_j\right)\right)_\mu$ for $\left|T_k\left(w_n\right) - \left(T_k\left(v_j\right)\right)_\mu\right| \leq \eta$ converge to $\nabla T_k(w) - \nabla\left(T_k\left(v_j\right)\right)_\mu$ weakly in $\left(L_\Psi\left(Q\right)\right)^N$, then

$$\begin{aligned}&\int_Q \mathbb{F}_n\left(x,t,w_n\right) \exp\left(G\left(w_n\right)\right) S_m\left(w_n\right) \nabla W_{\nu,\eta}^{n,j} dxdt \\ &\rightarrow \int_Q \mathbb{F}(x,t,w) S_m(w) \exp(G(w)) \nabla W_{\nu,\eta}^{j} dxdt, \text{ as } n \rightarrow +\infty\end{aligned}$$

Thanking to the modular convergence of $W_{\nu,\eta}^{j}$ as $j \rightarrow +\infty$ and let $\mu \longrightarrow \infty$, we obtain

$$\int_Q \mathbb{F}_n\left(x,t,w_n\right) S_m\left(w_n\right) \exp\left(G\left(w_n\right)\right) \nabla\left(W_{\nu,\eta}^{n,j}\right) dxdt = \epsilon(n,j,\mu) \quad \text{for any } m \geq 1. \tag{4.59}$$

Concerning (4.56):
For $n > m+1 > k$, we have

$$\nabla w_n S_m'\left(w_n\right) = \nabla T_{m+1}\left(w_n\right) \text{ a.e. in } Q.$$

The almost every where convergence of W_n implies that

$$\exp\left(G\left(w_n\right)\right) W_{\nu,\eta}^{n,j} \rightarrow \exp(G(w)) W_{\nu,\eta}^{j} \text{ in } L^\infty\left(Q\right) \text{ weak-* },$$

and since $\left(\mathbb{F}_n\left(x,t,T_{m+1}\left(w_n\right)\right)\right)_n'$ converge strongly in $E_\Phi\left(Q\right)$, then

$$\mathbb{F}_n\left(x,t,T_{m+1}\left(w_n\right)\right) \exp\left(G\left(w_n\right)\right) W_{\nu,\eta}^{n,j} \rightarrow \mathbb{F}\left(x,t,T_{m+1}(w)\right) \exp(G(w)) W_{\nu,\eta}^{j}$$

converge strongly in $E_\Phi\left(Q\right)$ as $n \rightarrow +\infty$.
Since $\nabla T_{m+1}\left(w_n\right) \rightarrow \nabla T_{m+1}(w)$ weakly in $\left(L_\Psi\left(Q\right)\right)^N$ as $n \rightarrow +\infty$ we obtain

$$\begin{aligned}&\int_{m \leq |w_n| \leq m+1} \mathbb{F}_n\left(x,t,T_{m+1}\left(w_n\right)\right) \nabla w_n S_m'\left(w_n\right) \exp\left(G\left(w_n\right)\right) W_{\nu,\eta}^{n,j} dxdt \\ &\qquad \rightarrow \int_{m \leq |w| \leq m+1} \mathbb{F}(x,t,w)) \nabla w \exp(G(w)) W_{\nu,\eta}^{j} dxdt\end{aligned}$$

as $n \rightarrow +\infty$ with the modular convergence of $W_{\nu,\eta}^{j}$ as $j \rightarrow +\infty$ and letting $\mu \rightarrow +\infty$ we get

$$\int_Q \mathbb{F}_n(x,t,w_n)\nabla w_n S'_m(w_n)\exp(G(w_n))W^{n,j}_{\nu,\eta}dxdt = \epsilon(n,j,\mu) \quad \text{for any } m\geq 1. \tag{4.60}$$

For (4.54):
One has

$$\begin{aligned}&\int_Q \varrho_n(x,t,w_n,\nabla w_n)S'_m(w_n)\nabla w_n \exp(G(w_n))\exp(G(w_n))W^{n,j}_{\nu,\eta}dxdt\\ &=\int_{m\leq|w_n|\leq m+1}\varrho_n(x,t,w_n,\nabla w_n)S'_m(w_n)\nabla w_n\exp(G(w_n))W^{n,j}_{\nu,\eta}dxdt\\ &\leq \eta C\int_{m\leq|w_n|\leq m+1}\varrho_n(x,t,w_n,\nabla w_n)\nabla w_n dxdt\end{aligned}$$

According to (4.40), we obtain

$$\int_Q \varrho_n(x,t,w_n,\nabla w_n)S'_m(w_n)\nabla w_n\exp(G(w_n))W^{n,j}_{\nu,\eta}dxds\leq\epsilon(n,\mu,m).$$

