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WEAK PERIODIC SOLUTION TO NONLINEAR VARIATIONAL PARABOLIC PROBLEMS HAVING NON LINEAR BOUNDARY CONDITIONS AND WITHOUT SIGN CONDITION

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Abstract In this research, we show the existence of a weak periodic solution for nonlinear variational parabolic problems having non linear boundary conditions and without sign condition on the non-linearity.

1 Introduction

In this study, we show that a weak periodic solution for nonlinear variational parabolic of the following type :

$$\begin{cases} \frac{\partial \varpi}{\partial t} - \Delta\varpi + H(x,t,\varpi,\nabla\varpi) = f & \text{in } Q \\ \varpi(x,0) = \varpi(x,T) & \text{in } \Omega \\ -\frac{\partial \varpi}{\partial \nu} = \beta(x,t)\varpi + h(x,t,\varpi) & \text{on } \partial\Omega\times]0,T[\end{cases} \tag{1.1}$$

with $\Omega \subset \mathbb{R}^N (N \geq 2)$ bounded open with smooth boundary denoted by $\partial\Omega$, $T \in \mathbb{R}_*^+$ is the period, $\Sigma = \partial\Omega\times]0,T[$, ν is the unit normal vector to $\partial\Omega$, $Q = \Omega\times]0,T[$, Δ denote the Laplacian operator, H is a Carathéodory function and $f \in L^2(Q)$.

Many physical, chemical and biological phenomena can be modeled mathematically by the problem (1.1). The existence of periodic solutions for problem (1.1) has been studied by many researchers ([2, 3, 5, 16, 6, 18, 4, 10]). In [2], by using the method of sub, super-solution and Schauder's fixed point theorem, Amann prove the existence of classical solution for the equation (1.1). In [16], Duel and Hess prove the existence results of a nonlinear parabolic problems. A large number of papers was devoted to the study the existence results of parabolic problems or elliptic problems under various assumptions and in different contexts: for a review on classical results see [8, 9, 7, 12, 13, 14].

In [18], Pao also study a class of coupled systems of non-linear parabolic equations under non-linear boundary conditions, including a combination of linear and non-linear conditions. Alaa and al [1] prove the existence results for some quasi-linear parabolic equations with data measures and the boundary condition is of Dirichlet type. In [5], the author studied the existence and uniqueness result to a class of non-linear parabolic equations having p(x)-growth conditions and L^1 data. In [3], Badii showed the existence of weak periodic solutions for the equation (1.1) with H is independent of the gradient. In [6], the authors have demonstrated the existence of weak periodic solution for problem (1.1) assuming that the non-linearity depends on the gradient and satisfies the sign condition.

The goal of this paper is to prove the existence of periodic solutions for parabolic problems of the form (1.1) without the sign condition:

$$H(x,t,s,r)s \geq 0$$

We have organized this paper as follows : Section 2, we start by making the structural assumptions on β, h, H and f and we present the functional framework that includes our work, at the end of this section, we define the notion of a periodic weak solution. In Section 3, we prove an existence result when the non-linearity is bounded. And in the fourth section, we establish the existence of a weak periodic solution to (1.1).

2 Assumptions and functional framework

2.1 Assumptions

Throughout this paper, we assume that :
Assumption (A1). $f \in L^2(Q)$ is a periodic function such that.
Assumption (A2). β is a periodic continuous function such that

$$0 < \beta_0 \leq \beta(x,t) \leq \beta_1, \quad \forall (x,t) \in \Sigma.$$

Assumption (A3). $h : \Sigma \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function periodic in time, $s \mapsto h(x,t,s)$ is non-decreasing with respect to s for a.e $(x,t) \in \Sigma$ and

$$h(x,t,s)s \geq 0 \tag{2.1}$$

$$|h(t,x,s)| \leq \xi(x,t) + |s| \tag{2.2}$$

where $\xi \in L^2(\Sigma)$.
Assumption (A4). $H : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that

$$H(x,t,s,\xi) \in L^1(Q) \quad \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N \text{ and a.e } (x,t) \in Q$$

Assumption (A5). For almost everywhere $(x,t) \in Q$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|H(x,t,s,\xi)| \leq g(s)|\xi|^2, \tag{2.3}$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$is a non-decreasing continuous function, $g \in L^1(\mathbb{R})$.

2.2 Functional framework and definition

In order to solve our problem, we must first introduce our functional framework for the periodic solutions of problem (1.1), we set

$$V := L^2\left(0,T;H^1(\Omega)\right)$$

and

$$V^* := L^2\left(0,T;\left(H^1(\Omega)\right)^*\right)$$

where $\left(H^1(\Omega)\right)^*$ is the topological dual space of $H^1(\Omega)$.We denote by $< .,. >$ the duality pairing between V and V^*, we define the standard norm of $L^2\left(0,T;H^1(\Omega)\right)$ as follows

$$\|\varpi\|_{L^2(0,T;H^1(\Omega))} := \left(\int_Q |\nabla \varpi(t,x)|^2 dtdx + \int_Q |\varpi(t,x)|^2 dtdx\right)^{\frac{1}{2}}$$

In this paper, we use the following norm

$$\|\varpi\|_V := \left(\int_Q |\nabla \varpi(t,x)|^2 dtdx + \int_\Sigma \beta(t,\sigma)|\varpi(t,\sigma)|^2 dtd\sigma\right)^{\frac{1}{2}}$$

which is equivalent to the standard norm of $L^2\left(0,T;H^1(\Omega)\right)$, we denote by ϖ the trace of ϖ on Σ. Consider the set

$$\mathcal{W} = \left\{\varpi \in V \mid \frac{\partial \varpi}{\partial t} \in V^* \text{ and } \varpi(0) = \varpi(T)\right\}$$

endowed with the norm

$$\|\varpi\|_{\mathcal{W}} = \|\varpi\|_V + \left\|\frac{\partial \varpi}{\partial t}\right\|_{V^*}$$

