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# GENERALIZATIONS OF $(\in, \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$-ANTI-FUZZY B-IDEALS OF BCI-ALGEBRAS

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**Abstract** In this paper, we introduce the concept of $(\in, \in \vee(\kappa^{*}, q_{\dot{\kappa}}))$-anti-fuzzy b-ideal (AFBI) in BCI-algebra and explores its properties. It can be extended the notion of $(\in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}), \in \vee(\dot{\kappa}^{*}, \mathrm{q}_{\dot{\kappa}}))$-AFBI is then presented and related properties are explored. We discuss the relations between $(\in, \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$-AFBI and $(\in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}), \in \vee(\dot{\kappa}^{*}, \mathrm{q}_{\dot{\kappa}}))$-AFBI. Furthermore, the concept of $(\in, \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$-level subset applied to anti-fuzzy b-ideal.

## 1 Introduction

The father of fuzzy set theory is Lotfi A. Zadeh. He introduced fuzzy sets in 1965 as an extension of classical set theory. Zadeh's [1] work on fuzzy sets provided a mathematical framework for dealing with uncertainty and vagueness in information and has since had significant applications in various fields, including artificial intelligence, control systems, decision-making, and pattern recognition. Imai et al. [2, 3] developed the thoughts of BCK/BCI-algebras. From that point forward, countless investigations of the hypothesis of BCK/BCI-algebras. Many researchers (See e.g., [4, 5, 6, 7, 8, 9, 10, 11, 29, 30, 31, 32, 33]) contemplated different parts of BCK/BCI-algebras in light of ideal hypothesis.

The possibility of quasi-coincidence with a fuzzy set, as expressed in [12], was basic in the improvement of different sorts of $(\alpha, \beta)$-fuzzy subgroups, as demarcated by Bhakat et al. [13]. Jun [14, 15] introduced $(\alpha, \beta)$-fuzzy ideals are a type of fuzzy ideal that incorporates two parameters: $\alpha$ and $\beta$. These parameters determine the degree to which elements satisfy the defining properties of the fuzzy ideal. Zhang et al. [16] introduced the notions of $(\in, \in \vee q)$-fuzzy p-ideal is a fuzzy set that satisfies certain conditions with respect to the $\in$, $\vee$, and $q$ operations. Here, "$q$" represents some additional operation or condition specific to the definition you are using. The precise conditions for an $(\in, \in \vee q)$-fuzzy p-ideal would depend on the specific definition or framework you are working with. Similarly, an $(\in, \in \vee q)$-fuzzy q-ideal satisfies certain conditions with respect to the $\in$, $\vee$, and $q$ operations, but the conditions for this type of fuzzy ideal may be different from those of a p-ideal. The concepts of $(\in, \in \vee q)$-interval-valued fuzzy ideals of BCI-algebras introduced by Mama et al. [18]. Al-Masarwah et al. [19], and Muhiuddin et al. [20] proposed a new system of $m$-polar $(\alpha, \beta)$-fuzzy ideals in BCK/BCI-algebras by extending the concept of fuzzy point to m-polar fuzzy sets. Numerous scientists have additionally expanded the fuzzy set hypothesis and related ideas to other algebras and different designs (see, for e.g., [10, 21, 22, 23, 24, 25, 26, 27]).

In this paper, we introduce the concept of $(\in, \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$-AFBI in BCI-algebra and explores its properties. This concept extends the notion of $(\in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}), \in \vee(\dot{\kappa}^{*}, \mathrm{q}_{\dot{\kappa}}))$-AFBIs are then presented and related properties are explored. We discuss the relations between $(\in, \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$-AFBI and $(\in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}), \in \vee(\dot{\kappa}^{*}, \mathrm{q}_{\dot{\kappa}}))$-AFBI. It examines how these two concepts are related and identifies their similarities and differences. Furthermore, the concept of $(\in, \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$-level subset is applied to anti-fuzzy b-ideal.

## 2 Preliminaries

**Definition 2.1.** [3, 4] An algebra $\tilde{\mathrm{E}} = (\tilde{\mathrm{E}}; \circledast, 0)$ of type $(2, 0)$ is a BCI-algebra if
(i) $((\dot{\omega} \circledast \dot{\rho}) \circledast (\omega \circledast \dot{\theta})) \circledast (\dot{\rho} \circledast \dot{\theta}) = 0$,
(ii) $(\dot{\omega} \circledast (\dot{\omega} \circledast \dot{\theta})) \circledast \dot{\theta} = 0$,
(iii) $\dot{\omega} \circledast \dot{\omega} = 0$,
(iv) $\dot{\omega} \circledast \dot{\theta} = 0$ and $\dot{\theta} \circledast \dot{\omega} = 0 \Rightarrow \dot{\omega} = \dot{\theta}, \forall \dot{\omega}, \dot{\theta}, \dot{\rho} \in \tilde{\mathrm{E}}$.