Concerning (4.57): as $S_m(r)\leq 1$, we obtain

$$\int_Q f_n S_m(w_n)\exp(G(w_n))W^{n,j}_{\nu,\eta}\quad dxdt\leq\epsilon(n,\eta), \tag{4.61}$$

For (4.53):

$$\begin{aligned}&\int_Q \varrho_n(x,t,w_n,\nabla w_n)S_m(w_n)\exp(G(w_n))\nabla W^{n,j}_{\nu,\eta}dxdt\\ &=\int_{\{[u_n|\leq k\}\cap\{0\leq T_k(w_n)-T_k(v_j)_\mu)\leq\eta\}}\varrho_n(x,t,T_k(w_n),\nabla T_k(w_n))S_m(w_n)\exp(G(w_n))\\ &\quad\times\left(\nabla T_k(w_n)-\nabla T_k(v_j)_\mu\right)dxdt\\ &\quad-\int_{\{|w_n|>k\}\cap\{0\leq T_k(w_n)-T_k(v_j)_\mu)\leq\eta\}}\varrho_n(x,t,w_n,\nabla w_n)S_m(w_n)\\ &\quad\times\exp(G(w_n))\nabla T_k(v_j)_\mu dxdt\end{aligned} \tag{4.62}$$

since $\varrho_n(x,t,T_{k+\eta}(w_n),\nabla T_{k+\eta}(w_n))$ is bounded in $(L_\Phi(Q))^N$, there exist $\varpi_{k+\eta}\in(L_\Phi(Q))^N$ such that

$$\varrho_n(x,t,T_{k+\eta}(w_n),\nabla T_{k+\eta}(w_n))\rightharpoonup\varpi_{k+\eta}\text{weakly in }(L_\Phi(Q))^N.$$

Then,

$$\begin{aligned}&\int_{\{|w_n|>k\}\cap\{0\leq T_k(w_n)-T_k(v_j)_\mu)\leq\eta\}}\varrho_n(x,t,w_n,\nabla w_n)S_m(w_n)\exp(G(w_n))\nabla T_k(v_j)_\mu dxdt\\ &=\int_{\{|w|>k\}\cap\{0\leq T_k(w)-T_k(v_j)\mu\rangle\leq\eta\}}S_m(w)\exp(G(w))\nabla T_k(v_j)_\mu\varpi_{k+\eta}dxdt+\epsilon(n)\end{aligned} \tag{4.63}$$

when we have used

$$S_m(w_n)\exp(G(w_n))\nabla T_k(v_j)_\mu\Big)\chi_{\{|w_n|>k\}\cap\{0\leq T_k(w_n)-T_k(v_j)_\mu)\leq\eta\}}$$

$$\to S_m(w)\exp(G(w))\nabla T_k(v_j)_\mu\Big)\chi_{\{|u|>k\}\cap\{0\leq T_k(w)-T_k(v_j)_\mu)\leq\eta\}}$$

strongly in $(E_\Psi(Q))^N$.

Let $j \to +\infty$, we can have

$$\int_{\{|u|>k\}\cap\{0\leq T_k(w)-T_k(v_j)\mu)\leq\eta\}} S_m(w)\exp(G(w))\nabla T_k\left(v_j\right)_\mu \varpi_{k+\eta}dxdt$$
$$= \int_{\{|w|>k\}\cap\{0\leq T_k(w)-T_k(w)_\mu)\leq\eta\}} S_m(w)\exp(G(w))\nabla T_k(w)_\mu\varpi_{k+\eta}dxdt + \epsilon(n,j)$$

we may have,

$$\int_{\{|w|>k\}\cap\{0\leq T_k(w)-T_k(w)_\mu)\leq\eta\}} S_m(w)\exp(G(w))\nabla T_k(w)_\mu\varpi_{k+\eta}dxdt = \epsilon(n,j,\mu)$$

By (4.52)-(4.63) we obtain

$$\int_{\{|w_n|\leq k\}\cap\{0\leq T_k(w_n)-T_k(v_j)\mu)|\leq\eta\}} \varrho_n(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right))S_m\left(w_n\right)\exp\left(G\left(w_n\right)\right)$$

$$\times\left(\nabla T_k\left(w_n\right)-\nabla T_k\left(v_j\right)_\mu\right)dxdt \leq C\eta+\epsilon(n,j,\mu,m),$$

we know that $\exp\left(G\left(w_n\right)\right)\geq 1$ and $S_m\left(w_n\right)=1$ for $|w_n|\leq k$ then,

$$\int_{\{|w_n|\leq k\}\cap\left\{0\leq T_k(w_n)-T_k(v_j)_\mu\right)|\leq\eta\}} \varrho_n(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right))\left(\nabla T_k\left(w_n\right)-\nabla T_k\left(v_j\right)_\mu\right)dxdt$$
$$\leq \epsilon(n,j,\mu,m). \tag{4.64}$$

