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STABILIZATION OF SEMILINEAR SYSTEMS IN BANACH SPACE

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Abstract In this paper, we aim to study the feedback stabilization of an infinite-dimensional semilinear system evolving in reflexive Banach state space.The concept of bounded control is also investigated in the realistic domain. Sufficient conditions for appropriate feedback control to ensure strong and weak stabilization are given.

1 Introduction

Let $(W, \|\cdot\|)$ be a reflexive Banach space, $\mathcal{A} : D(\mathcal{A}) \subset W \to W$ be an unbounded operator generates a C_0-semigroup of contractions $\mathcal{S}(s)$ on W.
Consider the following semilinear system:

$$\begin{cases} w'(s) = \mathcal{A}w(s) + \mathcal{N}w(s) + \mathbf{u}(s)\mathcal{B}w(s), & s \in J = [0, T], T > 0, \\ w(0) = w_0 \in W, \end{cases} \tag{1.1}$$

Here, $s \to \mathbf{u}(s)$ is a scalar control and $\mathcal{B}$ is a linear bounded operator. $\mathcal{N}$ is a non-linear operator such as $\mathcal{N}(0) = 0$. Thus 0 is an equilibrium for (1.1).
Stability of (1.1) with $\mathcal{N} = 0$ has been treated in various works (see [1], [17]). In [4], the authors has been proved that under the control:

$$\mathbf{u}(s) = -\langle w(s), \mathcal{B}w(s)\rangle, \tag{1.2}$$

the system (1.1) is weakly stabilizable for some compact operator $\mathcal{B}$ satisfying

$$\langle \mathcal{B}\mathcal{S}(\lambda)w, \mathcal{S}(\lambda)w\rangle = 0, \quad \forall\lambda \geq 0 \text{ implies } w = 0. \tag{1.3}$$

Moreover, under the condition (1.3), it has been proved that (1.2) ensures the strong stability for a class of semilinear systems (we refer to ([7], [8])). In [15], Ouzahra et al. proved the strong stability of (1.1) with $\mathcal{N} = 0$, under the following assumption:

$$\int_0^T \langle \mathcal{B}\mathcal{S}(\lambda)w, \mathcal{S}(\lambda)w\rangle d\lambda \geq \delta\|w\|^2, \quad \forall w \in W,\ T \geq 0\ , \delta > 0. \tag{1.4}$$

In [10], under (1.4), an exponential stabilization result has been established by using the constrained control defined by

$$\mathbf{u}(s) := \begin{cases} -\frac{\langle w(s), \mathcal{B}w(s)\rangle}{\|w(s)\|^2} & \text{if} \quad w(s) \neq 0, \\ \quad 0 & \text{if} \quad w(s) = 0, \end{cases}$$

In addition, the strong and weak stability of system (1.1) has been studied in the case of a Hilbert space by [18], and the regional exponential stabilization of (1.1) has been treated In [12] using a switching feedback. Then it has been proved that the control

$$\mathbf{u}(s) = \frac{-\mathcal{B}^\star i_\omega w(s)}{R_j(i_\omega w(s))},\ (j = 1; 2),\ i_\omega = \mathcal{X}_\omega^\star \mathcal{X}_\omega\ ,$$

where $\mathcal{X}_\omega$ is the restriction operator, $R_1(i_\omega w) = 1 + \|\mathcal{B}^\star i_\omega w\|$ and $R_2(i_\omega w) = \sup(1, \|\mathcal{B}^\star i_\omega w\|)$, guarantees the regional exponential stability if:

$$\int_0^T |\langle i_\omega \mathcal{B}\mathcal{S}(\lambda)w, \mathcal{S}(\lambda)w\rangle| d\lambda \geq \delta \|\mathcal{X}_\omega w\|^2, \ \forall w \in W, \ T \geq 0 \ , \delta > 0.$$

Furthermore, the authors in [11] studied the regional stabilization by using the switching control $\mathbf{u}(s) = -\tau sign(\langle w(s), i_\omega \mathcal{B}w(s)\rangle), \ \forall w \in W, \ \tau > 0$. Note that in the previous-mentioned works, the state space is a Hilbert space. Additionally, it was demonstrated that the majority of the above findings still hold when considering reflexive state space. Furthermore, the case of $\mathcal{N} = 0$ has been study in [6].
The aim of this paper is to study the stabilization of semilinear system (1.1) in reflexive Banach space.
The outline of the paper is as follows: In Sect. 2 we prove the existence, uniqueness, and regularity of the global solution (1.1). In Sect. 3, we study the problem of strong and weak-$\star$ stabilization. The Sect. 4, is devoted to illustrative examples.

2 Existence of mild solution

In the sequel, we assume that:

$(\mathcal{A}_1)$: The operator $\mathcal{A}$ is dissipative.

$(\mathcal{A}_2)$: The non-linear operator $\mathcal{N}$ is Lipschitz and dissipative such that

$$\|\mathcal{N}w\| \leq \frac{|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|}{1 + |\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|}. \tag{2.1}$$

$(\mathcal{A}_3)$: $\mathcal{I}$ is Lipschitz continuous,

where $\mathcal{I}$ is the duality mapping given by:

$$\mathcal{I}(w) = \{w^\star \in W^\star : \|w^\star\| = \|w\|, \langle w, w^\star\rangle = \|w\|^2\}, \qquad \forall w \in W.$$

With $\langle \cdot, \cdot\rangle$ is the duality pairing.
To get our stabilization results, we need the following lemma.

