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MAMAGEMENT OF FISHERIES WITH VARIABLE MARKET PRICE

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Abstract In this work, we present a mathematical fishery model, which is a set of five ordinary differential equations governing the fish stock. The first two ODEs correspond to the evolution of two populations moving and growing between two areas, and exploited by two fishing fleets represented by their fishing efforts. The evolution of the fishing efforts are represented by two other ODEs. The catch function is assumed to be non-linear with a price variation. The evolution of the price with respect to time is represented in the fifth equation by the difference between supply (which is the catch, in our case) and demand (which corresponds to a non-linear demand function). We suppose that the growth of fish and the evolution of fishing efforts follow a slow time scale, while the fish migration, the vessels movement, and the price variation occur at a fast time scale. Then, we use an aggregation of variables method to obtain what we call "a reduced model". The aggregated fishery model is analyzed mathematically, which shows that two main cases can occur under some conditions: A fish extinction case and a sustainable one. Finally, we prove that we can switch from an undesirable case to a desirable one by controlling the catch part in the model. Some numerical simulations are also given in this work.

1 Introduction

Bio-economic modeling has garnered considerable attention, from the point of view of managing renewable resources [8, 9], and also from the point of view of control theory and optimization [8, 15].

Fishing activity model, usually presented as a set of ordinary differential equations, considered two main variables: the fish stock represented by its biomass noted n and the fishing effort which is relative to its fishing vessels and noted by E. The fish stock equations describe the variation of biomass over time and are given by the difference between a production term represented by a logistic equation $g(n) = rn(1 - \frac{n}{k})$ and a catch term that is usually relative to a Schaefer function [22]. The fishing efforts equations describe the variation in time of fishing efforts and represent the difference between the cost and the benefices (i.e. the price is proportional to the catch function).

Furthermore, we introduce another variable in the model that represents the price of the resource. In many works, it is assumed that the resource price remains constant [17, 23, 24]. In reality, and due to the restriction of the fixed price, it is more realistic to suppose a variable price with respect to time. In our case, we consider a price depending on the harvesting function and the resource [6, 7]. As a consequence, the third ODEs equation represents the evolution of the resource's price in time, and is equal to the difference between two terms; the first one is supply which is presented by the catch and demand function [25].

In [3, 13, 21], authors investigate a bio-economic model coupled with a Schaefer fishing function $h(t) = qnE$ with a linear demand dependent on price; $D(p) = A - \alpha p$. They find special results. There exists a multi-stability represented by two different fisheries with different prices on the market. To summarize, employing a linear demand function results in a critical price point where the demand becomes negative, i.e., equal to zero. While opting for a nonlinear demand function ensures a positive demand persists even at significantly higher prices [14].

As an assumption of the linear fishing function which corresponds to the Schaefer function used [22], occur many restrictions, such as harvesting vessels are able to catch the fish stock in unlimited amounts. Nonetheless, the fishing fleets require certain time delays between successive fishing operations, with each delay allowing them to catch only limited fish biomass. So, In [18], authors prolonged this last assumption to the use of a non-linear harvesting function with saturation effect $h(t) = \frac{qnE}{D+n}$, see [5, 10]. An important consequence on the dynamic of a fishery arises from the modification made to the catch function. The main result that they obtain, is the existence of a stable limit cycle and a multi-stability can also occur.

Therefore, in the current study, we coupled two significant modifications. The first one relates to the type of the demand function and the second one is related to the use of a non-linear catch function. The demand function is supposed to be inversely proportional function to the price $D(p) = \frac{A}{p}$ with A is a positive parameter as cited in [14]. The catch function that we present corresponds to a Holling type II function $h(t) = \frac{qnE}{D+an}$ and depends to stock by a constant migration. The goal of this paper is to investigate the important consequence on fishery dynamics when we coupled a non-linear demand and a non-linear harvesting function, and how we can control our model to switch from an undesirable case to a sustainable one.

The organization of the current paper consists of five sections. Section 2 is dedicated to presenting a bio-economics model governing five ODEs with three main variables, n represents the fish stock, E is the fishing effort and p represents the price on the market of resources. Under the assumption of the existence of a two-time scale, this system is reduced by using an aggregation of variables method in section 3. We discuss and analyze mathematically this reduced model in section 4. In the last section, we focus on controlling our system to avoid an undesirable case obtained.

2 Presentation of the Fishery Model

2.1 Complete Model

Here we consider a spatial fishery model which describes the behavior of two distinct fish populations, each with its own density denoted as n_1 and n_2, residing in separate fishing areas. These populations are subjected to harvesting activities by two fishing efforts, each with their respective fishing fleets E_1 and E_2, see Figure 2.

We make the assumption that there are two distinct processes occurring at two different time scales. The overall density of fish and fishing effort occurs at a slow time scale as compared to the movement of fish and fishing boats between the two fishing patches following a fast one.