Every BCI-algebra $\tilde{\mathrm{E}}$ meets the following:
(1) $\dot{\omega} \circledast 0 = \dot{\omega}$,
(2) $(\dot{\omega} \circledast \dot{\theta}) \circledast \dot{\rho} = (\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta}$.
Define an order $\leq$ over $\tilde{\mathrm{E}}$ as $\dot{\omega} \leq \dot{\theta} \Leftrightarrow \dot{\omega} \circledast \dot{\theta} = 0$.
Let $\tilde{\mathrm{E}}$ be a BCI-algebra. Then $\Im : \tilde{\mathrm{E}} \to [0, 1]$ is a fuzzy subset (briefly, FSU) of $\tilde{\mathrm{E}}$.

**Definition 2.2.** [16] Let $\zeta \in (0, 1]$, $a \in \tilde{\mathrm{E}}$. Then

$$a_\zeta(\dot{\omega}) := \begin{cases} 0 & if\ \dot{\omega} \notin (a], \\ \zeta & if\ \dot{\omega} \in (a], \end{cases}$$

is an *ordered fuzzy point* (brief, OFP) $a_\zeta$, for all $\dot{\omega} \in \tilde{\mathrm{E}}$. Consequently, $a_\zeta$ is a FSU of $\tilde{\mathrm{E}}$. For a FS $\Im$ of $\tilde{\mathrm{E}}$, we write $a_\zeta \subseteq \Im$ as $a_\zeta \in \Im$ in the sequel. So, $a_\zeta \in \Im \Leftrightarrow \Im(a) \geq \zeta$.

**Definition 2.3.** [33] A FSU $\Im$ of $\tilde{\mathrm{E}}$ is an $(\in, \in \vee(\dot{\kappa}^*, \mathrm{q}))$-AFI of $\tilde{\mathrm{E}}$ if $\dot{\rho}_\zeta \in \Im$ and $\dot{\omega}_\eta \in \Im \Rightarrow (\dot{\rho} \circledast \dot{\omega})_{\zeta \vee \eta} \in \vee(\dot{\kappa}^*, q_{\dot{\kappa}})\Im$, for all $\zeta, \eta \in (0, 1]$ and $\dot{\rho}, \dot{\omega} \in \tilde{\mathrm{E}}$.

**Lemma 2.4.** *[33] Let $\Im$ be a FSU of $\tilde{\mathrm{E}}$. Then, $\dot{\rho}_\zeta \in \Im \Rightarrow 0_\zeta \in \vee(\dot{\kappa}^*, q_{\dot{\kappa}})\Im \Leftrightarrow \forall \dot{\rho} \in \tilde{\mathrm{E}}, \Im(\dot{\rho}) \vee (\frac{\dot{\kappa}^*}{2} - \frac{\dot{\kappa}}{2}) \geq \Im(0)$.*

**Lemma 2.5.** *[33] Let $\Im$ be an $(\in, \in \vee(\dot{\kappa}^*, q_{\dot{\kappa}}))$-AFI of $\tilde{\mathrm{E}}$ such that $\dot{\rho} \leq \dot{\omega}$. Then, $\Im(\dot{\omega}) \vee (\frac{\dot{\kappa}^*}{2} - \frac{\dot{\kappa}}{2}) \geq \Im(\dot{\omega})$.*

## 3 $(\in, \in \vee(\dot{\kappa}^*, \mathrm{q}_{\dot{\kappa}}))$-Anti Fuzzy b-Ideals

**Definition 3.1.** Let $a_\zeta$ be an OFP of $\tilde{\mathrm{E}}$ and $\dot{\kappa}^* \in (0, 1]$. Then, $a_\zeta$ is termed as $(\dot{\kappa}^*, q)$-*quasi-coincident* with a FS $\Im$ of $\tilde{\mathrm{E}}$, and can be written as $a_\zeta(\dot{\kappa}^*, q)\Im$, if $\dot{\kappa}^* < \Im(a) + \zeta$.