Lemma 2.1. *(Kato, T. (1967)): Let $s \in \mathbb{R} \to q(s) \in W$ be a function wich satisfies:*

(i) the function $s \to \|q(s)\|$ is almost everywhere differentiable on $\mathbb{R}$,

(ii) the weak derivative q' of q exists almost everywhere on $\mathbb{R}$.

Then,

$$\|q(s)\| \frac{d}{ds}\|q(s)\| = \langle q'(s), q^\star\rangle, \quad for \ \ almosteverywhere \ s \in \mathbb{R} \ \ and \ \ \forall q^\star \in \mathcal{I}(q(\cdot)).$$

In the theorem below, we present a strong result wich will be used to establish the stabilization result for system (1.1).

Theorem 2.2. *Suppose $(\mathcal{A}_1)$-$(\mathcal{A}_3)$ hold. Then under the control*

$$\mathbf{u}(s) = -\frac{\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle}{1 + |\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|},$$

(1) the system (1.1) possesses a unique global mild solution $w(s)$.

(2) the solution $w(s)$ satisfies

$$\int_0^T |\langle \mathcal{B}\mathcal{S}(\lambda)w_0, \mathcal{I}(\mathcal{S}(\lambda)w_0)\rangle| d\lambda \leq C\Big(\int_0^T \frac{|\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|^2}{1 + |\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|} d\lambda\Big)^{\frac{1}{2}}, \qquad \forall T > 0\cdot \tag{2.2}$$

Proof. We have

$$w'(s) = \mathcal{A}w(s) + \mathcal{N}w(s) + \mathbf{u}(s)\mathcal{B}w(s).$$

Then

$$\langle w'(s), \mathcal{I}(w(s))\rangle = \langle \mathcal{A}w(s), \mathcal{I}(w(s))\rangle + \langle \mathcal{N}w(s), \mathcal{I}(w(s))\rangle + \langle \mathbf{u}(s)\mathcal{B}w(s), \mathcal{I}(w(s))\rangle.$$

Since $\mathcal{A}$ and $\mathcal{N}$ are dissipatives, we deduce that

$$\frac{1}{2}\frac{d}{ds}\|w(s)\|^2 \leq \langle \mathbf{u}(s)\mathcal{B}w(s), \mathcal{I}(w(s))\rangle.$$

Then in order to make the function $\frac{1}{2}\|w(s)\|^2$ nonincreasing, we consider the control

$$\mathbf{u}(s) = -\frac{\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle}{1 + |\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|}. \tag{2.3}$$

This leads to the system

$$\begin{cases} w'(s) = \mathcal{A}w(s) + \mathcal{N}w(s) + \mathcal{F}(w(s)) \quad s \in J = [0, T], T > 0, \\ w(0) = w_0 \in W, \end{cases} \tag{2.4}$$

where

$$\mathcal{F}(w) = -\frac{\langle \mathcal{B}w, \mathcal{I}(w)\rangle}{1 + |\langle \mathcal{B}w, \mathcal{I}(w)\rangle|}Bw, \qquad w \in C([0,T], W) = \mathcal{C}.$$

System well posededness.

Let us show that $\mathcal{F}$ is a locally Lipschitz continuous function from $\mathcal{C}$ to W and $R \geq 0$ such that for all $w_1, w_2 \in \mathcal{C},\ \|w_1\| \leq R,\ \|w_2\| \leq R$, we have:

$$\begin{aligned} \big\|\mathcal{F}(w_1) - \mathcal{F}(w_2)\big\| &= \Big\|\frac{\langle \mathcal{B}w_1, \mathcal{I}(w_1)\rangle}{1 + |\langle \mathcal{B}w_1, \mathcal{I}(w_1)\rangle|}\mathcal{B}w_1 - \frac{\langle \mathcal{B}w_2, \mathcal{I}(w_2)\rangle}{1 + |\langle \mathcal{B}w_2, \mathcal{I}(w_2)\rangle|}\mathcal{B}w_2\Big\| \\ &= \Big\|\frac{\langle \mathcal{B}w_1, \mathcal{I}(w_1)\rangle}{1 + |\langle Bw_1, \mathcal{I}(w_1)\rangle|}(\mathcal{B}w_1 - \mathcal{B}w_2) \\ &+ \Big(\frac{\langle Bw_1, \mathcal{I}(w_1)\rangle}{1 + |\langle \mathcal{B}w_1, \mathcal{I}(w_1)\rangle|} - \frac{\langle \mathcal{B}w_2, \mathcal{I}(w_2)\rangle}{1 + |\langle \mathcal{B}w_2, \mathcal{I}(w_2)\rangle|}\Big)\mathcal{B}w_2\Big\| \\ &\leq \big\|\mathcal{B}\big\|\big\|w_1 - w_2\big\| + \mathcal{H}(w_1, w_2) \end{aligned} \tag{2.5}$$

where

$$\mathcal{H}(w_1, w_2) = \Big|\frac{\langle \mathcal{B}w_1, \mathcal{I}(w_1)\rangle}{1 + |\langle \mathcal{B}w_1, \mathcal{I}(w_1)\rangle|} - \frac{\langle \mathcal{B}w_2, \mathcal{I}(w_2)\rangle}{1 + |\langle \mathcal{B}w_2, \mathcal{I}(w_2)\rangle|}\Big|.$$