The evolution of fish stock is according to the famous logistic function $g(n) = rn(1 - \frac{n}{K})$ where r is the growth rate of fish biomass and K is its carrying capacity. This amount decreases according to the stock-effort harvesting function which represents the progression of harvesting efforts in each of the two zones. The fishing function in the two distinct zones is assumed to be dependent on the number of individuals present in both zone-1 and zone-2. Thus, the non-linear harvesting function in each area can be expressed as $h_i = \frac{q_i E_i n_i}{a_1 n_1 + a_2 n_2 + D}$ (where i=1,2). The last function indicates that an increase in the stock of zone-2 draws fishing vessels from zone-1 to zone-2, resulting in a rise in fishing efforts in zone-2 and a decrease in zone-1. From a mathematical standpoint, it can be observed that h_1 tends towards 0 as $n_2 \to \infty$, and conversely, h_2 tends towards 0 as $n_1 \to \infty$ as reported in [20].

We also incorporate the dynamics of fish pricing in our model, where the capture price fluctuates at a rapid time scale and is determined by the difference between a demand function $D(p)$ and a supply function solely determined by the instantaneous capture. In our study, we made the assumption that price fluctuation follows a nonlinear demand function. Specifically, we used a demand function represented by the equation $D(p) = \frac{A}{p}$, where A is a positive parameter as cited in [11, 14], see Figure 1.

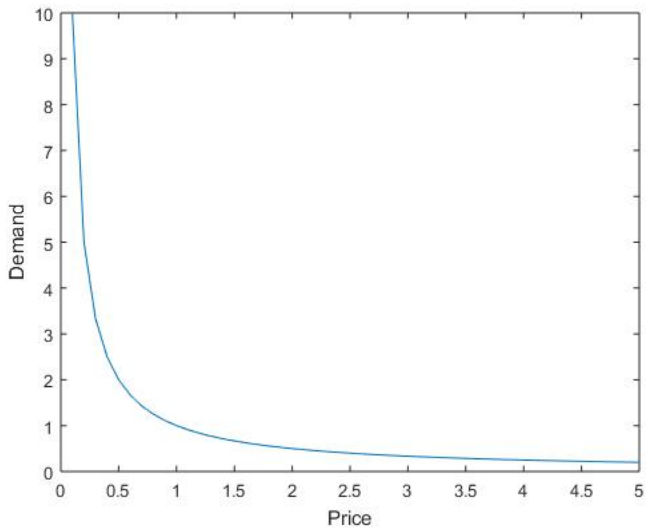

Figure 1. The graph of nonlinear demand function $D(p) = \frac{A}{p}$

These assumptions lead us to describe the complete system, which operates on a fast time scale $\tau = \frac{t}{\varepsilon}$ (compared to the slow time scale t where $\varepsilon \ll 1$), in the following way:

$$\begin{cases} \frac{\mathrm{d}n_1}{\mathrm{d}\tau} = (kn_2 - k^{'}n_1) + \varepsilon[r_1 n_1(1 - \frac{n_1}{k_1}) - \frac{q_1 n_1 E_1}{a_1 n_1 + a_2 n_2 + D}] \\ \\ \frac{\mathrm{d}n_2}{\mathrm{d}\tau} = (k^{'}n_1 - kn_2) + \varepsilon[r_2 n_2(1 - \frac{n_2}{k_2}) - \frac{q_2 n_2 E_2}{a_1 n_1 + a_2 n_2 + D}] \\ \\ \frac{\mathrm{d}E_1}{\mathrm{d}\tau} = [mE_2 - m^{'}E_1] + \varepsilon E_1(-c_1 + \frac{q_1 n_1}{a_1 n_1 + a_2 n_2 + D}p) \\ \\ \frac{\mathrm{d}E_2}{\mathrm{d}\tau} = [m^{'}E_1 - mE_2] + \varepsilon E_2(-c_2 + \frac{q_2 n_2}{a_1 n_1 + a_2 n_2 + D}p) \\ \\ \frac{\mathrm{d}p}{\mathrm{d}\tau} = \alpha(D(p) - [\frac{E_1 q_1 n_1}{a_1 n_1 + a_2 n_2 + D} + \frac{E_2 q_2 n_2}{a_1 n_1 + a_2 n_2 + D}]) \end{cases} \tag{2.1}$$

The variables r_1 and r_2 represent the intrinsic growth rate of two different stocks, denoted by n_1 and n_2 respectively. We assume that the two stocks inhabit different environments with distinct characteristics. Hence, the values of the carrying capacities, k_1 and k_2, and the catch-ability coefficient of the fishing fleet on each zone, denoted by q_1 and q_2, respectively, are different. For simplicity of calculation, we also assume that both the catch-ability and the handling time coefficients a_1 and a_2 are constant and equal to 1 for both zones and that does not affect the overall generality of the model. With these assumptions, the system described by equations (2.1) can be expressed in the following manner:

$$\begin{cases} \frac{\mathrm{d}n_1}{\mathrm{d}\tau} = (kn_2 - k^{'}n_1) + \varepsilon[r_1 n_1(1 - \frac{n_1}{k_1}) - \frac{n_1 E_1}{n_1 + n_2 + D}] \\ \\ \frac{\mathrm{d}n_2}{\mathrm{d}\tau} = (k^{'}n_1 - kn_2) + \varepsilon[r_2 n_2(1 - \frac{n_2}{k_2}) - \frac{n_2 E_2}{n_1 + n_2 + D}] \\ \\ \frac{\mathrm{d}E_1}{\mathrm{d}\tau} = [mE_2 - m^{'}E_1] + \varepsilon E_1(-c_1 + \frac{n_1}{n_1 + n_2 + D}p) \\ \\ \frac{\mathrm{d}E_2}{\mathrm{d}\tau} = [m^{'}E_1 - mE_2] + \varepsilon E_2(-c_2 + \frac{n_2}{n_1 + n_2 + D}p) \\ \\ \frac{\mathrm{d}p}{\mathrm{d}\tau} = \alpha(D(p) - [\frac{E_1 n_1}{n_1 + n_2 + D} + \frac{E_2 n_2}{n_1 + n_2 + D}]) \end{cases} \tag{2.2}$$

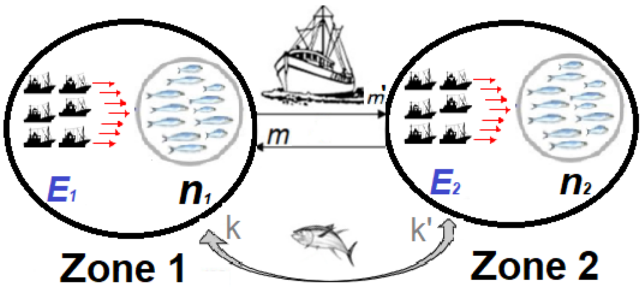

Figure 2. Fish migration and boats movement between the fishing zones

To investigate and examine the system's dynamics, we employ aggregation techniques as described in [1, 2, 19]. These techniques facilitate the consolidation of a multi-variable and multi-time scale system into a reduced model featuring a small set of global variables. As a result, the reduced model comprises a system of two ordinary differential equations that control the overall fish biomass and harvesting effort over a slower time scale.

2.2 Fast Model

Firstly, we set $\varepsilon = 0$, which means that we neglect the terms that appear at a slow time scale in all equations of (2.2). The fast model for fish will take the form as follows:

$$\begin{cases} \frac{\mathrm{d}n_1}{\mathrm{d}\tau} = (kn_2 - k^{'}n_1) \\ \\ \frac{\mathrm{d}n_2}{\mathrm{d}\tau} = (k^{'}n_1 - kn_2) \end{cases} \tag{2.3}$$

τ is the fast time scale, while t corresponds to the slow time scale, with $t = \varepsilon\tau$. The fast model is characterized by its conservative, and on the fast time scale, the total stock of the system, which is given by the sum of $n_1(t)$ and $n_2(t)$, maintains a constant value (i.e. $n(t) = n_1(t) + n_2(t)$).

Then, the fast model for fishing efforts takes the following form:

$$\begin{cases} \frac{\mathrm{d}E_1}{\mathrm{d}\tau} = [mE_2 - m^{'}E_1] \\ \\ \frac{\mathrm{d}E_2}{\mathrm{d}\tau} = [m^{'}E_1 - mE_2] \end{cases} \tag{2.4}$$

The total number of vessels is remain constant, so the fast model is also conservative. Furthermore, at a rapid time scale, the total stock denoted as $E(t) = E_1(t) + E_2(t)$ remains constant [4].

An elementary computation demonstrates that for any positive starting point, there is a singular stable equilibrium point with a positive value.

The following expressions represent the fast equilibria for fish and fishing efforts:

$$\begin{cases} n_1^* = v_1^* n, \qquad n_2^* = v_2^* n \\ \\ E_1^* = \mu_1^* E, \qquad E_2^* = \mu_2^* E \end{cases} \tag{2.5}$$

The fish proportions v_i^* and the corresponding fishing effort proportions μ_i^* in each zone i are fixed and determined by:

$$\begin{cases} v_1^* = \frac{k}{k+k'}, \qquad v_2^* = \frac{k^{'}}{k+k'} \\ \\ \mu_1^* = \frac{m}{m+m'}, \qquad \mu_2^* = \frac{m^{'}}{m+m'} \end{cases} \tag{2.6}$$

The non-trivial equilibrium values p^* are substituted for the catch effort price, which satisfies the equation:

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = \alpha(\frac{A}{p} - [\frac{E_1 n_1}{n_1 + n_2 + D} + \frac{E_2 n_2}{n_1 + n_2 + D}]) = 0 \tag{2.7}$$

This equation possesses a unique solution denoted as p^* and can be expressed as follows:

$$p^* = A\frac{an + D}{QEn} \tag{2.8}$$

The stability analysis of the fast equilibrium has shown that p^* is stable. (see Appendix 1)