Let $0 \leq \dot{\kappa} < \dot{\kappa}^* \leq 1$. Then an OFP$\dot{\rho}_\zeta$,
(1) if $\dot{\kappa}^* < \Im(\dot{\rho}) + \zeta + \dot{\kappa}$, then $\dot{\rho}_\zeta(\dot{\kappa}^*, q_{\dot{\kappa}})\Im$,
(2) if $\dot{\rho}_\zeta \in \Im$ or $\dot{\rho}_\zeta(\dot{\kappa}^*, q_{\dot{\kappa}})\Im$, $\dot{\rho}_\zeta \in \vee(\dot{\kappa}^*, q_{\dot{\kappa}})\Im$,
(3) if $\dot{\rho}_\zeta \alpha \Im$ does not hold for $\alpha \in \{(\dot{\kappa}^*, q_{\dot{\kappa}}), \in \vee(\dot{\kappa}^*, q_{\dot{\kappa}})\}$, then $\dot{\rho}_\zeta \overline{\alpha} \Im$.

**Definition 3.2.** A FSU $\Im$ of $\tilde{\mathrm{E}}$ is an $(\in, \in \vee q)$-AFBI of $\tilde{\mathrm{E}}$ if
(1) $\dot{\rho}_\zeta \in \Im \Rightarrow 0_\zeta \in \vee q \Im$
(2) $((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta})_\zeta \in \Im$ and $\dot{\theta}_\eta \in \Im \Rightarrow \dot{\omega}_{\zeta \vee \eta} \in \vee q \Im, \forall \dot{\omega}, \dot{\theta}, \dot{\rho} \in \tilde{\mathrm{E}}$ and $\zeta, \eta \in (0, 1]$.

**Example 3.3.** Take a BCI-algebra $\tilde{\mathrm{E}} = \{0, \dot{\omega}, \dot{\theta}, \dot{\rho}\}$ with $(\circledast)$ which is described in Table 1.

Table 1: Cayley table for $(\in, \in \vee(\dot{\kappa}^*, \mathrm{q}_{\dot{\kappa}}))$-AFBI.

| $\circledast$ | $0$ | $\dot{\omega}$ | $\dot{\theta}$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $\dot{\theta}$ |
| $\dot{\omega}$ | $\dot{\omega}$ | $0$ | $\dot{\theta}$ |
| $\dot{\theta}$ | $\dot{\theta}$ | $\dot{\theta}$ | $0$ |

Define a FSU $\Im$ on $\tilde{\mathrm{E}}$ as

$$\Im(\dot{\omega}) = \begin{cases} 0, & \text{if } \dot{\omega} = 0, \\ 0.5, & \text{if } \dot{\omega} \in \{\dot{\omega}, \dot{\theta}\}. \end{cases}$$

Then $\Im$ is an $(\in, \in \vee q)$-AFBI of $\tilde{\text{E}}$.

**Definition 3.4.** A FSU $\Im$ of $\tilde{\text{E}}$ is an $(\in, \in \vee(\dot{\kappa}^{*}, \text{q}_{\dot{\kappa}}))$-AFBI of $\tilde{\text{E}}$ if

(1) $\dot{\rho}_{\zeta} \in \Im \Rightarrow 0_{\zeta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$

(2) $((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta})_{\zeta} \in \Im$ and $\dot{\theta}_{\eta} \in \Im \Rightarrow \dot{\omega}_{\zeta\vee\eta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im,\ \forall\, \zeta, \eta \in (0,1]$, and $\dot{\omega}, \dot{\theta}, \dot{\rho} \in \tilde{\text{E}}$.

**Example 3.5.** Take a BCI-algebra $\tilde{\text{E}} = \{0, \dot{\omega}, \dot{\theta}, \dot{\rho}, \dot{\varpi}\}$ with $(\circledast)$ which is defined in Table 2.