We shall use the following definitions of a weak periodic solution for problem (1.1) in the following sense:

Definition 2.1. An weak periodic solution of (1.1) is a function ϖ such that

$$\begin{gathered}
\varpi \in V, H(x,t,\varpi,\nabla\varpi) \in L^1(Q), H(x,t,\varpi,\nabla\varpi)\varpi \in L^1(Q) \\
-<\varpi, \frac{\partial \phi}{\partial t}> + \int_Q \nabla\varpi\nabla\phi dxdt + \int_Q H(x,t,\varpi,\nabla\varpi)\phi dxdt + \int_\Sigma \beta(x,t)\varpi\phi dxdt + \int_\Sigma h(x,t,\varpi)\phi dxdt \\
= \int_Q f\phi dxdt. \\
\forall \phi \in \mathcal{W} \cap L^\infty(Q).
\end{gathered} \tag{2.4}$$

Throughout the rest of this paper, $C_i, i = 1, 2, ...$ denotes a positive constant.

3 An existence result when H is bounded

In this section, our aim is to show prove an existence result when H is bounded i.e.

$$|H(x,t,s,r)| \leqslant F(x,t). \tag{3.1}$$

a.e $(t,x) \in Q$, $\forall s \in \mathbb{R}$, and $\forall r \in \mathbb{R}^N$.
where $F \in L^2(Q)$ a nonnegative function

Theorem 3.1. *Assume that (A1)-(A4) and (3.1) hold true. Then the problem (1.1) has at least one solution ϖ in the sense of de Definition 2.1.*

Proof. To show the existence of a weak solution of (1.1), we show that the following nonlinear mapping has a fixed point

$$\mathcal{K} : V \longrightarrow V$$
$$w \longmapsto \mathcal{K}(w) = \varpi$$

where ϖ is the unique weak periodic solution of

$$\begin{cases} \frac{\partial \varpi}{\partial t} - \Delta\varpi + H(x,t,w,\nabla w) = f & \text{in } Q \\ \varpi(0) = \varpi(T) & \text{in } \Omega \\ -\frac{\partial \varpi}{\partial \nu} = \beta(x,t)\varpi + h(x,t,\varpi) & \text{on } \Sigma \end{cases} \tag{3.2}$$

By using Badii's theorem (see [3]), we get that there exists $u \in \mathcal{W} \subset V$ a unique weak periodic solution of (3.2) such that

$$\begin{aligned}
-<\varpi, \frac{\partial \varphi}{\partial t}> + \int_Q \nabla\varpi\nabla\varphi dxdt + \int_\Sigma h(x,t,\varpi)\varphi dxdt + \int_\Sigma \beta(x,t)\varpi\varphi dxdt \\
+ \int_Q H(x,t,w,\nabla w)\varphi dxdt = \int_Q f\varphi dxdt
\end{aligned} \tag{3.3}$$
$$\forall \varphi \in \mathcal{W}.$$

This mean that the mapping $\mathcal{K}$ is well defined.
In order to apply Schauder's fixed point theorem to prove the existence of a fixed point of $\mathcal{K}$, let's check the conditions of Schauder's theorem.

- **Continuity of** $\mathcal{K}$. Let $w_n \in V$ be a sequence such that

  $$w_n \to w \text{ strongly in V}$$

  and let ϖ_n the weak periodic solution of the problem

$$\begin{aligned}
-<\varpi_n, \frac{\partial \varphi}{\partial t}> + \int_Q \nabla\varpi_n\nabla\varphi dxdt + \int_\Sigma h(x,t,\varpi_n)\varphi dxdt + \int_\Sigma \beta(x,t)\varpi_n\varphi dxdt \\
+ \int_Q H(x,t,w_n,\nabla w_n)\varphi dxdt = \int_Q f\varphi dxdt.
\end{aligned} \tag{3.4}$$

Choosing $\varphi = \varpi_n$ as a test function in (3.4), we obtain

$$\begin{aligned} - < \varpi_n, \frac{\partial \varpi_n}{\partial t} > + \int_Q |\nabla \varpi_n|^2 \, dxdt + \int_\Sigma h(x,t,\varpi_n)\,\varpi_n dxdt + \int_\Sigma \beta(x,t)\varpi_n^2 dxdt \\ + \int_Q H(x,t,w_n,\nabla w_n)\,\varpi_n dxdt = \int_Q f \varpi_n dxdt \end{aligned} \tag{3.5}$$

By using (A2) , (A3), (3.1), the periodicity and the Young's inequality ,we get

$$\begin{aligned} \int_Q |\nabla \varpi_n|^2 \, dxdt + \int_\Sigma \beta(x,t)\,|\varpi_n|^2 \, dxdt &\leqslant \int_Q |f\varpi_n| \, dxdt + \int_Q |F\varpi_n| \, dxdt \\ &\leqslant \frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)} + \epsilon\,\|\varpi_n\|^2_{L^2(Q)} + \frac{1}{2\varepsilon}\|F\|^2_{L^2(Q)}. \\ &\leqslant \frac{1}{2\varepsilon}\left[\|f\|^2_{L^2(Q)} + \|F\|^2_{L^2(Q)}\right] + \epsilon C_1\,\|\varpi_n\|^2_V \end{aligned}$$