By making use of the function $\mathcal{F}(w) = \dfrac{w}{|w| + 1}$, we get

$$\begin{aligned} \mathcal{H}(w_1, w_2) &\leq \big|\langle \mathcal{B}w_1, \mathcal{I}(w_1)\rangle - \langle \mathcal{B}w_2, \mathcal{I}(w_2)\rangle\big| \\ &= |\langle \mathcal{B}w_1, \mathcal{I}(w_1) - \mathcal{I}(w_2)\rangle + \langle \mathcal{B}w_1 - \mathcal{B}w_2, \mathcal{I}(w_2)\rangle| \\ &\leq \|\mathcal{B}\|w_1\|\|\mathcal{I}(w_1) - \mathcal{I}(w_2)\| + \|\mathcal{B}\|\|w_1 - w_2\|\mathcal{I}(w_2)\|\|\| \end{aligned}$$

Using the fact that $\mathcal{I}$ is Lipschitz we get:

$$\|\mathcal{I}(w_1) - \mathcal{I}(w_2)\| \leq L\|w_1 - w_2\|,$$

where $L > 0$.
Moreover, we have

$$\|\mathcal{I}(w_2)\| = \|w_2\|.$$

Then

$$\begin{aligned}\mathcal{H}(w_1,w_2) &\leq L\|\mathcal{B}\|\|w_1\|\|w_1-w_2\| + \|\mathcal{B}\|\|w_2\|\|w_1-w_2\| \\ &\leq L\|\mathcal{B}\|R\|w_1-w_2\| + \|\mathcal{B}\|R\|w_1-w_2\| \\ &= (1+L)\|\mathcal{B}\|R\|w_1-w_2\|.\end{aligned} \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\begin{aligned}\|\mathcal{F}(w_1)-\mathcal{F}(w_2)\| &\leq \|\mathcal{B}\|\|w_1-w_2\| + (1+L)\|\mathcal{B}\|R\|w_1-w_2\| \\ &= \|\mathcal{B}\|\Big(1+R(1+L)\Big)\|w_1-w_2\|.\end{aligned}$$

We deduce that $\mathcal{F}$ is locally Lipschitz.
Since the function $w \to \mathcal{N}(w)+\mathcal{F}(w)$ is locally Lipschitz on W. Then the system (2.4) admits a unique mild solution defined on a maximal interval $[0, s_{max}]$ (see [16], which is continuous with respect to the initial state given by the following variation of constants formula:

$$w(s) = \mathcal{S}(s)w_0 + \int_0^s \mathcal{S}(s-\lambda)[\mathcal{N}w(\lambda) + Fw(\lambda)]d\,\lambda. \tag{2.7}$$

Since $\mathcal{A}$ and $\mathcal{N}$ are dissipatives, we have

$$\frac{1}{2}\frac{d}{ds}\|w(s)\|^2 \leq -\frac{\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle^2}{1+|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|}. \tag{2.8}$$

Integrating (2.8) over $[0, s]$, we get :

$$\|w(s)\|^2 - \|w_0\|^2 \leq -2\int_0^s \frac{\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle^2}{1+|\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|}d\lambda. \tag{2.9}$$

So for all $w_0 \in W$, we have

$$\|w(s)\| \leq \|w_0\|, \qquad \forall s \in [0, s_{max}[. \tag{2.10}$$

Hence for each $w_0 \in W$, the solution $w(s)$ is global (i.e. $s_{max} = +\infty$).

□

Let us show the estimate (2.2)

Proof. For all $w_0 \in W$ and $s \geq 0$, we have

$$\langle \mathcal{B}\mathcal{S}(s)w_0, \mathcal{I}(\mathcal{S}(s)w_0)\rangle = \langle \mathcal{B}\mathcal{S}(s)w_0 - \mathcal{B}w(s), \mathcal{I}(\mathcal{S}(s)w_0)\rangle + \langle \mathcal{B}w(s), \mathcal{I}(\mathcal{S}(s)w_0) - \mathcal{I}(w(s))\rangle + \langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle \tag{2.11}$$

From (2.11), and using (2.10), $\mathcal{B}$ is bounded, we deduce

$$|\langle \mathcal{B}\mathcal{S}(s)w_0, \mathcal{I}(\mathcal{S}(s)w_0)\rangle| \leq \|\mathcal{B}\|\|w_0\|\|\mathcal{S}(s)w_0 - w(s)\| + \|\mathcal{B}\|\|w_0\|\|\mathcal{I}(\mathcal{S}(s)w_0) - \mathcal{I}(w(s))\| + |\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|. \tag{2.12}$$

Using the fact that $\mathcal{I}$ is Lipshitz, we obtain

$$|\langle \mathcal{B}\mathcal{S}(s)w_0, \mathcal{I}(\mathcal{S}(s)w_0)\rangle| \leq \|\mathcal{B}\|\|w_0\|\|\mathcal{S}(s)w_0 - w(s)\| + L\|\mathcal{B}\|\|w_0\|\|\mathcal{S}(s)w_0 - w(s)\| + |\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|. \tag{2.13}$$

From (2.7), we have

$$\mathcal{S}(s)w_0 - w(s) = \int_0^s \mathcal{S}(s-\lambda)\frac{\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle}{1+\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle}\mathcal{B}w(\lambda)d\lambda - \int_0^s \mathcal{S}(s-\lambda)\mathcal{N}w(\lambda)d\lambda \tag{2.14}$$