Now, let us prove that:

$$\int_Q \left[\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right)\right)-\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k(w)\right)\right]\left[\nabla T_k\left(w_n\right)-\nabla T_k(w)\right]dxdt \to 0 \tag{4.65}$$

Setting for $s>0, Q^s=\{(x,t)\in Q:|\nabla T_k(w)|\leq s\}$ and $Q_j^s=\{(x,t)\in Q:|\nabla T_k\left(v_j\right)|\leq s\}$ and denoting by χ^s and χ_j^s the characteristic functions of Q^s and $\bar{Q}_j^s$ respectively, we deduce that letting $0<\delta<1$, define

$$\Theta_{n,k}=\left(\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right)\right)-\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k(w)\right)\right)\left(\nabla T_k\left(w_n\right)-\nabla T_k(w)\right)$$

For $s>0$, we have

$$\begin{aligned}0 &\leq \int_{Q^s}\Theta_{n,k}^\delta dxdt\\ &= \int_{Q^s}\Theta_{n,k}^\delta\chi_{\left|T_k(w_n)-T_k(v_j)_\mu\right|\leq\eta)}dxdt\\ &\quad+\int_{Q^s}\Theta_{n,k}^\delta\chi_{\left|T_k(w_n)-T_k(v_j)_\mu\right|>\eta)}dxdt.\end{aligned}$$

By using the Holder inequality on the first term of the right-side hand we can have,

$$\begin{aligned}\int_{Q^s}\Theta_{n,k}^\delta\chi_{\left|T_k(w_n)-T_k(v_j)_\mu\right|\leq\eta)}dxdt &\leq \left(\int_{Q^s}\Theta_{n,k}\chi_{|T_k(w_n)-T_k(v_j)\mu|\leq\eta)}dxdt\right)^\delta\left(\int_{Q^s}dxdt\right)^{1-\delta}\\ &\leq C_1\left(\int_{Q^s}\Theta_{n,k}\chi_{|T_k(w_n)-T_k(v_j)\mu|\leq\eta)}dxdt\right)^\delta\end{aligned}$$

By applying the Holder inequality, on the second term of the right-side hand we get,

$$\int_{Q^s}\Theta_{n,k}^\delta\chi_{|T_k(w_n)-T_k(v_j)\mu|>\eta)}dxdt \leq \left(\int_{Q^s}\Theta_{n,k}dxdt\right)^\delta\left(\int_{|T_k(w_n)-T_k(v_j)\mu|>\eta)}dxdt\right)^{1-\delta}$$

since $\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right)\right)$ is bounded in $\left(L_\Phi\left(Q_T\right)\right)^N$, While $\nabla T_k\left(w_n\right)$ is bounded in $\left(L_\Psi\left(Q_T\right)\right)^N$ then,

$$\int_{Q^s}\Theta_{n,k}^\delta\chi_{|T_k(w_n)-T_k(v_j)\mu|>\eta)}dxdt\leq C_2\,\text{meas}\left\{(x,t)\in Q_T:\left|T_k\left(w_n\right)-T_k\left(v_j\right)_\mu\right|>\eta\right\}^{1-\delta}.$$

We obtain,

$$\int_{Q^s}\Theta_{n,k}^\delta dxdt\leq C_1\left(\int_{Q^*}\Theta_{n,k}\chi_{\left|T_k(w_n)-T_k(v_j)_\mu\right|\leq\eta)}dxdt\right)^\delta$$
$$+C_2\,\text{meas}\left\{(x,t)\in Q_T:\left|T_k\left(w_n\right)-T_k\left(v_j\right)_\mu\right|>\eta\right\}^{1-\delta}.$$

Secondly,

$$\int_{Q^s}\Theta_{n,k}\chi_{\left|T_k(w_n)-T_k(v_j)_\mu\right|\leq\eta)}dxdt$$

$$\leq\int_{\left|T_k(w_n)-T_k(v_j)_\mu\right|\leq\eta)}\left(\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right)\right)-\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k(w)\chi_s\right)\right)$$
$$\times\left(\nabla T_k\left(w_n\right)-\nabla T_k(w)\chi_s\right)dxdt$$