3 Aggregated model

To obtain the aggregated model, one would substitute the rapid and steady-state equilibrium values for the price (2.8), as well as the movements of the fish and fleets (2.5), into the complete system (2.2).
This lead to this stable model:

$$\begin{cases} \frac{\mathrm{d}n}{\mathrm{d}t} = n[r(1 - \frac{n}{K}) - \frac{QE}{an+D}] \\ \\ \frac{\mathrm{d}E}{\mathrm{d}t} = -cE + A \end{cases} \tag{3.1}$$

where,

$$\begin{cases} r = v_1^* r_1 + v_2^* r_2 \\ \\ Q = \mu_1^* v_1^* + \mu_2^* v_2^* \\ \\ c = c_1 \mu_1^* + c_2 \mu_2^* \\ \\ a = v_1^* + v_2^* \\ \\ \frac{r}{K} = \frac{r_1 v_1^{*2}}{k_1} + \frac{r_2 v_2^{*2}}{k_2} \end{cases} \tag{3.2}$$

3.1 Positivity and Boundedness of Reduced System

Theorem 3.1. *Every possible solution $(n(t), E(t))$ of the reduced system (3.1) initiating in $\mathbf{R}_+^2$ are positive for all $t \geq 0$.*

Proof. The first equation of the aggregated model (3.1) implies that $n = 0$ is an invariant subset. Thus, for all $t \geq 0$, $n(t) > 0$ if $n(0) > 0$. From the second equation of (3.1), when $E = 0$, E is increasing. Thus, for all $t \geq 0$, $E(t) > 0$ if $E(0) \geq 0$.

Theorem 3.2. *All solutions of the aggregated model of (3.1) are uniformly bounded.*

Proof. Considering

$$W(t) = W(n(t), E(t)) = n(t) + E(t) \tag{3.3}$$

with
$W(0,0) = 0$, $W(n(t), E(t)) > 0$
The time derivative of equation (3.3), as it evolves over the solutions of the reduced system (3.1), can be presented as:

$$\begin{aligned}\frac{dW}{dt} &= nr(1-\frac{n}{K}) - \frac{QnE}{an+D} - Ec + A \\ &\leq nr(1-\frac{n}{K}) - Ec + A \\ &\leq nr(1-\frac{n}{K}) + cn - c(n+E) + A\end{aligned}$$

So the expression $[nr(1-\frac{n}{K}) + cn]$ reaches a maximum value of $\frac{K(r+c)^2}{4r}$ then:

$$\begin{aligned}\frac{dW}{dt} &\leq \frac{K(r+c)^2}{4r} - c(n+E) + A \\ &\leq \mu - cW\end{aligned}$$

where $\mu = \frac{K(r+c)^2}{4r} + A$ then

$$\frac{dW}{dt} + cW \leq \mu$$

By utilizing the differential inequality theorem proposed in [12], we acquire:

$$0 < W(t) < W(0)e^{-ct} + \frac{\mu}{c}(1-e^{-ct})$$

and for $t \to \infty$, we have

$$W \leq \frac{\mu}{c}$$

Finally, all solutions of the reduced model (3.1) which initiate at positive conditions are confined in the compact

$$\Omega = \{(n,E) \in \mathbf{R}_+^2; n(t) + E(t) \leq \frac{\mu}{c}\}$$

4 Analysis of the Aggregated Model

4.1 Existence of Equilibria

Upon consideration of the reduced system (3.1), we can identify the n-nullclines as being composed of $n = 0$ and $E = \frac{r}{QK}(K-n)(an+D)$, while the E-nullclines are comprised of $E = \frac{A}{c}$. From this, we can derive the equilibrium points as being $(0, \frac{A}{c})$, as well as an interior point (n^*, E^*) where $E^* = \frac{A}{c}$ and n^* represent a positive solution to the subsequent quadratic equation.

$$g(n) = an^2 + n(D-aK) + K(\frac{AQ}{rc} - D) \tag{4.1}$$

To determine the number of positive interior equilibria of system (3.1), we must identify the number of positive roots of equation (4.1) on the interval $(0, K)$, where $g(K) = \frac{AQ}{c} > 0$, and $g(0) = K(\frac{AQ}{rc} - D)$. Based on this, we can describe the existence of the equilibria as follows:

- If $g(0) < 0$, then there exists a positive root n^* that can be expressed as follows:

$$n^* = \frac{aK - D}{2a} + \frac{\sqrt{(aK-D)^2 - 4aK(\frac{AQ}{rc} - D)}}{2a}$$

As a result, a single positive interior equilibrium point, denoted as (n^*, E^*), is obtained.

- If $g(0) > 0$ and $(aK + D)^2 > \frac{4AQKa}{rc}$ then, there exists two positive roots n_1^* and n_2^* are given by

$$n_1^* = \frac{aK - D}{2} - \frac{\sqrt{(aK-D)^2 - 4aK(\frac{AQ}{rc} - D)}}{2a}$$

$$n_2^* = \frac{aK - D}{2a} + \frac{\sqrt{(aK-D)^2 - 4aK(\frac{AQ}{rc} - D)}}{2a}$$

with

$$0 < n_1^* < \frac{aK - D}{2a} < n_2^* < K$$

So, we get two positive interior equilibria (n_1^*, E^*) and (n_2^*, E^*).