Using (2.10), $\mathcal{A}_2$, (2.14) and the fact that $\|\mathcal{S}(s)\| \leq 1, \forall s \geq 0$, we obtain

$$\|\mathcal{S}(s)w_0 - w(s)\| \leq \|\mathcal{B}\|\|w_0\| \int_0^s \frac{|\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|}{1+|\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|}d\lambda + \int_0^s \frac{|\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|}{1+|\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|}d\,\lambda. \tag{2.15}$$

Then

$$|\langle \mathcal{B}\mathcal{S}(s)w_0, \mathcal{I}(\mathcal{S}(s)w_0)\rangle| \leq \|\mathcal{B}\|\|w_0\|(1+\|\mathcal{B}\|\|w_0\|)(1+L)\int_0^s \frac{|\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|}{1+|\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|}d\lambda + |\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle| \tag{2.16}$$

Integrating (2.16) over $[0,T]$, which give:

$$\begin{aligned}\int_0^T |\langle \mathcal{B}\mathcal{S}(s)w_0, \mathcal{I}(\mathcal{S}(s)w_0)\rangle| ds &\leq \|\mathcal{B}\|\|w_0\|(1+\|\mathcal{B}\|\|w_0\|)(1+L)T\int_0^T \frac{|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|}{1+|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|}ds \\ &+ \int_0^T |\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle| ds \\ &\leq \|\mathcal{B}\|\|w_0\|(1+\|\mathcal{B}\|\|w_0\|)(1+L)T\int_0^T \frac{|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|}{1+|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|}ds \\ &+ \left[1+\|\mathcal{B}\|\|w_0\|^2\right]\int_0^T \frac{|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|}{1+|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|}ds\end{aligned}$$

Using the Holder inequality, we obtain that

$$\begin{aligned}\int_0^T |\langle \mathcal{B}\mathcal{S}(s)w_0, \mathcal{I}(\mathcal{S}(s)w_0)\rangle| ds &\leq \|\mathcal{B}\|\|w_0\|(1+\|\mathcal{B}\|\|w_0\|)(1+L)T^{\frac{3}{2}}\Big(\int_0^T \frac{|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|^2}{1+|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|}ds\Big)^{\frac{1}{2}} \\ &+ \sqrt{T}\left[1+\|\mathcal{B}\|\|w_0\|^2\right]\Big(\int_0^T \frac{|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|^2}{1+|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|}ds\Big)^{\frac{1}{2}} \\ &\leq C\Big(\int_0^T \frac{|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|^2}{1+|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|}ds\Big)^{\frac{1}{2}}.\end{aligned}$$

Where $C = \|\mathcal{B}\|\|w_0\|(1+\|\mathcal{B}\|\|w_0\|)(1+L)T^{\frac{3}{2}} + \sqrt{T}\left[1+\|\mathcal{B}\|\|w_0\|^2\right]$.
Which achieves the proof. □

3 Stabilization results

In this section, we will be interested in the stability results for (1.1).
Let us recall the definition of weak and strong stabilization.

Definition 3.1 ([2],[3]). The system (1.1) is strongly (resp. weakly) partially stabilizable if there exists a feedback control $f(w(s))$ such that the corresponding mild solution w(s) of the system (1.1) satisfies the properties:

(1) if there exists a feedback control $\mathbf{u}(s) = \mathbf{u}(w(s))$ such that for all initial state w_0 in W, the corresponding mild solution of (1.1) is defined on $\mathbb{R}^+$, the origin is a Lyaponov stable equilibrium point and $w(s) \to 0$ as $s \to +\infty$.

(2) if there exists a feedback control $\mathbf{u}(s) = \mathbf{u}(w(s))$ such that for all initial state w_0 in W, the corresponding mild solution of (1.1) is defined on $\mathbb{R}^+$, the origin is a Lyaponov stable equilibrium point and $w(s) \rightharpoonup 0$ as $s \to +\infty$.

Theorem 3.2. *Let $\mathcal{A}$ generate a semigroup $\mathcal{S}(s)$ of contractions on W, such as $(\mathcal{A}_3)$ holds, then the feedback* (2.3) *stabilizes* (1.1).

Proof. Let $w(s)$ denote the corresponding solution of (2.4). For $s \geq 0$ we define the function

$$\sigma \to v(\sigma) = \int_s^\sigma \mathcal{S}(\sigma-\lambda)\mathbf{u}(\lambda)\mathcal{B}w(\lambda) + \mathcal{S}(\sigma-\lambda)\mathcal{N}w(\lambda)d\lambda.$$

Applying the variation of constant formula with $w(s)$ as the initial state, we get

$$w(\sigma) = \mathcal{S}(\sigma-s)w(s) + v(\sigma), \qquad \forall \sigma \in [s, s+T].$$

Since $\mathcal{S}(s)$ is a semigroup of contractions, then

$$\|w(\sigma)\| \leq \|w(s)\| + \|v(\sigma)\|$$

Furthermore:

$$\begin{aligned} \|v(\sigma)\| &\leq \|\mathcal{B}\| \int_s^\sigma \|w(\lambda)\| d\lambda + \int_s^\sigma \|\mathcal{N}w(\lambda) - \mathcal{N}w(0)\| d\lambda \\ &\leq \|\mathcal{B}\| \int_s^\sigma \|w(\lambda)\| d\lambda + K \int_s^\sigma \|w(\lambda)\| d\,\lambda \\ &\leq (\|\mathcal{B}\| + K) \int_s^\sigma \|w(\lambda)\| d\,\lambda \end{aligned}$$