For each $s>r,r>0$, one has

$$0\leq\int_{Q^r\cap\left\{\left|T_k(w_n)-T_k(v_j)_\mu\right|\leq\eta)\right\}}\left(\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right)\right)-\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k(w)\right)\right)$$
$$\times\left(\nabla T_k\left(w_n\right)-\nabla T_k(w)\right)dxdt$$
$$\leq\int_{Q^s\cap\{|T_k(w_n)-T_k(v_j)\mu|\leq\eta)\}}\left(\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right)\right)-\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k(w)\right)\right)$$
$$\times\left(\nabla T_k\left(w_n\right)-\nabla T_k(w)\right)dxdt$$

$$=\int_{Q^e\cap\{|T_k(w_n)-T_k(v_j)\mu|\leq\eta)\}}\left(\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right)\right)-\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k(w)\chi_s\right)\right)$$
$$\times\left(\nabla T_k\left(w_n\right)-\nabla T_k(w)\chi_s\right)dxdt$$
$$\leq\int_{Q\cap\{|T_k(w_n)-T_k(v_j)\mu|\leq\eta)\}}\left(\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right)\right)-\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k(w)\chi^s\right)\right)$$
$$\times\left(\nabla T_k\left(w_n\right)-\nabla T_k(w)\chi^s\right)dxdt$$
$$=\int_{\left|T_k(w_n)-T_k(v_j)_\mu\right|\leq\eta}\left(\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right)\right)-\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)$$
$$\times\left(\nabla T_k\left(w_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)dxdt$$
$$+\int_{|T_k(w_n)-T_k(v_j)\mu|\leq\eta}\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k\left(w_n\right)\right)\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k(w)\chi^s\right)dxdt$$

$$+\int_{\left|T_k(w_n)-T_k(v_j)_\mu\right|\leq\eta}\left(\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)-\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k(w)\chi^s\right)\right)$$

$$\nabla T_k\left(w_n\right)dxdt$$

$$-\int_{\left|T_k(w_n)-T_k(v_j)_\mu\right|\leq\eta}\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\nabla T_k\left(v_j\right)\chi_j^s dxdt$$

$$+\int_{|T_k(w_n)-T_k(v_j)\mu|\leq\eta}\varrho\left(x,t,T_k\left(w_n\right),\nabla T_k(w)\chi^s\right)\nabla T_k(w)\chi^s dxdt$$
$$=I_1(n,j,s)+I_2(n,j)+I_3(n,j)+I_4(n,j,\mu)+I_5(n,\mu)$$

We go to the limit as n, j, μ, and $s \to +\infty$

$$I_1 = \int_{|T_k(w_n) - T_k(v_j)\mu| \leq \eta} \varrho\left(x, t, T_k\left(w_n\right), \nabla T_k\left(w_n\right)\right)\left(\nabla T_k\left(w_n\right) - \nabla T_k\left(v_j\right)_\mu\right) dxdt$$

$$- \int_{|T_k(w_n) - T_k(v_j)\mu| \leq \eta} \varrho\left(x, t, T_k\left(w_n\right), \nabla T_k\left(w_n\right)\right)\left(\nabla T_k\left(v_j\right)\chi_j^s - \nabla T_k\left(v_j\right)_\mu\right) dxdt$$

$$- \int_{\left|T_k(w_n) - T_k(v_j)_\mu\right| \leq \eta} \varrho\left(x, t, T_k\left(w_n\right), \nabla T_k\left(v_j\right)\chi_j^s\right)\right)\left(\nabla T_k\left(w_n\right) - \nabla T_k\left(v_j\right)\chi_j^s\right)) dxdt$$

Thanks to (4.64), we have

$$\begin{aligned}
&\int_{|T_k(w_n) - T_k(v_j)| \leq \eta} \varrho\left(x, t, T_k\left(w_n\right), \nabla T_k\left(w_n\right)\right)\left(\nabla T_k\left(w_n\right) - \nabla T_k\left(v_j\right)_\mu\right) dxdt \\
&\leq C\eta + \epsilon(n, m, j, s) - \int_{|w| > k \cap |T_k(w) - T_k(v_j)\mu| \leq \eta} \varrho\left(x, t, T_k(w), 0\right) \nabla T_k\left(v_j\right)_\mu dxdt \\
&\leq C\eta + \epsilon(n, m, j, \mu)
\end{aligned}$$