4.2 Stability of Equilibria and Discussion

At any given point, the Jacobian matrix can be expressed as:

$$Jac_{(n,E)} = \begin{pmatrix} r - \frac{2rn}{K} - QD\frac{E}{(an+D)^2} & \frac{-Qn}{an+D} \\ 0 & -c \end{pmatrix} \tag{4.2}$$

- **Local Stability of the Over-Exploitation Equilibrium**. The Jacobian matrix at the over-exploitation equilibrium $(0, \frac{A}{c})$ can be expressed as

$$Jac_{(0,\frac{A}{c})} = \begin{pmatrix} r - \frac{QA}{cD} & 0 \\ 0 & -c \end{pmatrix}$$

The matrix above has eigenvalues that are given by: $\lambda_1 = r - \frac{QA}{cD}$ and $\lambda_2 = -c < 0$ then $(0, \frac{A}{c})$ is a stable equilibrium if and only if $g(0) > 0$.

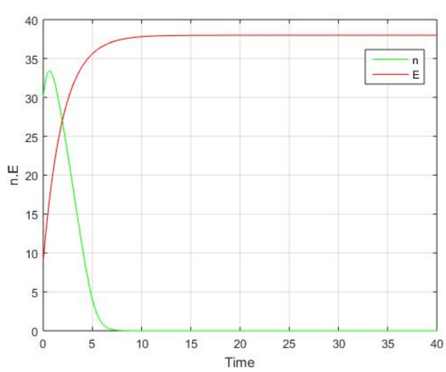

Figure 3. Case of a stable over-exploitation with parameters values $Q = 1$, $A = 9$, $c = 0.5$, $K = 40$, $r = 0.5$, $D = 20$, $n(0) = 30$, $E(0) = 10$.

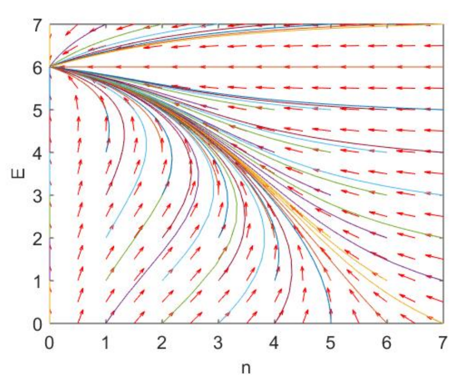

Figure 4. Phase plane for the case corresponds to over-exploitation which obtained by $Q = 2$, $A = 3$, $c = 0.5$, $K = 5$, $r = 1.9$, $D = 5$.

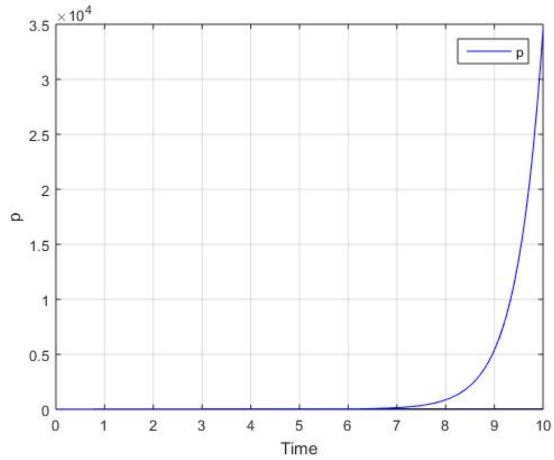

Figure 5. Price in time

Interpretation 1.

$g(0) > 0$ is equivalent to $c < \frac{AQ}{rD}$, indicates that a low cost of fishing results in unsustainable levels of the fish population, leading to over-exploitation, causing fish extinction, see Figures 3 and 4. At extinction, the fishing effort tends to $\frac{A}{c}$ and not to zero. Hence, as the fish population approaches the brink of extinction, there is a substantial increase in the price per unit of fish biomass, as demonstrated in Figure 5. As a result, despite the diminishing fish stocks, the fishing industry is able to maintain a profitable income due to the elevated market price. This can partly explain why fishing fleets continue to exploit the resource until the point of extinction.

- **Local Stability of the Interior Equilibrium**.
  The Jacobian matrix at (n^*, E^*) yields

$$Jac_{(n^*,E^*)} = \begin{pmatrix} -\frac{rn^*}{K} + \frac{Qan^*E^*}{(an^*+D)^2} & \frac{-Qn^*}{an^*+D} \\ 0 & -c \end{pmatrix}$$

The eigenvalues are respectively given by: $\lambda_2 = -c < 0$ and $\lambda_1 = -\frac{rn^*}{K} + \frac{Qan^*E^*}{(an^*+D)^2}$

Proposition 1.

We can classify the possible equilibria of the system as follows:

- **Case 1:** When $g(0) < 0$, there exists only one positive equilibrium point denoted by (n^*, E^*), which is locally asymptotically stable (LAS).

- **Case 2:** If $g(0) > 0$ and $(aK + D)^2 > \frac{4AQKa}{rc}$, there are two positive equilibria, denoted by (n_1^*, E^*) and (n_2^*, E^*). The equilibrium point (n_1^*, E^*) is unstable, while (n_2^*, E^*) is LAS.