Table 2: Cayley table for $(\in, \in \vee(\dot{\kappa}^{*}, \text{q}_{\dot{\kappa}}))$-AFBI.

| $\circledast$ | $0$ | $\dot{\omega}$ | $\dot{\theta}$ | $\dot{\rho}$ | $\dot{\varpi}$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $\dot{\rho}$ | $\dot{\rho}$ |
| $\dot{\omega}$ | $\dot{\omega}$ | $0$ | $\dot{\omega}$ | $\dot{\varpi}$ | $\dot{\rho}$ |
| $\dot{\theta}$ | $\dot{\theta}$ | $\dot{\theta}$ | $0$ | $\dot{\rho}$ | $\dot{\rho}$ |
| $\dot{\rho}$ | $\dot{\rho}$ | $\dot{\rho}$ | $\dot{\rho}$ | $0$ | $0$ |
| $\dot{\varpi}$ | $\dot{\varpi}$ | $\dot{\rho}$ | $\dot{\varpi}$ | $\dot{\omega}$ | $0$ |

Define a FUS $\Im$ on $\tilde{\text{E}}$ as

$$\Im(\dot{\omega}) = \begin{cases} 0.1, & \text{if } \dot{\omega} = 0, \\ 0.2, & \text{if } \dot{\omega} \in \{\dot{\omega}, \dot{\varpi}\}, \\ 0, & \text{if } \dot{\omega} = \dot{\theta}, \\ 0.5, & \text{if } \dot{\omega} = \dot{\rho}. \end{cases}$$

Take $\dot{\kappa} = 0.3$ and $\dot{\kappa}^{*} = 0.7$. Then $\Im$ is an $(\in, \in \vee(\dot{\kappa}^{*}, \text{q}_{\dot{\kappa}}))$-AFBI of $\tilde{\text{E}}$.

**Lemma 3.6.** *In* $\tilde{\text{E}}$*, every* $(\in, \in \vee q)$*-AFBI is* $(\in, \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$*-AFBI.*

*Proof.* Let $\Im$ be an $(\in, \in \vee q)$-AFBI of $\tilde{\text{E}}$. Let $\dot{\rho}_{\zeta} \in \Im$ for $\dot{\rho} \in \tilde{\text{E}}$ and $\zeta \in (0,1]$. Then, by speculation, $0_{\zeta} \in \vee q\Im \Rightarrow \zeta \geq \Im(0)$ or $1 \geq \Im(0) - \zeta$. Thus, $\zeta \geq \Im(0)$ or $\dot{\kappa}^{*} > \Im(0) - \dot{\kappa} - \zeta$. Therefore, $0_{\zeta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$. Next, take any $((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta})_{\zeta} \in \Im$ and $\dot{\theta}_{\eta} \in \Im$. So, $\dot{\omega}_{\zeta\vee\eta} \in \vee q\Im \Rightarrow \zeta \vee \eta \geq \Im(\dot{\omega})$ or $1 > \Im(\dot{\omega}) - \zeta \vee \eta$. Therefore, $\zeta \vee \eta \geq \Im(\dot{\omega})$ or $\dot{\kappa}^{*} > \Im(\dot{\omega}) - \dot{\kappa} - \zeta \vee \eta$. Thus, $\dot{\omega}_{\zeta\vee\eta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$. Hence, $\Im$ is an $(\in, \in \vee(\dot{\kappa}^{*}, \text{q}_{\dot{\kappa}}))$-AFBI of $\tilde{\text{E}}$. □

Generally, the reverse of Lemma 3.4 is invalid, as shown in the succeeding example.

**Example 3.7.** Consider $\tilde{\text{E}} = \{0, \dot{\omega}, \dot{\theta}, \dot{\rho}\}$ as a BCI-algebra with $(\circledast)$ described in Table 3.

Table 3: Cayley table for $(\in, \in \vee(\dot{\kappa}^{*}, \text{q}_{\dot{\kappa}}))$-AFBI.

| $\circledast$ | $0$ | $\dot{\omega}$ | $\dot{\theta}$ | $\dot{\rho}$ |
|---|---|---|---|---|
| $0$ | $0$ | $\dot{\rho}$ | $\dot{\theta}$ | $\dot{\omega}$ |
| $\dot{\omega}$ | $\dot{\omega}$ | $0$ | $\dot{\rho}$ | $\dot{\theta}$ |
| $\dot{\theta}$ | $\dot{\theta}$ | $\dot{\omega}$ | $0$ | $\dot{\rho}$ |
| $\dot{\rho}$ | $\dot{\rho}$ | $\dot{\theta}$ | $\dot{\omega}$ | $0$ |

Define a FSU $\Im$ on $\tilde{\text{E}}$ as

$$\Im(\dot{\omega}) = \begin{cases} 0.3, & \text{if } \dot{\omega} = 0, \\ 0.5, & \text{if } \dot{\omega} = \dot{\omega}, \\ 0.6, & \text{if } \dot{\omega} \in \{\dot{\theta}, \dot{\rho}\}. \end{cases}$$