By using the equivalence of norms in V, we have

$$\|\varpi_n\|_{L^2(Q)} \leq C_1 \|\varpi_n\|_V,$$

then

$$\begin{aligned} \int_Q |\nabla \varpi_n|^2 \, dxdt + \int_\Sigma \beta(x,t)\,|\varpi_n|^2 \, dxdt &\leqslant \int_Q |f\varpi_n| \, dxdt + \int_Q |F\varpi_n| \, dxdt \\ &\leqslant \frac{1}{2\varepsilon}\left[\|f\|^2_{L^2(Q)} + \|F\|^2_{L^2(Q)}\right] + \epsilon C_1\,\|\varpi_n\|^2_V \end{aligned}.$$

By choosing $\varepsilon < \dfrac{1}{C_1}$, we get

$$\|\varpi_n\|_V \leq C_2 \tag{3.6}$$

By using (3.4) and (3.6) we conclude that $\left(\frac{\partial \varpi_n}{\partial t}\right)$ is bounded in the V^*.
Which implies that ϖ_n is bounded in $\mathcal{W}$, i.e.

$$\|\varpi_n\|_{\mathcal{W}} \leq C_3,$$

As a result, there exists a subsequence still denoted by ϖ_n such that

$$\varpi_n \rightharpoonup \varpi \text{ weakly in } V$$

By using Aubin's Theorem [19], we have

$$\varpi_n \longrightarrow \varpi \text{ in } L^2(Q) \text{ and a.e. in } Q$$

Moreover, the trace theorem, see Morrey [15], implies that

$$\varpi_n \longrightarrow \varpi \text{ in } L^2(\Sigma) \text{ and a.e. in } \Sigma$$

Now we prove that the sequence $\nabla \varpi_n$ converges strongly to $\nabla \varpi$ in $L^2(Q)$.
From (3.5), one has

$$\begin{aligned} \int_Q |\nabla \varpi_n|^2 \, dxdt = \int_Q f\varpi_n dxdt - \int_\Sigma h(x,t,\varpi_n)\,\varpi_n dxdt - \int_\Sigma \beta(x,t)\varpi_n^2 dxdt \\ - \int_Q H(x,t,w_n,\nabla w_n)\,\varpi_n dxdt, \end{aligned} \tag{3.7}$$

by passing to the limit in (3.7), we get

$$\begin{aligned} \lim_{n\mapsto+\infty} \int_Q |\nabla \varpi_n|^2 \, dxdt = \int_Q f\varpi dxdt - \int_\Sigma h(x,t,\varpi)\varpi dxdt - \int_\Sigma \beta(x,t)\varpi^2 dxdt \\ - \int_Q H(x,t,w,\nabla w)\varpi dxdt \end{aligned} \tag{3.8}$$

On the other hand, Taking $\varphi = \varpi$ as a test function in (3.4), we obtain

$$-<\varpi_n, \frac{\partial \varpi}{\partial t}> + \int_Q \nabla\varpi_n \nabla\varpi dxdt = \int_Q f\varpi dxdt - \int_\Sigma h\left(x,t,\varpi_n\right)\varpi dxdt - \int_\Sigma \beta(x,t)\varpi_n\varpi dxdt - \int_Q H\left(x,t,w_n,\nabla w_n\right)\varpi dxdt \tag{3.9}$$

When we pass to the limit $n \to +\infty$,, we get

$$\int_Q |\nabla\varpi|^2 dxdt = \int_Q f\varpi dxdt - \int_\Sigma h(x,t,\varpi)\varpi dxdt - \int_\Sigma \beta(x,t)\varpi^2 dxdt - \int_Q H(x,t,w,\nabla w)\varpi dxdt \tag{3.10}$$

By comparing (3.8) and (3.10), we obtain

$$\lim_{n\to+\infty} \int_Q \left|\nabla\varpi_n\right|^2 dxdt = \int_Q |\nabla\varpi|^2 dxdt$$

which implies that the mapping $\mathcal{K}$ is continuous.

- **Compactness of** $\mathcal{K}$. Let (w_n) be a bounded sequence in V and we denote $\varpi_n = \mathcal{K}(w_n)$, as above, we have (up to a subsequence)

$$\begin{aligned} w_n &\to w \text{ weakly in } V, \\ \varpi_n &\to \varpi \text{ weakly in } V, \\ \frac{\partial\varpi_n}{\partial t} &\to \frac{\partial\varpi}{\partial t} \text{ weakly in } V^*, \\ \varpi_n &\to \varpi \text{ strongly in } L^2(Q) \text{ and a.e in } Q, \\ \varpi_n &\to \varpi \text{ strongly in } L^2(\Sigma) \text{ and a.e in } \Sigma. \end{aligned}$$

To show compactness of $\mathcal{K}$, it suffices to prove the strong convergence of $(\nabla\varpi_n)$ in $L^2(Q)$. Let us first note that

$$\int_Q \left|\nabla\varpi_n - \nabla\varpi\right|^2 dxdt = \int_Q \nabla\varpi_n\left(\nabla\varpi_n - \nabla\varpi\right)dxdt - \int_Q \nabla\varpi\left(\nabla\varpi_n - \nabla\varpi\right)dxdt$$

Using the weak convergence of (ϖ_n) in V, we get

$$\lim_{n\mapsto+\infty} \int_Q \nabla\varpi\left(\nabla\varpi_n - \nabla\varpi\right) = 0$$

On the other hand, Taking $\varphi = \varpi$ as a test function in (3.4), we obtain

$$\int_Q \nabla\varpi_n\nabla\varpi dxdt = \int_Q f\varpi dxdt - <\frac{\partial\varpi_n}{\partial t}, \varpi> - \int_\Sigma h\left(x,t,\varpi_n\right)\varpi dxdt - \int_\Sigma \beta(x,t)\varpi_n\varpi dxdt - \int_Q H\left(x,t,w_n,\nabla w_n\right)\varpi dxdt \tag{3.11}$$