Thus

$$\|w(\sigma)\| \leq \|w(s)\| + (\|\mathcal{B}\| + K) \int_s^\sigma \|w(\lambda)\| d\,\lambda$$

Using the Gronwall inequality, we get

$$\|w(\sigma)\| \leq \|w(s)\| \exp[(\|\mathcal{B}\| + K)T], \qquad \forall \sigma \in [s, s+T]. \tag{3.1}$$

From the expression:

$$\begin{aligned} \langle \mathcal{B}\mathcal{S}(\sigma - s)w(s), \mathcal{I}(\mathcal{S}(\sigma - s)w(s))\rangle &= \langle \mathcal{B}w(\sigma) - \mathcal{B}w(\sigma), \mathcal{I}(\mathcal{S}(\sigma - s)w(s))\rangle \\ &= -\langle \mathcal{B}w(\sigma), \mathcal{I}(\mathcal{S}(\sigma - s)w(s))\rangle \\ &+ \langle \mathcal{B}w(\sigma), \mathcal{I}(\mathcal{S}(\sigma - s)w(s)) - \mathcal{I}(w(\sigma))\rangle + \langle \mathcal{B}w(\sigma), \mathcal{I}(w(\sigma))\rangle \end{aligned}$$

It follows that

$$\begin{aligned} |\langle \mathcal{B}\mathcal{S}(\sigma - s)w(s), \mathcal{I}(\mathcal{S}(\sigma - s)w(s))\rangle| &\leq \|\mathcal{B}\| \|w(\sigma)\| \mathcal{I}(\mathcal{S}(\sigma - s)w(s))\| \\ &+ \|\mathcal{B}\| \|w(\sigma)\| \|\mathcal{I}(\mathcal{S}(\sigma - s)w(s)) - \mathcal{I}(w(\sigma))\| \\ &+ |\langle \mathcal{B}w(\sigma), \mathcal{I}(w(\sigma))\rangle| \end{aligned}$$

Using the fact that $\mathcal{I}$ is Lipschitz and $\mathcal{I}(\mathcal{S}(\sigma - s)w(s)) = \mathcal{S}(\sigma - s)w(s)$, we deduce that

$$\begin{aligned} |\langle \mathcal{B}\mathcal{S}(\sigma - s)w(s), \mathcal{I}(\mathcal{S}(\sigma - s)w(s))\rangle| &\leq \|\mathcal{B}\| \|w(\sigma)\| \|w(s)\| + \|\mathcal{B}\| \|w(\sigma)\| L \|\mathcal{S}(\sigma - s)w(s) - w(\sigma)\| \\ &+ |\langle \mathcal{B}w(\sigma), \mathcal{I}(w(\sigma))\rangle| \\ &\leq \|\mathcal{B}\| \|w(\sigma)\| \|w(s)\| + \|\mathcal{B}\| \|w(\sigma)\| L \|w(\sigma)\| \\ &+ |\langle \mathcal{B}w(\sigma), \mathcal{I}(w(\sigma))\rangle| \\ &= \|\mathcal{B}\| \|w(\sigma)\| [\|w(s)\| + L\|w(\sigma)\|] + |\langle \mathcal{B}w(\sigma), \mathcal{I}(w(\sigma))\rangle| \end{aligned}$$

Using (3.1) we deduce that

$$\begin{aligned} |\langle \mathcal{B}\mathcal{S}(\sigma - s)w(s), \mathcal{I}(\mathcal{S}(\sigma - s)w(s))\rangle| &\leq \|\mathcal{B}\| \Big(\|\mathcal{B}\| + K\Big) T \|w(s)\|^2 \left[1 + L \exp\left[\Big(\|\mathcal{B}\| + K\Big) T\right]\right] \\ &+ |\langle \mathcal{B}w(\sigma), \mathcal{I}(w(\sigma))\rangle| \\ &\leq \Big(\|\mathcal{B}\|^2 + K\|\mathcal{B}\|\Big) T \|w(s)\|^2 \left[1 + L \exp\left[\Big(\|\mathcal{B}\|^2 + \Big) T\right]\right] \\ &+ |\langle \mathcal{B}w(\sigma), \mathcal{I}(w(\sigma))\rangle|. \end{aligned} \tag{3.2}$$

By integrating (3.2) over $[s, s+T]$ we obtain:

$$\int_s^{s+T} |\langle \mathcal{B}\mathcal{S}(\sigma - s)w(s), \mathcal{I}(\mathcal{S}(\sigma - s)w(s))\rangle| ds \leq \int_s^{s+T} \Big(\|\mathcal{B}\|^2 + K\|\mathcal{B}\|\Big) T^2 \|y(0)\|^2 \Big[1 + L \exp\Big(\|\mathcal{B}\| + K\Big) T\Big] + |\langle \mathcal{B}w(\sigma), \mathcal{I}(w(\sigma))\rangle| d\sigma.$$

Remarking that

$$\int_{s}^{s+T} |\langle \mathcal{B}w(\sigma), \mathcal{I}(w(\sigma))\rangle| d\sigma \to 0 \quad as \;\; s \to +\infty.$$

which give:

$$\|w(s)\| \to 0 \quad as \;\; s \to +\infty.$$

□

3.1 Weak stabilization

In this section, we discuss the weak stabilization for system (1.1).