Proof. It is easy to see that (n^*, E^*) is a stable node when $g(0) < 0$ since $n_1^* < \frac{aK-D}{2a} < n_2^* < n^*$. However, when $g(0) > 0$, the equilibrium (n_1^*, E^*) has a triangular Jacobian with one negative and one positive eigenvalue, making it a saddle point. On the other hand, the equilibrium (n_2^*, E^*) always has a triangular Jacobian with two negative eigenvalues, resulting in a stable node.

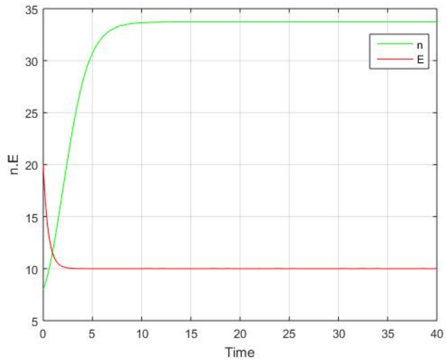

Figure 6. Illustration of the case 1. The unique stable interior equilibrium with parameter values $Q = 1$, $A = 20$, $c = 2$, $K = 40$, $r = 1$, $D = 30$, $n(0) = 8$, $E(0) = 20$.

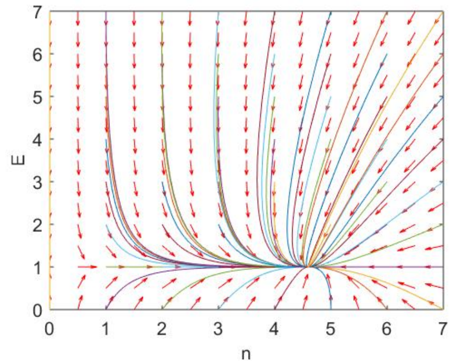

Figure 7. Phase plane for the case corresponds to a unique stable equilibrium obtained by $Q = 2$, $A = 3$, $c = 0.5$, $K = 5$, $r = 1.9$, $D = 5$.

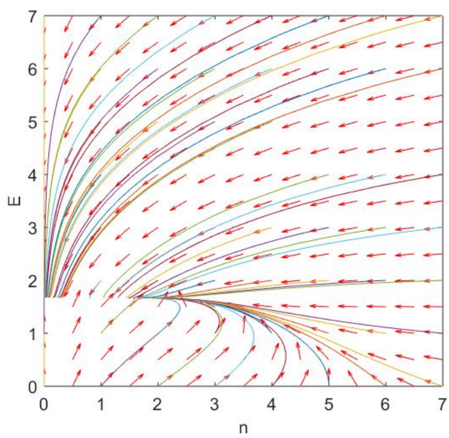

Figure 8. Illustration of case 2 with parameters values $c = 0.6$, $Q = 1.9$, $A = 1$, $K = 5$, $r = 1$, $D = 3$.

Interpretation 2.

$g(0) < 0$ is equivalent to $c > \frac{AQ}{rD}$, this suggests that when the government increases taxes on boat owners to cover costs, the rate of harvesting will decrease. Figure 6 illustrates the case of a sustainable fishery. The fish biomass and the fishing fleet tend to a positive value in Figure 7. The fish stock is still abundant and not at risk of extinction. Figure 8 illustrates case 2 which represents the multi-stability which means the existence of three equilibria, a stable one corresponds to the over-exploitation that leads to fish extinction and a second one is a saddle point, and a thirty-one represented a sustainable fishery which means that the trajectory converges to the first and to the second one.

5 Introduction of a Control Parameter

As illustrated in the preceding section, the dynamics of the model can result in either a stable equilibrium or a stable Fish Extinction state, depending on the parameter values. Regarding the sustainable management of the fishery, it is desirable to avoid significant fluctuations in both the total fish stock and the fishing effort. Introducing a control parameter, denoted by u, to the model could be beneficial as cited in [16]. This parameter is a real constant that satisfies the inequality $0 < u < 1$. A potential method for a coastal state to regulate its fishery is by limiting the technical capabilities of fishing vessels. This could be accomplished by imposing restrictions on fishing techniques and thus curtailing the catch of fleets. In our original model, the catch-abilities were assumed to be equal to 1. However, to incorporate the effect of limiting catch-abilities, we introduce u as a catch-ability term that is uniform for both fishing fleets. To accomplish this, we multiply the catch terms $\frac{E_i n_i}{n_1+n_2+D}$ by the parameter u in every equation of (2.2). The resulting system can be expressed as follows:

$$\begin{cases} \frac{dn_1}{d\tau} = (kn_2 - k^{'}n_1) + \varepsilon[r_1 n_1(1-\frac{n_1}{k_1}) - \frac{un_1E_1}{n_1+n_2+D}] \\ \frac{dn_2}{d\tau} = (k^{'}n_1 - kn_2) + \varepsilon[r_2 n_2(1-\frac{n_2}{k_2}) - \frac{un_2E_2}{n_1+n_2+D}] \\ \frac{dE_1}{d\tau} = [mE_2 - m^{'}E_1] + \varepsilon E_1(-c_1 + \frac{un_1}{n_1+n_2+D}p) \\ \frac{dE_2}{d\tau} = [m^{'}E_1 - mE_2] + \varepsilon E_2(-c_2 + \frac{un_2}{n_1+n_2+D}p) \\ \frac{dp}{d\tau} = \alpha(D(p) - [\frac{uE_1n_1}{n_1+n_2+D} + \frac{uE_2n_2}{n_1+n_2+D}]) \end{cases} \tag{5.1}$$