Tanks to (3.5) and (3.11), we have

$$\begin{aligned} \int_Q \nabla\varpi_n\left(\nabla\varpi_n - \nabla\varpi\right)dxdt &= \int_Q f\left(\varpi_n - \varpi\right)dxdt - <\frac{\partial\varpi_n}{\partial t}, \varpi_n - \varpi> \\ &\quad - \int_\Sigma h\left(x,t,\varpi_n\right)\left(\varpi_n - \varpi\right)dxdt - \int_\Sigma \beta(x,t)\varpi_n\left(\varpi_n - \varpi\right)dxdt \\ &\quad - \int_Q H\left(x,t,w_n,\nabla w_n\right)\left(\varpi_n - \varpi\right)dxdt \end{aligned} \tag{3.12}$$

By using (3.1), we have

$$\int_Q |H(x,t,w_n,\nabla w_n)(\varpi_n-\varpi)|\,dxdt \leqslant \|F\|_{L^2(Q)}\|\varpi_n-\varpi\|_{L^2(Q)}.$$

Hence

$$\lim_{n\mapsto+\infty}\int_Q |H(x,t,w_n,\nabla w_n)(\varpi_n-\varpi)| = 0.$$

On the other hand, since $\left(\frac{\partial \varpi_n}{\partial t}\right)$ converge to $\left(\frac{\partial \varpi}{\partial t}\right)$ weakly in V^*, we have

$$\lim_{n\mapsto+\infty} < \frac{\partial \varpi_n}{\partial t}, \varpi_n-\varpi >= \lim_{n\mapsto+\infty} < \frac{\partial \varpi_n}{\partial t}, \varpi_n > - < \frac{\partial \varpi}{\partial t}, \varpi >$$

By using the periodicity of ϖ_n and the periodicity of ϖ, we obtain

$$\lim_{n\mapsto+\infty} < \frac{\partial \varpi_n}{\partial t}, \varpi_n-\varpi >= 0$$

Now, we can pass to the limit in (3.12) to get

$$\lim_{n\mapsto+\infty}\int_Q \nabla\varpi_n(\nabla\varpi_n-\nabla\varpi)\,dxdt = 0$$

Then $(\nabla\varpi_n)$ converges strongly in $L^2(Q)$. Which implies that the mapping $\mathcal{K}$ is compact.

- $\mathcal{K}$ **send the ball of** V **of** R **radius to itself.** Our objective in this step is to find a constant $R>0$ such that $\mathcal{K}(B(0,R))\subset B(0,R)$ where $B(0,R)$ is the ball of V with radius R.
 Let $w\in V$ and $\varpi=\mathcal{K}(w)$. By taking ϖ as test function in (3.3),we get

$$\|\varpi\|_V \leqslant \left(\|F\|_{L^2(Q)}+\|f\|_{L^2(Q)}\right) := R.$$

Finally, by applying Schauder's fixed, there exists $\varpi\in V$ such that $\varpi=\mathcal{K}(\varpi)$. □

4 Main result

Before presenting our main result, we recall that for all $k>0$, the truncation function $T_k:\mathbb{R}\mapsto\mathbb{R}$ is defined as follows

$$T_k(s)=\begin{cases} k & \text{if} \quad s>k\\ s & \text{if} \quad |s|\le k\\ -k & \text{if} \quad s<-k.\end{cases}$$

Now, let's present our main result.

Theorem 4.1. *Assume that (A1)-(A5) hold true. Then, there exists at least one weak periodic solution u of problem (1.1)*

Proof. The proof of theorem 4.1 is divided into four steps.
Step 1. A priori estimates. For $n>0$, let H_n be a sequence such that :

$$H_n(x,t,s,\xi)=\frac{H(x,t,s,\xi)}{1+\frac{1}{n}|H(x,t,s,\xi)|}$$

it is easy to verify that

$$|H_n(x,t,s,\xi)|\le H(x,t,s,\xi) \text{ and } |H_n(x,t,s,\xi)|\le n.$$

Consider the approximate problem:

$$\begin{cases} \varpi_n \in \mathcal{W} \\ \int\limits_Q \frac{\partial \varpi_n}{\partial t}\phi dxdt + \int\limits_Q \nabla\varpi_n \nabla\phi dxdt + \int\limits_Q H_n\left(x,t,\varpi_n,\nabla\varpi_n\right)\phi dxdt \\ \quad + \int\limits_\Sigma \beta(x,t)\varpi_n\phi dxdt + \int\limits_\Sigma h\left(x,t,\varpi_n\right)\phi dxdt = \int\limits_Q f\phi dxdt \\ \forall\phi\in\mathcal{W}. \end{cases} \tag{4.1}$$

Since H_n is bounded, by applying Theorem 3.1, there exists ϖ_n a weak periodic solution of the approximate problem (4.1).
Now, consider the function G such that

$$G(s) = \int\limits_0^s g(r)dr$$

where the function g appears in (2.3).
Then, by choosing $\phi = \exp\left(G\left(\varpi_n\right)\right)\varpi_n^+$ as a test function in (4.1), one has

$$\begin{aligned} &\int\limits_Q \frac{\partial\varpi_n^+}{\partial t}\exp\left(G\left(\varpi_n^+\right)\right)\varpi_n^+ dxdt + \int\limits_Q \nabla\varpi_n^+\nabla\left(\exp\left(G\left(\varpi_n^+\right)\right)\varpi_n^+\right)dxdt \\ &+ \int\limits_\Sigma \beta(x,t)(\varpi_n^+)^2\exp\left(G\left(\varpi_n^+\right)\right)dxdt + \int\limits_\Sigma h\left(x,t,\varpi_n^+\right)(\varpi_n)^+\exp\left(G\left(\varpi_n^+\right)\right)dxdt \\ &= -\int\limits_Q H_n\left(x,t,\varpi_n^+,\nabla\varpi_n^+\right)\exp\left(G\left(\varpi_n^+\right)\right)\varpi_n^+ dxdt + \int\limits_Q f\exp\left(G\left(\varpi_n^+\right)\right)\varpi_n^+ dxdt \end{aligned} \tag{4.2}$$