Take $\dot{\kappa} = 0.01$ and $\dot{\kappa}^{*} = 0.81$. Then $\Im$ is an $(\in, \in \vee(\dot{\kappa}^{*}, \text{q}_{\dot{\kappa}}))$-AFBI of $\tilde{\text{E}}$ but not an $(\in, \in \vee q)$-AFBI because $(\dot{\rho} \circledast (0 \circledast \dot{\omega}))_{\zeta=0.5} \in \Im$, $0_{\eta=0.5} \in \Im$ but $(\dot{\rho} \circledast \dot{\omega})_{\zeta\vee\eta=0.5} \overline{\in \vee q}\Im$.

**Lemma 3.8.** *Let* $\Im$ *be a FSU of* $\tilde{\text{E}}$*. Then,* $((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta})_{\zeta} \in \Im$ *and* $\dot{\theta}_{\eta} \in \Im \Rightarrow \dot{\omega}_{\zeta\vee\eta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im \Leftrightarrow$ $\Im(\dot{\omega}) \leq \Im((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta}) \vee \Im(\dot{\theta}) \vee (\frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2})$.

*Proof.* On the contrary, let us assume that $\Im(\dot\omega) > \Im((\dot\omega \circledast \dot\rho) \circledast \dot\theta) \vee \Im(\dot\theta) \vee (\frac{\dot\kappa^{\circledast}}{2} - \frac{\dot\kappa}{2})$ for some $\dot\omega, \dot\theta, \dot\rho \in \tilde{\mathrm{E}}$. Let $\zeta \in (0, (\frac{\dot\kappa^{\circledast}}{2} - \frac{\dot\kappa}{2})]$ be such that $\Im(\dot\omega) > \zeta \geq \Im((\dot\omega \circledast \dot\rho) \circledast \dot\theta) \vee \Im(\dot\theta) \vee (\frac{\dot\kappa^{\circledast}}{2} - \frac{\dot\kappa}{2})$. Then, $((\dot\omega \circledast \dot\rho) \circledast \dot\theta))_\zeta \in \Im$ and $\dot\theta_\zeta \in \Im$, but $\dot\theta_\zeta \overline{\in \vee(\dot\kappa^{\circledast}, q_{\dot\kappa})} \Im$, which is not possible. Thus, $\Im(\dot\omega) \leq \Im((\dot\omega \circledast \dot\rho) \circledast \dot\theta) \vee \Im(\dot\theta) \vee (\frac{\dot\kappa^{\circledast}}{2} - \frac{\dot\kappa}{2})$.

Conversely, suppose $((\dot\omega \circledast \dot\rho) \circledast \dot\theta)_\zeta \in \Im$ and $\dot\theta_\eta \in \Im, \forall \zeta, \eta \in (0,1]$. Then, $\Im((\dot\omega \circledast \dot\rho) \circledast \dot\theta) \leq \zeta$ and $\Im(\dot\theta) \leq \eta$. Thus,

$$\Im(\dot\omega) \leq \Im((\dot\omega \circledast \dot\rho) \circledast \dot\theta) \vee \Im(\dot\theta) \vee \frac{\dot\kappa^{\circledast}}{2} - \frac{\dot\kappa}{2} \leq \zeta \vee \eta \vee \frac{\dot\kappa^{\circledast}}{2} - \frac{\dot\kappa}{2}.$$

Now, if $\zeta \vee \eta \geq (\frac{\dot\kappa^{\circledast}}{2} - \frac{\dot\kappa}{2})$, then $\Im(\dot\omega) \leq \zeta \vee \eta$. Hence, $\dot\omega_{\zeta\vee\eta} \in \Im$. Otherwise, if $\zeta \vee \eta < (\frac{\dot\kappa^{\circledast}}{2} - \frac{\dot\kappa}{2})$, then $\Im(\dot\omega) \leq (\frac{\dot\kappa^{\circledast}}{2} - \frac{\dot\kappa}{2})$. So,

$$\Im(\dot\omega) + \zeta \vee \eta < \frac{\dot\kappa^{\circledast}}{2} - \frac{\dot\kappa}{2} + \frac{\dot\kappa^{\circledast}}{2} - \frac{\dot\kappa}{2} = \dot\kappa^{\circledast} - \dot\kappa \Rightarrow \dot\omega_{\zeta\vee\eta}(\dot\kappa^{\circledast}, q_{\dot\kappa})\Im.$$

Therefore, $\dot\omega_{\zeta\vee\eta} \in \vee(\dot\kappa^{\circledast}, q_{\dot\kappa})\Im$. □

Combining Lemma 2.4 with Lemma 3.8 yields the next theorem.