The second term of the right-hand side tends to

$$\int_{|T_k(w) - T_k(v_j)| \leq \eta} \varpi_k\left(\nabla T_k\left(v_j\right)\chi_j^s - \nabla T_k\left(v_j\right)_\mu\right) dxdt,$$

since $\varrho\left(x, t, T_k\left(w_n\right), \nabla T_k\left(w_n\right)\right)$ is bounded in $\left(L_\Phi\left(Q\right)\right)^N$, there exist some $\varpi_k \in \left(L_\Phi\left(Q\right)\right)^N$ such that (for a subsequence still denoted by w_n

$$\varrho\left(x, t, T_k\left(w_n\right), \nabla T_k\left(w_n\right)\right) \to \varpi_k \quad \text{in } \left(L_\Psi\left(Q\right)\right)^N \text{ for } \quad \sigma\left(\Pi L_\Phi, \Pi E_\Psi\right).$$

In view of

$$\begin{aligned}
&\left(\nabla T_k\left(v_j\right)\chi_j^s - \nabla T_k\left(v_j\right)_\mu\right)\chi_{|T_k(w_n) - T_k(v_j)\mu| \leq \eta} \\
&\to \left(\nabla T_k\left(v_j\right)\chi_j^s - \nabla T_k\left(v_j\right)_\mu\right)\chi_{|T_k(w) - T_k(v_j)| \leq \eta}
\end{aligned}$$

strongly in $\left(E_\Psi\left(Q\right)\right)^N$ as $n \to +\infty$.

The third term of the right-hand side tends to

$$\int_{\left|T_k(w) - T_k(v_j)_\mu\right| \leq \eta} \varrho\left(x, t, T_k(w), \nabla T_k\left(v_j\right)\chi_j^s\right)\left(\nabla T_k(w) - \nabla T_k\left(v_j\right)\chi_j^s\right) dxdt.$$

since

$$\begin{aligned}
&\varrho\left(x, t, T_k\left(w_n\right), \nabla T_k\left(v_j\right)\chi_j^s\right)\right)\chi_{\left|T_k(w_n) - T_k(v_j)_\mu\right| \leq \eta} \\
&\to \varrho\left(x, t, T_k(w), \nabla T_k\left(v_j\right)\chi_j^s\right)\right)\chi_{\left|T_k(w) - T_k(v_j)_\mu\right| \leq \eta}
\end{aligned}$$

in $\left(E_\Phi\left(Q\right)\right)^N$. while

$$\left(\nabla T_k\left(w_n\right) - \nabla T_k\left(v_j\right)\chi_j^s\right)\right) \to \left(\nabla T_k(w) - \nabla T_k\left(v_j\right)\chi_j^s\right)\right)$$

in $\left(L_\Psi\left(Q\right)\right)^N$ for $\sigma\left(\Pi L_\Phi, \Pi E_\Psi\right)$ Passing to limit as $j \to +\infty$ and $\mu \to +\infty$ and using Lebesgue's theorem, we have

$$I_1 \leq C\eta + \epsilon(n, j, s, \mu)$$

For what concerns I_2, by letting $n \to +\infty$, we have

$$I_2 \to \int_{\left|T_k(w) - T_k(v_j)_\mu\right| \leq \eta)} \varpi_k\left(\nabla T_k\left(v_j\right)\chi_j^s - \nabla T_k(w)\chi^s\right) dxdt$$

since $\varrho\left(x, t, T_k\left(w_n\right), \nabla T_k\left(w_n\right)\right) \rightarrow \varpi_k$ in $\left(L_\Phi(Q)\right)^N$, for $\sigma\left(\Pi L_\Phi, \Pi E_\Psi\right)$, while

$$\begin{aligned}&\left(\nabla T_k\left(v_j\right) \chi_j^s-\nabla T_k(w) \chi^s\right) \chi_{\left|T_k\left(w_n\right)-T_k\left(v_j\right)_\mu\right| \leq \eta} \\ &\rightarrow\left(\nabla T_k\left(v_j\right) \chi_j^s-\nabla T_k(w) \chi^s\right) \chi_{\left|T_k(w)-T_k\left(v_j\right)_\mu\right| \leq \eta},\end{aligned}$$

strongly in $\left(E_\Psi(Q)\right)^N$.