The periodicity of ϖ_n^+ leads to

$$\int\limits_Q \frac{\partial\varpi_n^+}{\partial t}\exp\left(G\left(\varpi_n^+\right)\right)\varpi_n^+ dxdt = \int\limits_\Omega \theta_n(\varpi_n^+(x,T))dx - \int\limits_\Omega \theta_n(\varpi_n^+(x,0))dx = 0$$

where

$$\theta_n(r) = \int\limits_0^r s\exp(G(s))ds$$

Thanks to (2.1) and (2.3) we have

$$\begin{aligned} &\int\limits_Q |\nabla\varpi_n^+|^2 g(\varpi_n^+)\exp\left(G\left(\varpi_n^+\right)\right)\varpi_n^+ dxdt + \int\limits_Q |\nabla\varpi_n^+|^2\exp\left(G\left(\varpi_n^+\right)\right)dxdt + \int\limits_\Sigma \beta(x,t)(\varpi_n^+)^2\exp\left(G\left(\varpi_n^+\right)\right) \\ &\leq \int\limits_Q g\left(\varpi_n^+\right)|\nabla\varpi_n^+|^2\exp\left(G\left(\varpi_n^+\right)\right)|\varpi_n^+|dxdt \\ &\quad + \int\limits_Q f\exp\left(G\left(\varpi_n^+\right)\right)\varpi_n^+ dxdt \end{aligned}$$

The fact that $\exp(G(\varpi_n^+)) \geq 1$ implies that

$$\int\limits_Q |\nabla\varpi_n^+|^2 dxdt + \int\limits_\Sigma \beta(x,t)(\varpi_n^+)^2 dxdt \leq \int\limits_Q f\exp\left(G\left(\varpi_n^+\right)\right)\varpi_n^+ dxdt$$

Using the inequality $G(\varpi_n) \leq \|g\|_{L^1(\mathbb{R})}$, the Young inequality, give us

$$\begin{aligned} &\int\limits_Q |\nabla\varpi_n^+|^2 dxdt + \int\limits_\Sigma \beta(x,t)|\varpi_n^+|^2 dxdt \\ &\leq \exp(\|g\|_{L^1(\mathbb{R})})\left(\frac{\varepsilon}{2}\int\limits_Q |\varpi_n^+|^2 dxdt + \frac{1}{4\varepsilon}\int\limits_Q |f|^2 dxdt\right). \end{aligned}$$

Since

$$\|\varpi_n^+\|_{L^2(Q)} \leq C_1\|\varpi_n^+\|_V,$$

then

$$\|\varpi_n^+\|_V^2 \leq \tfrac{1}{4\varepsilon}\exp(\|g\|_{L^1(\mathbb{R})}) \int_Q |f|^2 dxdt + \varepsilon C_4\|\varpi_n^+\|_V^2$$

We choose $\varepsilon < \dfrac{1}{C_4}$ to obtain

$$\|\varpi_n^+\|_V \leq C_5 \tag{4.3}$$

Now, we choose $\phi = \exp\left(-G\left(\varpi_n\right)\right)\varpi_n^-$ as a test function in (4.1) . We get

$$\begin{aligned}
&\int_Q \frac{\partial \varpi_n^-}{\partial t}\exp\left(-G\left(\varpi_n^-\right)\right)\varpi_n^- dxdt - \int_Q \nabla\varpi_n^-\nabla\left(\exp\left(-G\left(\varpi_n^-\right)\right)u_n^-\right)dxdt \\
&+\int_\Sigma \beta(x,t)(\varpi_n^-)^2\exp\left(-G\left(\varpi_n^-\right)\right)dxdt + \int_\Sigma h\left(x,t,\varpi_n^-\right)\varpi_n^-\exp\left(-G\left(\varpi_n^-\right)\right)dxdt \\
&+\int_Q H_n\left(x,t,\varpi_n^-,\nabla\varpi_n^-\right)\exp\left(-G\left(\varpi_n^-\right)\right)\varpi_n^- dxdt = \int_Q f\exp\left(-G\left(\varpi_n^-\right)\right)\varpi_n^- dxdt
\end{aligned} \tag{4.4}$$

By using th periodicity of ϖ_n^- and (2.3), the equality (4.4) can be written as follows

$$\begin{aligned}
&\int_Q |\nabla\varpi_n^-|^2\exp\left(-G\left(\varpi_n^-\right)\right)dxdt \\
&+\int_\Sigma \beta(x,t)(\varpi_n^-)^2\exp\left(-G\left(\varpi_n^-\right)\right)dxdt + \int_\Sigma h\left(x,t,\varpi_n^-\right)\varpi_n^-\exp\left(-G\left(\varpi_n^-\right)\right)dxdt \\
&\geq \int_Q f\exp\left(-G\left(\varpi_n^-\right)\right)\varpi_n^- dxdt
\end{aligned}$$

By using the same technique which we have followed to show (4.3), we prove that

$$\|\varpi_n^-\|_V \leq C_6 \tag{4.5}$$

Combining (4.3) and (4.5) we conclude that

$$\|\varpi_n\|_V \leq C_7 \tag{4.6}$$

By using (2.3), we have

$$\begin{aligned}
\int_Q |H_n(x,t,\varpi_n,\nabla\varpi_n)|dxdt &\leq \|g\|_\infty \int_Q |\nabla\varpi_n|^2 dxdt \\
&\leq C_8
\end{aligned} \tag{4.7}$$