In this case, the aggregated system is:

$$\begin{cases} \frac{dn}{dt} = n[r(1-\frac{n}{K}) - \frac{uQE}{an+D}] \\ \frac{dE}{dt} = -cE + A \end{cases} \tag{5.2}$$

The Jacobian matrix of system (5.2) is given by:

$$Jac_{(n,E)} = \begin{pmatrix} r - \frac{2rn}{K} - uQD\frac{E}{(an+D)^2} & \frac{-uQn}{an+D} \\ 0 & -c \end{pmatrix} \tag{5.3}$$

The values of the eigenvalues for the equilibrium point $(0, \frac{A}{c})$ are expressed by: $\lambda_1 = r - \frac{uAQ}{cD}$ and $\lambda_2 = -c < 0$. The Fish extinction $(0, \frac{A}{c})$ is a stable equilibrium if and only if $c < \frac{uAQ}{rD}$ that means when the cost per unit of fishing is small enough, a large number of fishing fleet attract the density of stock until it's extinction.

To maintain an unstable equilibrium at $(0, \frac{A}{c})$, where $c > \frac{uAQ}{rD}$, and to ensure that increasing the cost of government by levying taxes on boat owners leads to a reduction in the harvesting rate, it is essential to keep the control parameter below a specific threshold value:

$$0 < u < \frac{rDc}{AQ} \tag{5.4}$$

If $H = \frac{rDc}{AQ} > 1$, the equilibrium $(0, \frac{A}{c})$ is inherently unstable without any control measures. Conversely, if $H < 1$, controlling the system is crucial, and the parameter u should be set as outlined in equation (5.4).

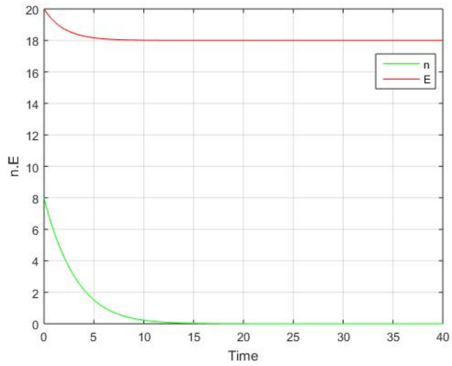

Figure 9. Case without control of a stable over-exploitation with parameters values $Q = 1$, $A = 9$, $c = 0.5$, $K = 40$, $r = 0.5$, $D = 20$, $n(0) = 8$, $E(0) = 20$.

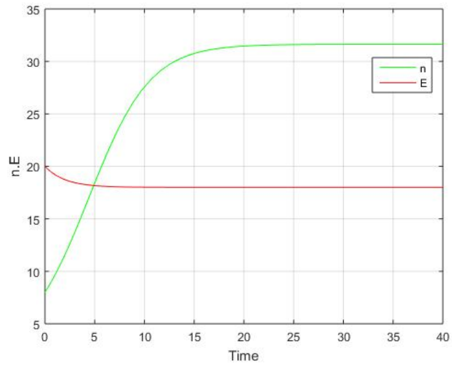

Figure 10. Illustration of the case when we add a control parameter, u=0.3.

First of all, we examine the scenario where an over-exploitation equilibrium exists, as depicted in Figure 9, without the inclusion of any control parameter. In Figure 10, our system is modified to include a control parameter after a time interval of t to establish a sustainable equilibrium, thus preventing the Fish Extinction scenario. As shown, the stock is seen to be recovering and converging towards the stable state of the model.

6 Conclusion and Perspective

Our aim was to create a bio-economic model that describes the dynamics of a fishery operating in two zones, with two distinct fishing fleets, based on a system of five ordinary differential equations. The model describes the dynamics of the two local fish stocks, the fishing efforts employed in each harvesting zone, and the variation in market price. Our model assumes that the market price is determined by the difference between the demand, represented by the function D(p), and the supply, which is proportional to the catch. The non-linear demand function proposed is inversely proportional to the price which has an important consequence on the dynamics of fisheries. Our study considers two distinct time scales: a fast scale occurs at the rapid movements between fishing zones, and a slower scale governs the evolution of fish populations and the variation of revenue for the fishing fleets. To account for this difference, we use both time scales to construct an aggregated model. It is focused on describing the dynamics of total harvesting efforts, and total fish stocks at the slower time scale.

Upon analyzing the aggregated system, we identified two distinct cases of interest: an over-exploitation equilibrium that ultimately leads to fish extinction and a sustainable one. In the latter case, the harvesting boats continue to exploit the stock despite its dwindling numbers due to the high market price that arises when the fish population becomes scarce. Although this behavior may seem reasonable, it poses a significant risk of extinction. To mitigate this risk, we propose the introduction of a control parameter that can be used to destabilize the Fish Extinction equilibrium and help the fish stock reach a sustainable state. This control parameter allows for better management of fishing efforts and avoids the extinction of the fish stock. Numerical simulations are illustrated to verify analytical results.