Theorem 3.3. *Assume that $\mathcal{A}$ generates a C_0-semigroup $\mathcal{S}(s)$ of contractions, $\mathcal{B}$ be a linear bounded operator and for all sequence $(w_n) \subset W$ such that $w_n \rightharpoonup w$ in W, we have*

$$[\langle \mathcal{B}\mathcal{S}(s)w_n, \mathcal{I}(\mathcal{S}(s)w_n)\rangle \to 0, \quad as \;\; n \to +\infty \quad for \;\; all \;\; s > 0] \Longrightarrow [w = 0]. \tag{3.3}$$

Then, the system (1.1) is weakly stabilisable by the feedback (2.3) .

Proof. Let (s_n) be a sequence of real numbers such that $s_n \to +\infty$ as $n \to +\infty$. It follows from (2.10), that $\|w(s_n)\|$ is bounded, then there exists a subsequence $(s_{\phi(n)})$ of (s_n) such that

$$w(s_{\phi(n)}) \rightharpoonup w' \in W, \quad as \;\; n \to +\infty. \tag{3.4}$$

By (2.2) and using the superposition property, we obtain for any $T > 0$,

$$\int_0^T | < \mathcal{B}\mathcal{S}(s)w(s_{\phi(n)}), \mathcal{I}(\mathcal{S}(\lambda)w(s_{\phi(n)}))\rangle| d\lambda \le C\Big(\int_{s_{\phi(n)}}^{T+s_{\phi(n)}} |\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|^2 d\lambda\Big)^{\frac{1}{2}}.$$

Remarking that

$$\int_{s_{\phi(n)}}^{T+s_{\phi(n)}} |\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|^2 d\lambda \to 0, \quad as \;\; n \to +\infty.$$

Then

$$\int_0^T | < B\mathcal{S}(\lambda)w(s_{\phi(n)}), \mathcal{I}(\mathcal{S}(\lambda)w(s_{\phi(n)}))\rangle| d\lambda \to 0, \quad as \;\; n \to +\infty.$$

Hence, by the dominated convergence theorem, we have

$$\int_0^T \lim_{n\to+\infty} | < B\mathcal{S}(\lambda)w(s_{\phi(n)}), \mathcal{I}(\mathcal{S}(\lambda)w(s_{\phi(n)}))\rangle| d\lambda = 0.$$

We conclude that

$$\lim_{n\to+\infty} | < B\mathcal{S}(\lambda)w(s_{\phi(n)}), \mathcal{I}(\mathcal{S}(\lambda)w(s_{\phi(n)}))\rangle| = 0, \qquad \forall\lambda \ge 0.$$

Since $T > 0$ is arbitrary, this implies from (3.3) that $w' = 0$. Moreover by proceeding similarly, we can show that 0 is the unique weak limit point of the sequence $w(s_n)$. Consequently, the control $\mathbf{u}(s)$ stabilises the system (1.1) in the sense of the weak topology. □

Theorem 3.4. *(Strong Stabilization) Assume that $\mathcal{A}$ generates a C_0-semigroup $\mathcal{S}(s)$ of contractions, and $\mathcal{B}$ be a linear bounded operator, and there exist $T > 0$ and $\delta > 0$ such that*

$$\int_0^T |\langle \mathcal{B}\mathcal{S}(s)w, \mathcal{I}(\mathcal{S}(s)w)\rangle| ds \ge \delta\|w\|^2 \quad \forall w \in W, \tag{3.5}$$

then control (2.3) strongly stabilises system (1.1), and we have the decay estimate:

$$\|w(s)\| = O(s^{\frac{1}{2}}), \qquad as \;\; s \to +\infty.$$

Proof. Using theorem (2.2), we have

$$\int_0^T |\langle \mathcal{B}\mathcal{S}(s)w_0, \mathcal{I}(\mathcal{S}(t)w_0)\rangle| ds \le C\Big(\int_0^T \frac{|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|^2}{1+|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|} ds\Big)^{\frac{1}{2}}.$$

This inequality together with (3.5), gives

$$\delta\|w_0\| \le C\Big(\int_0^T \frac{|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|^2}{1+|\langle \mathcal{B}w(s), \mathcal{I}(w(s))\rangle|} ds\Big)^{\frac{1}{2}}.$$

The last inequality holds for any w_0, then replacing w_0 by $w(s)$, we deduce

$$\delta\|w(s)\| \le C\Big(\int_0^T \frac{|\langle \mathcal{B}w(s+\lambda), \mathcal{I}(w(s+\lambda))\rangle|^2}{1+|\langle \mathcal{B}w(s+\lambda), \mathcal{I}(w(s+\lambda))\rangle|} d\lambda\Big)^{\frac{1}{2}}.$$

Thus

$$\delta\|w(s)\| \le C\Big(\int_s^{s+T} \frac{|\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|^2}{1+|\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|} d\lambda\Big)^{\frac{1}{2}}.$$

Setting $u_k = \|w(kT)\|^2$, $k \in \mathbb{N}$, we get:

$$\delta u_k \le C\Big(\int_{kT}^{(k+1)T} \frac{|\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|^2}{1+|\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|} d\lambda\Big)^{\frac{1}{2}}. \tag{3.6}$$

Using (2.9), we obtain:

$$2\int_{kT}^{(k+1)T} \frac{\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle^2}{1+|\langle \mathcal{B}w(\lambda), \mathcal{I}(w(\lambda))\rangle|} d\lambda \le u_k - u_{k+1}. \tag{3.7}$$