Which proves that $H_n(x,t,\varpi_n,\nabla\varpi_n)$ is boubded in $L^1(Q)$.
From (4.1) and (4.6) we get that $\frac{\partial\varpi_n}{\partial t}$ is bounded in the $V^*+L^1(Q)$. Therefore ϖ_n in is bounded $\mathcal{W}$.
By using the compactness result of of [19], we conclude that the sequence (ϖ_n) is relatively compact in $L^2(Q)$, which implies that there exists a subsequence still denoted by (ϖ_n) , such that

$$\varpi_n \to \varpi \text{ strongly in } L^2(Q) \text{ and a.e in } Q.$$

Furthermore, by using the trace theorem (see [15] Theorem 3.4.1)we have

$$\varpi_n \to \varpi \text{ strongly in } L^2(\Sigma) \text{ and a.e in } \Sigma.$$

Step 2. Almost everywhere convergence of the gradients
To show that $\nabla\varpi_n \to \nabla\varpi$ almost everywhere in Q, it suffices to show that $(\nabla\varpi_n)$ is a Cauchy sequence in measure. This consists in showing that

$$\forall\delta > 0, \forall\varepsilon > 0, \exists n_0 \text{ such that } \forall m,n \geq n_0 \quad meas\left\{(x,t)/|\nabla\varpi_n - \nabla\varpi_m| \geq \delta\right\} \leq \varepsilon.$$

Let $\forall\delta > 0$ and $\forall\varepsilon > 0$. For $k > 0$ and $\eta > 0$ we have

$$\{(x,t)/|\nabla\varpi_n - \nabla\varpi_m| \geq \delta\} \subset \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

where

$$\Gamma_1 = \{(x,t)/|\nabla\varpi_n| \geq k\}, \quad \Gamma_2 = \{(x,t)/|\nabla\varpi_n| \geq k\}, \quad \Gamma_3 = \{(x,t)/|\varpi_n - \varpi_n| \geq \eta\}$$
$$\Gamma_4 = \{(x,t)/|\nabla\varpi_n - \nabla\varpi_n| \geq \delta, |\nabla\varpi_n| \leq k, |\nabla\varpi_m| \leq k, |\varpi_n - \varpi_n| \leq \eta\}$$

Since $(\nabla\varpi_n)$ is bounded in $L^2(Q)$, then

$$k\mathbf{meas}(\Gamma_1) \leq \int\limits_{\Gamma_1} |\nabla\varpi_n| dxdt \leq \int\limits_{Q} |\nabla\varpi_n| dxdt \leq (\mathbf{meas}(Q))^{\frac{1}{2}} \|\nabla\varpi_n\|_{L^2(Q)} \leq C_9$$

hence for k large enough, we have

$$\mathbf{meas}((\Gamma_1) \leq \frac{C_9}{k} \leq \varepsilon \tag{4.8}$$

and by the same way we have

$$\mathbf{meas}((\Gamma_2) \leq \varepsilon \tag{4.9}$$

Let us fix k such that $\mathbf{meas}((\Gamma_1) \leq \varepsilon$ and $\mathbf{meas}((\Gamma_2) \leq \varepsilon$.
For $\mathbf{meas}(\Gamma_3)$, we have

$$\eta\mathbf{meas}(\Gamma_3) \leq \int\limits_{\Gamma_3} |\varpi_n - \varpi_m| dxdt \leq \int\limits_{Q} |\varpi_n - \varpi_m| dxdt \leq (\mathbf{meas}(Q))^{\frac{1}{2}} \|\varpi_n - \varpi_m\|_{L^2(Q)}$$

Since (ϖ_n) converge strongly in $L^2(Q)$, the sequence (ϖ_n) is a Cauchy sequence in $L^2(Q)$ hence for a given η, there exists is n_0 such that for $n, m \geq n_0$, we have

$$\mathbf{meas}(\Gamma_3) \leq \varepsilon \tag{4.10}$$

Now it suffices to bound $\mathbf{meas}(\Gamma_4)$, and to choose η. we consider the map

$$\begin{array}{rccc} A: & Q \times \mathbb{R}^N & \mapsto & \mathbb{R}^N \\ & (x,t,\xi) & \mapsto & \xi \end{array}$$

Remark we have

$$[A(x,t,\xi_1) - A(x,t,\xi_2)](\xi_1 - \xi_2) = |\xi_1 - \xi_2|^2 > 0 \text{ for } \xi_1 \neq \xi_2$$

Since the set $F = \{(\xi_1, \xi_2) \text{ such that } |\xi_1| \leq k, |\xi_2| \leq k, \text{ and } |\xi_1 - \xi_2| \geq \delta)\}$ is compact and A is continuous with respect to ξ for almost all $(x,t) \in Q$, then there exists $\Lambda(x,t) > 0$ such that

$$\Lambda(x,t) = \min_{(\xi_1,\xi_2)\in F} [A(x,t,\xi_1) - A(x,t,\xi_2)](\xi_1 - \xi_2).$$

Since $\Lambda(x,t) > 0$ then to bound $\mathbf{meas}(\Gamma_4)$, it is sufficient to show that ther existe $\varepsilon' > 0$ such that

$$\int\limits_{\Gamma_4} \Lambda(x,t) dxdt \leq \varepsilon'$$

By definition of Λ , A and γ_4, we have

$$\begin{aligned} \int\limits_{\Gamma_4} dxdt &\leq \int\limits_{\Gamma_4} |\nabla\varpi_n - \nabla\varpi_m|^2 1_{\{|\varpi_n - \varpi_m| \leq \eta\}} dxdt \\ &\leq \int\limits_{Q} |T_\eta(\varpi_n - \varpi_m)|^2 dxdt \end{aligned}$$