Passing to limit $j \rightarrow+\infty$, and using Lebesgue's theorem, we have

$$I_2=\epsilon(n, j).$$

Similar ways as above give

$$\begin{aligned}& I_3=\epsilon(n, j). \\ & I_4=\int_{\left|T_k(w)-T_k(w)_\mu\right| \leq \eta)} \varrho\left(x, t, T_k(w), \nabla T_k(w)\right) \nabla T_k(w) d x d t+\epsilon(n, j, \mu, s, m). \\ & I_5=\int_{\left|T_k(w)-T_k(w)_\mu\right| \leq \eta)} \varrho\left(x, t, T_k(w), \nabla T_k(w)\right) \nabla T_k(w) d x d t+\epsilon(n, j, \mu, s, m).\end{aligned}$$

Finally, we obtain,

$$\int_{Q^*} \Theta_{n, k} d x d t \leq C_1(\epsilon(n, \mu, \eta, m))^\delta+C_2(\epsilon(n, \mu,))^{1-\delta}.$$

By passing to the limit sup over n, j, μ and s

$$\int_{Q^r}\left[\left(\varrho\left(x, t, T_k\left(w_n\right), \nabla T_k\left(w_n\right)\right)-\varrho\left(x, t, T_k\left(w_n\right), \nabla T_k(w)\right)\right)\left(\nabla T_k\left(w_n\right)-\nabla T_k(w)\right)\right]^\delta d x d t=\epsilon(n).$$

Then, $\nabla w_n \rightarrow \nabla w$ a.e. in Q^r, and as r is arbitrary,

$$\nabla w_n \rightarrow \nabla w, \quad \text { a.e. in } \quad Q.$$

Step 6: Equi-integrability of $\mathbb{H}$

We shall prove that $\mathbb{H}_n\left(x, t, w_n, \nabla w_n\right) \rightarrow \mathbb{H}(x, t, w, \nabla w)$ strongly in $L^1(\Omega)$.
Consider $\vartheta_0\left(w_n\right)=\int_0^{w_n} \rho(s) \chi_{\{s>h\}} d s$ and multiply (4.8) by $\exp \left(G\left(w_n\right)\right) \vartheta_0\left(w_n\right)$, we get

$$\begin{aligned}&\int_{\Omega} \widetilde{T}_h\left(w_n\right)(T) d x+\int_Q \varrho\left(x, t, w_n, \nabla w_n\right) \nabla\left(\exp \left(G\left(w_n\right)\right) \gtrapprox_0\left(w_n\right)\right) d x d t \\ &+\int_Q \mathbb{F}_n\left(x, t, w_n, \nabla w_n\right) \nabla\left(\exp \left(G\left(w_n\right)\right) \vartheta_0\left(w_n\right)\right) d x d t \\ &\left.+\int_Q \mathbb{H}_n\left(x, t, w_n, \nabla w_n\right) \exp \left(G\left(w_n\right)\right) \vartheta_0\left(w_n\right)\right) d x d t \\ &\leq\left(\int_h^{+\infty} \rho(s) d x\right) \exp \left(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha^{\prime}}\right)\left[\|f\|_{L^1(Q)}+\left\|b\left(w_0\right)\right\|_{L^1(\Omega)}\right]. \\ &\text { where } \widetilde{T}_h(r)=\int_0^r \vartheta_0(s) \exp (G(s)) d s \geq 0,\end{aligned}$$

by the similar idea used in previous step we can obtain

$$\int_{\left\{w_n>h\right\}} \rho\left(w_n\right) \Psi\left(x, \nabla w_n\right) d x d t \leq C\left(\int_h^{+\infty} \rho(s) d x\right).$$

As $\rho \in L^1(\mathbb{R})$, we have

$$\lim _{h \rightarrow \infty} \sup _{n \in \mathbb{N}} \int_{\left\{w_n>h\right\}} \rho\left(w_n\right) \Psi\left(x, \nabla w_n\right) d x d t=0$$

By the similar idea as above, let $\vartheta_0(w_n) = \int_{w_n}^{0} \rho(s)\chi_{\{s<-h\}} dx$ in (4.8) we have also

$$\lim_{h\to\infty} \sup_{n\in\mathbb{N}} \int_{\{w_n<-h\}} \rho(w_n) \Psi(x, \nabla w_n)\, dxdt = 0$$

this implies that

$$\lim_{h\to\infty} \sup_{n\in\mathbb{N}} \int_{\{|w_n|>h\}} \rho(w_n) \Psi(x, \nabla w_n)\, dxdt = 0. \tag{4.66}$$

Let $D \subset \Omega$ then

$$\begin{aligned}\int_D \rho(w_n) \Psi(x, \nabla w_n)\, dxdt \leq{}& \max_{\{|w_n|\leq h\}} (\rho(x)) \int_{D\cap\{|w_n|\leq h\}} \Psi(x, \nabla w_n)\, dxdt \\ &+ \int_{D\cap\{|w_n|>h\}} \rho(w_n) \Psi(x, \nabla w_n)\, dxdt.\end{aligned}$$

Consequently $\rho(w_n) \Psi(x, \nabla w_n)$ is equi-integrable. Then $\rho(w_n) \Psi(x, \nabla w_n) \longrightarrow \rho(w)\Psi(x, \nabla w)$ strongly in $L^1(\mathbb{R})$. By (1.6) we get

$$\mathbb{H}_n(x, t, w_n, \nabla w_n) \to \mathbb{H}(x, t, w, \nabla w) \text{ strongly in } L^1(Q). \tag{4.67}$$

Step 7: Passing to the limit.