As perspectives, there exist presently a case of over-exploitation and it will be interesting to look for the effect of the surface size on marine protected areas (MPA) to destabilize the over-exploitation case. The size of a marine protected area (MPA) can have a significant effect on its ability to stabilize a fishery that is experiencing over-exploitation. Larger MPAs have a greater capacity to support and protect fish populations, as well as the habitats and ecosystems they depend on.

Appendix

1. Calculation of the fast equilibria

At the fast time :

$$D(p) - [\frac{E_1 q_1 n_1}{n_1 + n_2 + D} + \frac{E_2 q_2 n_2}{n_1 + n_2 + D}] = 0 \tag{6.1}$$

$$\begin{aligned}
D(p) - [\frac{E_1 q_1 n_1}{n_1 + n_2 + D} + \frac{E_2 q_2 n_2}{n_1 + n_2 + D}] = 0 \quad \Leftrightarrow \quad & D(p) = \frac{E_1 q_1 n_1}{n_1 + n_2 + D} + \frac{E_2 q_2 n_2}{n_1 + n_2 + D} \\
& \Leftrightarrow \frac{A}{p} = \frac{QnE}{an + D} \\
& \Leftrightarrow A(an + D) = pQnE \\
& \Leftrightarrow p^* = \frac{A(an + D)}{QnE}
\end{aligned}$$

2. Derivation of the reduced model

We replace the fast equilibria, and we will derive the subsequent system:

$$\begin{aligned}
\frac{\mathrm{d}n_1}{\mathrm{d}\tau} &= \varepsilon[r_1 n_1^*(1 - \frac{n_1^*}{k_1}) - \frac{E_1^* q_1 n_1^*}{n_1^* + n_2^* + D}] \\
\frac{\mathrm{d}n_2}{\mathrm{d}\tau} &= \varepsilon[r_2 n_2^*(1 - \frac{n_2^*}{k_2}) - \frac{E_2^* q_1 n_2^*}{n_1^* + n_2^* + D}] \\
\frac{\mathrm{d}E_1}{\mathrm{d}\tau} &= \varepsilon E_1^*(-c_1 + \frac{q_1 n_1^*}{n_1^* + n_2^* + D} p^*) \\
\frac{\mathrm{d}E_2}{\mathrm{d}\tau} &= \varepsilon E_2^*(-c_2 + \frac{q_2 n_2^*}{n_1^* + n_2^* + D} p^*).
\end{aligned}$$

Since $n = n_1 + n_2$ we have $\frac{\mathrm{d}n}{\mathrm{d}\tau} = \frac{\mathrm{d}n_1}{\mathrm{d}\tau} + \frac{\mathrm{d}n_2}{\mathrm{d}\tau}$. By introducing this timescale $\tau = \frac{t}{\varepsilon}$, we can derive a slow timescale equation that governs the dynamics of the total fish stock:

$$\begin{aligned}
\frac{\mathrm{d}n}{\mathrm{d}t} &= r_1 n_1^*(1 - \frac{n_1^*}{k_1}) + r_2 n_2^*(1 - \frac{n_2^*}{k_2}) - [\frac{E_1^* q_1 n_1^*}{n_1^* + n_2^* + D} + \frac{E_2^* q_2 n_2^*}{n_1^* + n_2^* + D}] \\
&= n(1 - \frac{n}{K})(r_1 v_1^* + r_2 v_2^*) - \frac{EQn}{an + D}
\end{aligned}$$

We denote r = $r_1 v_1^* + r_2 v_2^*$, Q=$q_1 v_1^* \mu_1^* + q_2 v_2^* \mu_2^*$ and $a = v_1^* + v_2^*$. The equation governing the fish stock at the slow timescale can be expressed as:

$$\frac{\mathrm{d}n}{\mathrm{d}t} = n[r(1 - \frac{n}{K}) - \frac{EQ}{an + D}]$$

The dynamics of the fishing effort E at the slow timescale are governed by the following equation

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= E_1^*(-c_1+\frac{q_1n_1^*}{n_1^*+n_2^*+D}p^*)+E_2^*(-c_2+\frac{q_2n_2^*}{n_1^*+n_2^*+D}p^*)\\ &= p^*(\frac{E_1^*q_1n_1^*}{n_1^*+n_2^*+D}+\frac{E_2^*q_2n_2^*}{n_1^*+n_2^*+D})-(E_1^*c_1+E_2^*c_2)\\ &= (-c+p^*\frac{Qn}{an+D})E\end{aligned}$$

Eventually, at a slow pace, we arrive at the following:

$$\frac{\mathrm{d}E}{\mathrm{d}t}=(-c+p^*\frac{Qn}{an+D})E$$

We denote c = $c_1\mu_1^*+c_2\mu_2^*$ and the value of p^* is determined by both n and E, and it can be expressed as $p^*=\frac{A(an+D)}{QnE}$
Finally, we get

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-cE+A$$

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