From (4.1) , we have

$$\begin{aligned}&\int_Q \frac{\partial \varpi_n}{\partial t}\phi dxdt+\int_Q \nabla\varpi_n\nabla\phi dxdt+\int_Q H_n\left(x,t,\varpi_n,\nabla\varpi_n\right)\phi dxdt\\&\quad+\int_\Sigma \beta(x,t)\varpi_n\phi dxdt+\int_\Sigma h\left(x,t,\varpi_n\right)\phi dxdt=\int_Q f\phi dxdt\end{aligned}.$$

and

$$\begin{aligned}&\int_Q \frac{\partial \varpi_m}{\partial t}\phi dxdt+\int_Q \nabla\varpi_m\nabla\phi dxdt+\int_Q H_m\left(x,t,\varpi_m,\nabla\varpi_m\right)\phi dxdt\\&\quad+\int_\Sigma \beta(x,t)\varpi_m\phi dxdt+\int_\Sigma h\left(x,t,\varpi_m\right)\phi dxdt=\int_Q f\phi dxdt\end{aligned}.$$

Hence

$$\begin{aligned}&\int_Q \frac{\partial(\varpi_n-\varpi_m)}{\partial t}\phi dxdt+\int_Q \nabla(\varpi_n-\varpi_m)\nabla\phi dxdt+\int_\Sigma \beta(x,t)(\varpi_n-\varpi_m)\phi dxdt\\&\quad+\int_Q \left[H_n\left(x,t,\varpi_n,\nabla\varpi_n\right)-H_m\left(x,t,\varpi_m,\nabla\varpi_m\right)\right]\phi dxdt\\&\quad+\int_\Sigma [h\left(x,t,\varpi_n\right)-h\left(x,t,\varpi_m\right)]\phi dxdt=0\end{aligned}. \tag{4.11}$$

By choosing $\phi=T_\eta(\varpi_n-\varpi_m)\in V\cap L^\infty(Q)$ in (4.11) and we use the periodicity of $(\varpi_n-\varpi_m)$, we get

$$\begin{aligned}&\int_Q |\nabla T_\eta(\varpi_n-\varpi_m)|^2dxdt+\int_\Sigma \beta(x,t)(\varpi_n-\varpi_m)T_\eta(\varpi_n-\varpi_m)dxdt\\&\quad\leq\eta\int_Q \left|H_n\left(x,t,\varpi_n,\nabla\varpi_n\right)-H_m\left(x,t,\varpi_m,\nabla\varpi_m\right)\right|dxdt\\&\quad+\eta\int_\Sigma \left|h\left(x,t,\varpi_n\right)-h\left(x,t,\varpi_m\right)\right|dxdt.\end{aligned}.$$

By combining (2.2), (4.7) and the fact $sT_\eta(s)\geq 0$, we obtain

$$\int_Q |\nabla T_\eta(\varpi_n-\varpi_m)|^2dxdt \quad\leq 2\eta C_8+\eta\int_\Sigma \left(2\xi(x,t)+|\varpi_n|+|\varpi_m|\right)dxdt.$$

According to Hölder's inequality, we have

$$\begin{aligned}\int_Q |\nabla T_\eta(\varpi_n-\varpi_m)|^2dxdt &\leq 2\eta C_8+\eta(\mathbf{meas}(\Sigma))^{\frac12}\left[\|\xi\|_{L^2(\Sigma)}+\|\varpi_n\|_{L^2(\Sigma)}+\|\varpi_m\|_{L^2(\Sigma)}\right]\\&\leq \eta C_{10}\end{aligned}.$$

Hence for η small enough, one has

$$\int_{\Gamma_4} \varLambda(x,t)dxdt\leq\varepsilon'$$

which implies

$$\mathbf{meas}(\Gamma_4)\leq\varepsilon \tag{4.12}$$

We combine (4.8), (4.9), (4.10) and (4.12), we conclude that $\forall n,m\geq n_0$, we have

$$\mathbf{meas}(\{(x,t)/|\nabla\varpi_n-\nabla\varpi_m|\geq\delta\}\leq 4\varepsilon$$

which implies that the sequence $(\nabla\varpi_n)$ converge in measure to $\nabla\varpi$, and therfore The sequence $(\nabla\varpi_n)$ converges almost everywhere to $(\nabla\varpi)$ (up to a subsequence).

Step 3. Strong convergence of ϖ_n.

To prove that (ϖ_n) converges strongly in V, it suffices to show that

$$\nabla\varpi_n \to \nabla\varpi \text{ strongly in } (L^2(Q))^N.$$

First we recall the following lemma.

Lemma 4.2. *Let $\theta(s) = s\exp(\delta s^2)$, we have*

$$\theta(0) = 0 \qquad , \qquad \theta(s) > 0 \text{ for all } s \in \mathbb{R}.$$

Moreover if $\delta \geq \frac{y^2}{4x^2}$, then

$$x\theta'(s) - y|\theta(s)| \geq \frac{x}{2} \tag{4.13}$$

Now,coming back to the equation (4.11) and choosing $\phi = \theta(\varpi_n - \varpi_m)$ as a test function, we obtain

$$\begin{aligned}
&\int_Q \frac{\partial(\varpi_n - \varpi_m)}{\partial t}(\varpi_n - \varpi_m)\exp(\delta(\varpi_n - \varpi_m)^2)dxdt + \int_Q |\nabla(\varpi_n - \varpi_m)|^2\theta'(\varpi_n - \varpi_m)dxdt \\
&+ \int_\Sigma \beta(x,t)(\varpi_n - \varpi_m)(\varpi_n - \varpi_m)\exp(\delta(\varpi_n - \varpi_m)^2)dxdt \\
&\quad + \int_Q \left[H_n(x,t,\varpi_n,\nabla\varpi_n) - H_m(x,t,\varpi_m,\nabla\varpi_m)\right](\varpi_n - \varpi_m)\exp(\delta(\varpi_n - \varpi_m)^2)dxdt \\
&\quad + \int_\Sigma [h(x,t,\varpi_n) - h(x,t,\varpi_m)](\varpi_n - \varpi_m)\exp(\delta(\varpi_n - \varpi_m)^2)dxdt = 0
\end{aligned}.$$