We establish that $w \geq \Lambda$ a.e. in Q according to (4.20) and (4.28) we obtain

$$0 \leq \int_Q T_n(w_n - \Lambda)^- \, dxdt \leq \frac{c_1}{n}$$

Let $n \longrightarrow +\infty$ we obtain

$$\int_Q (w - \Lambda)^- dxdt = 0$$

then

$$(w - \Lambda)^- = 0 \text{ a.e. in } Q.$$

We pass Now to the limit in (4.68) in order to prove that w satisfies (4.2)

Let $v \in W_0^1 L_\Psi(Q) \cap L^\infty(Q)$ such that $\frac{\partial v}{\partial t} \in W^{-1,x} L_\Phi(Q) + L^1(Q)$, then by theorem 2.1 we can take

$$\bar{v} = v \text{ on } Q$$
$$\bar{v} \in W^{1,x} L_\Psi(\Omega\times\mathbb{R}) \cap L^1(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R})$$
$$\frac{\partial \bar{v}}{\partial t} \in W^{-1,x} L_\Phi(Q) + L^1(Q)$$

and there exists $v_j \in \mathcal{D}(\Omega\times\mathbb{R})$ such that

$$v_j \to \bar{v} \quad \text{in} \quad W_0^{1,x} L_\Psi(\Omega\times\mathbb{R}) \quad \text{and} \quad \frac{\partial v_j}{\partial t} \to \frac{\partial \bar{v}}{\partial t} \in W^{-1,x} L_\Phi(Q) + L^1(Q).$$

for the modular convergence in $W_0^1 L_\Psi(Q)$, with

$$\|v_j\|_{L^\infty(Q)} \leq (N+2)\|v\|_{L^\infty(Q)}.$$

By taking $T_k(w_n - v_j)$, as a test function in (4.8) we obtain

$$\begin{cases} \int_0^\tau < \dfrac{\partial b_n(w_n)}{\partial s}, T_k(w_n - v_j) > ds + \int_Q \varrho_n(x, s, w_n, \nabla w_n) \nabla T_k(w_n - v_j)\, dxds \\ + \int_Q \mathbb{F}_n(x, s, w_n) \nabla T_k(w_n - v_j)\, dxds + \int_Q T_n(w_n - \Lambda)^- sh_{\frac{1}{n}}(w_n) T_k(w_n - v_j)\, dxds \\ + \int_Q^T \mathbb{H}_n(x, s, w_n, \nabla w_n) \nabla T_k(w_n - v_j)\, dxds = \int_Q f_n T_k(w_n - v_j)\, dxds \end{cases} \tag{4.68}$$

Now, we pass to the limit as in (4.68), when $n, j \longrightarrow +\infty$:
Firstly, we can write

$$
\begin{aligned}
\int_0^\tau < \frac{\partial b_n(w_n)}{\partial s}, T_k\left(w_n-v_j\right) > ds = & \int_0^\tau < \frac{\partial\left(b_n(w_n)-v_j\right)}{\partial s}, T_k\left(w_n-v_j\right) > ds \\
& + \int_0^\tau < \frac{\partial v_j}{\partial s}, T_k\left(b_n(w_n)-v_j\right) > ds \\
= & S_k\left(b_n(w_n)(\tau)-v_j(\tau)\right)-S_k\left(b_n(w_n)(0)-v_j(0)\right) \\
& + \int_0^\tau < \frac{\partial v_j}{\partial s}, T_k\left(w_n-v_j\right) > ds
\end{aligned}
$$

As $n, j \rightarrow +\infty$ we can have

$$
\begin{aligned}
\int_0^\tau < \frac{\partial b_n(w_n)}{\partial s}, T_k\left(w_n-v_j\right) > ds \rightarrow & \int_\Omega S_k\left(b_n(w_n)(\tau)-v(\tau)\right) dx-\int_\Omega S_k\left(b_n(w_n)(0)-v(0)\right) dx \\
& + \int_0^\tau < \frac{\partial v}{\partial s}, T_k(b(w)-v) > ds
\end{aligned}
$$

– We follow the same idea used in [5] to show that

$$
\begin{aligned}
& \liminf_{j \rightarrow \infty} \liminf_{n \rightarrow \infty} \int_Q \varrho\left(x, s, w_n, \nabla w_n\right) \nabla T_k\left(w_n-v_j\right) dx ds \\
\geq & \int_Q \varrho(x, s, w, \nabla w) \nabla T_k(w-v) dx ds
\end{aligned}
$$