Hence, (3.6) and (3.7) yield

$$2\delta^2 u_k^2 \le C^2(u_k - u_{k+1}) \qquad \forall k \ge 0.$$

Since u_k is a positive and decreasing sequence, we deduce that:

$$u_{k+1} + \beta u_{k+1}^2 \le u_k, \qquad \forall k \ge 0, \tag{3.8}$$

with $\beta = 2\dfrac{\delta^2}{C^2}$. We now apply the following lemma from [14]

Lemma 3.5. *Let h denote a positive increasing function such that $h(0) = 0$ and set*

$$h'(\nu) = \nu - (I+h)^{-1}(\nu) \qquad \nu \in [0, +\infty[,$$

where I denotes the identity function. Let $\{\nu_k\}_{k=0}^{k=\infty}$ be a sequence of positive numbers such that

$$\nu_{k+1} + h(\nu_{k+1}) \le \nu_k, \qquad k \ge 0.$$

Then $\nu_k \le x(k)$, where x is the solution of:

$$\begin{cases} x'(s) + h'(x(s)) = 0 & s > 0, \\ x(0) = \nu_0. \end{cases} \tag{3.9}$$

It follows from equation (3.1) and lemma (3.5), applied to the sequence $\nu_k = u_k$ and $h(\nu) = \beta\nu^2$ that:

$$u_k \le x(k), \qquad k \ge 0$$

Since $x(s)$ decreases, we get $x(s) \ge 0$, $\forall s \ge 0$. Furthermore, it is easy to see that h' is an increasing function such that

$$0 \le h'(\nu) \le h(\nu), \qquad \forall \nu \ge 0.$$

Then $-\beta x(s)^2 \le x'(s) \le 0$, which implies that

$$x(s) = O(\frac{1}{s}), \qquad as \;\; s \to +\infty.$$

Thus from lemma (3.5), we obtain that $u_k = O(\dfrac{1}{k})$. This implies, since $\|w(s)\|$ decreases, that

$$\|w(s)\|^2 = O(\frac{1}{s}), \qquad as \;\; s \to +\infty.$$

Which achieves the proof. □

4 Application

In this section, we present examples illustrating the efficiency of the obtained results.

Example 4.1. Let us consider the following semilinear equation, and $\Omega =]0,2[$.

$$\begin{cases} \frac{\partial w}{\partial s}(\cdot,s) = \mathcal{A}w(x,s) + \mathcal{N}w(x,s) + v(s)\mathcal{B}w(x,s) & \text{in} \quad \Omega\times]0,+\infty[, \\ w(0,\cdot) = 0 \in W, & s \in]0,+\infty[, \\ w(x,0) = w_0 & \text{in} \quad \Omega, \end{cases} \tag{4.1}$$

where $W = \{w \in L^2(\Omega)/\|w\| \leq 1\}$, $\mathcal{A}w = -w$, $\mathcal{B}w = w$, $\mathcal{N}w = -\dfrac{\langle \mathcal{B}w, \mathcal{I}(w)\rangle}{1+\langle \mathcal{B}w, \mathcal{I}(w)\rangle}w, \quad \forall w \in W$
The operator $\mathcal{A}$ generates a semigroup of contractions on $L^2(\Omega)$ given by $\mathcal{S}(s)w_0 = e^{-s}w_0$.
The operator $\mathcal{A}$ is dissipative inded

$$\langle \mathcal{A}w, w\rangle = -\|w\|^2 \leq 0 \qquad \forall w \in W.$$

For all $w \in W$, we have

$$\langle \mathcal{N}w, w\rangle = -\frac{\langle w, \mathcal{I}(w)\rangle}{1+\langle w, \mathcal{I}(w)\rangle}\langle w, w\rangle = \frac{-\|w\|^2}{1+\langle w, \mathcal{I}(w)\rangle} \leq 0,$$

and

$$\|\mathcal{N}w\| = \| - \frac{\langle \mathcal{B}w, \mathcal{I}(w)\rangle}{1+\langle \mathcal{B}w, \mathcal{I}(w)\rangle}w\| \leq \frac{|\langle \mathcal{B}w, \mathcal{I}(w)\rangle|}{1+|\langle \mathcal{B}w, \mathcal{I}(w)\rangle|}\|w\| \leq \frac{|\langle \mathcal{B}w, \mathcal{I}(w)\rangle|}{1+|\langle \mathcal{B}w, \mathcal{I}(w)\rangle|}$$

Then the condition $\mathcal{A}_2$ holds.
For $w \in W$, and $T = 2$, we obtain

$$\begin{aligned} \int_0^2 |\langle \mathcal{B}\mathcal{S}(s)w, \mathcal{I}(\mathcal{S}(s)w)\rangle| ds &= \int_0^2 e^{-2s} ds \int_\Omega |w(x)|^2 dx \\ &= \int_0^2 e^{-2s} ds \|w\|^2_{L^2(\Omega)} \\ &\geq \beta \|w\|^2_{L^2(\Omega)}. \end{aligned}$$

Where $\beta = \int_0^2 e^{-2s} ds > 0$.
Then the condition (3.5) is verified. According to theorem (3.4), the control

$$v(s) = -\frac{\|w(s)\|^2}{1+\|w(s)\|^2},$$

strongly stabilises the system (4.1) with the decay estimate

$$\|w(s)\| = O(s^{\frac{1}{2}}), \qquad s \to +\infty.$$

Example 4.2. We consider the following semilinear :

$$\begin{cases} \frac{\partial w}{\partial s}(\cdot,s) = \frac{\partial^2}{\partial x^2}w(x,s) + \mathcal{N}w(x,s) + v(s)w(x,s) & \text{in} \quad \Omega\times]0,+\infty[, \\ w(0,\cdot) = 0 \in W, & s \in]0,+\infty[, \\ w(x,0) = w_0 & \text{in} \quad \Omega. \end{cases} \tag{4.2}$$