Since $\beta(x,t) > 0$ and $s \mapsto h(x,t,s)$ is nondecreasing, then by using the periodicity of $(\varpi_n - \varpi_m)$, we have

$$\begin{aligned}
&\int_Q |\nabla(\varpi_n - \varpi_m)|^2\theta'(\varpi_n - \varpi_m)dxdt \\
&\leq \int_Q |H_n(x,t,\varpi_n,\nabla\varpi_n) - H_m(x,t,\varpi_m,\nabla\varpi_m)|\,|\varpi_n - \varpi_m|\exp(\delta(\varpi_n - \varpi_m)^2)dxdt
\end{aligned}.$$

We use the growth condition (2.3),we get

$$\begin{aligned}
\int_Q |\nabla(\varpi_n - \varpi_m)|^2\theta'(\varpi_n - \varpi_m)dxdt &\leq \int_Q \left(g(\varpi_n)|\nabla\varpi_n|^2 + g(\varpi_m)|\nabla\varpi_m|^2\right)|\theta(\varpi_n - \varpi_m)|dxdt \\
&\leq \|g\|_\infty \int_Q \left(|\nabla\varpi_n|^2 + |\nabla\varpi_m|^2\right)|\theta(\varpi_n - \varpi_m)|dxdt
\end{aligned}.$$

Wich gives us

$$\int_Q \left(\theta'(\varpi_n - \varpi_m) - \|g\|_\infty|\theta(\varpi_n - \varpi_m)|\right)|\nabla\varpi_n - \nabla\varpi_m|^2dxdt \leq 2\|g\|_\infty \int_Q |\nabla\varpi_n|.|\nabla\varpi_m||\theta(\varpi_n - \varpi_m)|$$

By choosing $\delta \geq \frac{\|g\|_\infty}{4}$, the inequality (4.13) implies that

$$\frac{1}{2}\int_Q |\nabla\varpi_n - \nabla\varpi_m|^2dxdt \leq 2\|g\|_\infty \int_Q |\nabla\varpi_n|.|\nabla\varpi_m||\theta(\varpi_n - \varpi_m)|dxdt.$$

Since $\nabla u_n \to \nabla u$ a.e in Ω and $\nabla u_n \rightharpoonup \nabla u$ weakly in V, by using Fatou's lemma, we can pass to the limit when m tends to ∞, we obtain

$$\frac{1}{2}\int_Q |\nabla\varpi_n - \nabla\varpi|^2 dxdt \quad \leq \quad 2\|g\|_\infty \int_Q |\nabla\varpi_n|.|\nabla\varpi||\theta(\varpi_n - \varpi)|dxdt \ .$$

So, by applying Lebesgue's convergence theorem, we get

$$\lim_{n\mapsto+\infty}\int_Q |\nabla\varpi_n - \nabla\varpi|^2 dxdt = 0$$

Hence

$$\nabla\varpi_n \to \nabla\varpi \text{ strongly in } (L^2(Q))^N. \tag{4.14}$$

Step 4. equi-integrability of $H_n(x,t,\varpi_n,\nabla\varpi_n)$.
We shall prove that $H_n(x,t,\varpi_n,\nabla\varpi_n) \mapsto H(x,t,\varpi,\nabla\varpi)$ strongly in $L^1(Q)$ by using Vitali's theorem.
Since $\varpi_n \mapsto \varpi$ a.e in Q and $\nabla\varpi_n \mapsto \nabla\varpi$ a.e in Q, then

$$H_n(x,t,\varpi_n,\nabla\varpi_n) \mapsto H(x,t,\varpi,\nabla\varpi) \text{ a.e in } Q,$$

hence it suffies to prove that $H_n(x,t,\varpi_n,\nabla\varpi_n)$ is equi-integrable in $L^1(Q)$. Let $E \subset Q$ be a measurable subset of Q. Let $\varepsilon > 0$
From (2.3), one has

$$\begin{aligned}\int_E |H_n(x,t,\varpi_n,\nabla\varpi_n)|dxdt \quad &\leq \quad \int_E g(\varpi_n)|\nabla\varpi_n|^2 dxdt \\ &\leq \quad \|g\|_\infty \int_E g|\nabla\varpi_n|^2 dxdt\end{aligned} \tag{4.15}$$

From (4.14),we have $(|\nabla\varpi_n|^2)$ is equi-integrable in $L^1(Q)$,then there exists $\eta > 0$ such as ,

$$|E| < \eta \implies \|g\|_\infty \int_E |\nabla\varpi_n|^2 dxdt \leq \varepsilon$$

which implies that

$$|E| < \eta \implies \int_E |H_n(x,t,\varpi_n,\nabla\varpi_n)|dxdt \leq \varepsilon.$$

Hence $H_n(x,t,\varpi_n,\nabla\varpi_n)$ is equi-integrable in $L^1(Q)$. Hence passing to the limit as $n \mapsto +\infty$ we obtain that u is a solution of the problem (2.4). The proof of Theorem 4.1 is now complete. □

5 Conclusion

In this paper, we have shown the existence of a weak periodic solution for nonlinear variational parabolic problems having non linear boundary conditions and without sign condition on the non-linearity taking into account the assumptions **A1-A5**. We used Schauder's fixed point theorem to prove an auxiliary result when the non-linearity is bounded. By applying to the auxiliary result and using the method of truncation, we proved our main result .
At the end, we point out that this result can be developed in future works by reducing the number of assumptions.

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