–For $n \geq k+(N+2)\|v\|_{L^{\infty}(Q)}$

$$
\mathbb{F}_n\left(x, s, w_n\right) \nabla T_k\left(w_n-v_j\right)=\mathbb{F}\left(x, s, T_{k+(N+2)\|v\|_{L^{\infty}(Q)}}\left(w_n\right)\right) \nabla T_k\left(w_n-v_j\right)
$$

The pointwise convergence of w_n to w as $n \longrightarrow +\infty$ and (1.7) then

$$
\begin{aligned}
& \mathbb{F}\left(x, s, T_{k+(N+2)\|v\|_{L \infty}\left(Q_T\right)}\left(w_n\right)\right) \nabla T_k\left(w_n-v_j \rightharpoonup\right. \\
& \mathbb{F}\left(x, s, T_{k+(N+2)\|v\|_{L^{\infty}\left(Q_T\right)}}(w)\right) \nabla T_k\left(w-v_j\right)
\end{aligned}
$$

weakly for $\sigma\left(\Pi L_v, \Pi L_{\Phi}\right)$.
Y the same idea, we get

$$
\begin{aligned}
\lim_{j \rightarrow \infty} & \int_Q \mathbb{F}\left(x, s, T_{k+(N+2)\|v\|_{L^{\infty}(Q)}}(w)\right) \nabla T_k\left(w-v_j\right) dx ds \\
= & \int_Q \mathbb{F}\left(x, s, T_{k+(N+2)\|v\|_{L^{\infty}(Q)}}(w)\right) \nabla T_k(w-v) dx ds \\
& = \int_Q \mathbb{F}(x, s, w) \nabla T_k(w-v) dx ds
\end{aligned}
$$

Limit of $\mathbb{H}_n\left(x, s, w_n, \nabla w_n\right) T_k\left(w_n-v_j\right)$:
Since $\mathbb{H}_n\left(x, s, w_n, \nabla w_n\right)$ converge strongly to $\mathbb{H}(x, t, w, \nabla w)$ in $L^1(Q)$. and the point wise convergence of w_n to w as $n \rightarrow +\infty$, we can show that $\mathbb{H}_n\left(x, s, w_n, \nabla w_n\right) T_k\left(w_n-v_j\right)$ converge to $\mathbb{H}(x, s, w, \nabla w) T_k\left(w-v_j\right)$ in $L^1(Q)$ and

$$
\lim_{j \rightarrow \infty} \int_Q \mathbb{H}(x, s, w, \nabla w) T_k\left(w-v_j\right) dx ds=\int_Q \mathbb{H}(x, s, w, \nabla w) T_k(w-v) dx ds
$$

Since f_n converge strongly to f in $L^1(Q)$, and

$$
T_k\left(w_n-v_j\right) \rightarrow T_k\left(w-v_j\right) \text{ weakly* in } L^{\infty}(Q),
$$

we have

$$\int_Q f_n T_k\left(w_n - v_j\right) dxds \to \int_Q f T_k\left(w - v_j\right) dxds,$$

as $n \to \infty$ and also we have

$$\int_Q f T_k\left(w - v_j\right) dxds \to \int_Q f T_k(w - v) dxds,$$

as $j \to \infty$.

Finally we know that

$$\int_Q T_n\left(w_n - \Lambda\right)^- sh_{\frac{1}{n}}\left(w_n\right) T_k\left(w_n - v_j\right) dxds \geq 0,$$

thus

$$\begin{cases} \int_\Omega S_k(b(w(\tau)) - v(\tau))dx + \int_0^\tau < \dfrac{\partial v}{\partial s}, T_k(b(w) - v) > ds \\ + \int_Q \varrho(x,s,w,\nabla w)\nabla T_k(w - v)dxds + \int_Q \mathbb{H}(x,s,w,\nabla w)T_k(w - v)dxds \\ + \int_Q \mathbb{F}(x,s,w)\nabla T_k(w - v)dxds \leq \int_Q f T_k(w - v)dxds - \int_\Omega S_k\left(b(w_0) - v(x,0)\right)dx \end{cases}$$

which justifies the desired result.

5 Conclusion

In this work, we have shown that the main problem admits a solution (the precise meaninig being (4.1) and (4.2)) based on the method of penalization. The result obtained in this paper will no doubt inspire researchers to develop it by dealing with the uniqueuess of the solution to the problem or by reducing the number of conditions.

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