Where $W = \{w \in L^2(\Omega)/\|w\| \leq 1\}$, $\mathcal{A} = \frac{\partial^2}{\partial x^2}$ and $\mathcal{S}(s)w(x) = \sum_{i=1}^{i=\infty} \exp(\lambda_i s)\langle w, \xi_i\rangle \xi_i(x)$, where

$$\lambda_i = -i^2\pi^2 \qquad \textit{and} \;\; \xi_i(x) = \sqrt{2}\sin(i\pi x).$$

Then $\mathcal{A}$ generates a semigroup of contractions $\{\mathcal{S}(s)\}$ on W, so that $\mathcal{A}$ is dissipative.
The operator $\mathcal{N}w = -\dfrac{\langle w, \mathcal{I}(w)\rangle}{1+\langle w, \mathcal{I}(w)\rangle} w$ is lipshitz and dissiptive, and

$$\|\mathcal{N}w\| \leq \frac{\big|\langle \mathcal{B}w, \mathcal{I}(w)\rangle\big|}{1+\big|\langle \mathcal{B}w, \mathcal{I}(w)\rangle\big|}, \qquad \forall w \in W.$$

Moreover, we have

$$\langle \mathcal{B}\mathcal{S}(s)w, \mathcal{I}(\mathcal{S}(s)w)\rangle = \|\mathcal{S}(s)w\|^2 \qquad \forall s \geq 0 \quad and \;\; w \in W.$$

Thus using the fact that $\mathcal{S}(s)$ is compact, we deduce that the (3.3) is satisfied. Then by using theorem (3.3), the control $\mathbf{u}(s) = -\dfrac{\langle \mathcal{B}w, \mathcal{I}(w)\rangle}{1+|\langle \mathcal{B}w, \mathcal{I}(w)\rangle|} = -\dfrac{\|w(s)\|^2}{1+\|w(s)\|^2}$ weakly stabilizes the system (4.2) in W.

5 Conclusion

This work has proposed a feedback control that ensures the weak and strong stability of an infinite-dimensional semilinear systems evolving in a reflexive Banach space. This work gives an openinig to others questions, this is the case of establishing similar results for semilinears systems in a non reflexive Banach space.

References

[1] I. Aksikas, J. Winkin, D. Dochain, Asymptotic stability of infinite-dimensional semilinear systems: Application to a nonisothermal reactor, *Systems Control Lett.* **56**, 122–132 (2007).

[2] J. M. Ball, Strongly continuous semigroups, weak solutions, and the variation of constants formula, *Proceedings of the American Mathematical Society* **63(2)**, 370–373 (1977) .

[3] J. M. Ball, On the asymptotic behavior of generalized processes, with applications to nonlinear evolution equations, *Journal of differential equations* **27(2)**, 224–265 (1978).

[4] J.M. Ball and M. Slemrod, Feedback stabilization of distributed semilinear control systems, *J Appl Math Opti.* **5**, 169–179 (1979).

[5] V. Barbu, *Nonlinear differential in Banach spaces*, Springer (2009).

[6] A. Benzaza and M. Ouzahra, Weak and strong stabilization of bilinear systems in a Banach space, *International Journal of Control* **92(2)**, 1–15 (2018).

[7] L. Berrahmoune, Stabilization and decay estimate for distributed bilinear systems, *Systems and Control Letters* **36**, 167–171 (1999).

[8] L. Berrahmoune, Y. Elboukfaoui, M. Erraoui, Remarks on the feedback stabilization of systems affine in control, *Eur. J. Control* **7**, 17–28 (2001).

[9] L. Berrahmoune, Stabilization and decay estimate for distributed bilinear systems, *Systems Control Letters*, **36(3)**, 167–171 (2009).

[10] M. S, Chen, Exponential stabilization of a constrained bilinear system, *Automatica* **34**, 989–992 (1998).

[11] A. El Alami, I. El Harraki, A. Boutoulout, Regional Feedback Stabilization for Infinite Semilinear Systems, *J. Dyn Control Syst.* **24**, 343–354 (2018).

[12] I. El Harraki, A. El Alami, A. Boutoulout, M. Serhani, *Regional stabilization for semilinear parabolic systems*, *MA J Math Control Inf.* **34(3)**, 961–971 (2017).

[13] T. Kato, Nonlinear semigroups and evolution equations, *Journal of the Mathematical Society of Japan*, **19(4)**, 508–520 (1984).

[14] I. Lasiecka, D. Tataru, Uniform boundary stabilization of semilinear wave equation with nonlinear boundary conditions, *J. Di. Integ. Eq.* **6**, 507–533 (1993).

[15] M. Ouzahra, Strong stabilization with decay estimate of semilinear systems, *Systems and Control Letters* **57**, 813–815 (2008).

[16] A. Pazy, Semi-groups of Linear Operators and Applications to Partial Differential Equations, *Springer Verlag, New York* (1983).

[17] J.P. Quinn, Stabilization of bilinear systems by quadratic feedback control, *J. Math. Anal. Appl.* **75**, 66–80 (1980).

[18] Z. Rabie and A. El Alami, Strong and weak stabilization of semi-linear parabolic systems, *IMA Journal of Mathematical Control and Information*, **37(1)**, 50–63 (2020).

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