**Theorem 3.9.** *A FSU $\Im$ of $\tilde{\mathrm{E}}$ is an $(\in, \in \vee(\dot\kappa^{\circledast}, q_{\dot\kappa})) - AFBI$ of $\tilde{\mathrm{E}} \Leftrightarrow$*
*(1) $\Im(0) \leq \Im(\dot\rho) \vee (\frac{\dot\kappa^{\circledast}}{2} - \frac{\dot\kappa}{2})$*
*(2) $\Im(\dot\omega) \leq \Im((\dot\omega \circledast \dot\rho) \circledast \dot\theta) \vee \Im(\dot\theta) \vee (\frac{\dot\kappa^{\circledast}}{2} - \frac{\dot\kappa}{2}), \forall \dot\omega, \dot\theta, \dot\rho \in \tilde{\mathrm{E}}$.*

**Theorem 3.10.** *Every $(\in, \in \vee(\dot\kappa^{\circledast}, q_{\dot\kappa}))$-AFBI of $\tilde{\mathrm{E}}$ is an $(\in, \in \vee(\dot\kappa^{\circledast}, q_{\dot\kappa}))$-AFI.*

*Proof.* Suppose $\Im$ is an $(\in, \in \vee(\dot\kappa^{\circledast}, q_{\dot\kappa}))$-AFBI of $\tilde{\mathrm{E}}$. Then, $\forall \dot\omega, \dot\theta, \dot\rho \in \tilde{\mathrm{E}}$,

$$\Im(\dot\omega) \leq \Im((\dot\omega \circledast \dot\rho) \circledast \dot\theta) \vee \Im(\dot\theta) \vee \tfrac{\dot\kappa^{\circledast}}{2} - \tfrac{\dot\kappa}{2}.$$

Substitute $\dot\rho$ by 0, to obtain

$$\Im(\dot\omega) \leq \Im((\dot\omega \circledast 0) \circledast \dot\theta)) \vee \Im(\dot\theta) \vee \tfrac{\dot\kappa^{\circledast}}{2} - \tfrac{\dot\kappa}{2}.$$

Thus, $\Im(\dot\omega) \leq \Im(\dot\omega \circledast \dot\theta) \vee \Im(\dot\theta) \vee (\frac{\dot\kappa^{\circledast}}{2} - \frac{\dot\kappa}{2})$. □

Usually, the converse of Theorem 3.10 is invalid, as shown in the subsequent example.

**Example 3.11.** Consider a BCI-algebra $\tilde{\mathrm{E}} = \{0, \dot\omega, \dot\theta, \dot\rho, \dot\varpi\}$ with $(\circledast)$ defined in Table 4.

Table 4: Cayley table for binary $\circledast$.

| $\circledast$ | $0$ | $\dot\omega$ | $\dot\theta$ | $\dot\rho$ | $\dot\varpi$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $\dot\varpi$ | $\dot\rho$ | $\dot\theta$ |
| $\dot\omega$ | $\dot\omega$ | $0$ | $\dot\varpi$ | $\dot\rho$ | $\dot\theta$ |
| $\dot\theta$ | $\dot\theta$ | $\dot\theta$ | $0$ | $\dot\varpi$ | $\dot\rho$ |
| $\dot\rho$ | $\dot\rho$ | $\dot\rho$ | $\dot\theta$ | $0$ | $\dot\varpi$ |
| $\dot\varpi$ | $\dot\varpi$ | $\dot\varpi$ | $\dot\rho$ | $\dot\theta$ | $0$ |

Define a $FST$ on $\tilde{\mathrm{E}}$ as

$$\Im(\dot\omega) = \begin{cases} 0, & \text{if } \dot\omega = 0 \\ 0.2, & \text{if } \dot\omega = \dot\omega \\ 0.3, & \text{if } \dot\omega \in \{\dot\theta, \dot\rho, \dot\varpi\} \end{cases}$$

Put $\dot\kappa = 0.5$, $\dot\kappa^{\circledast} = 0.5$. Then $\Im$ is an $(\in, \in \vee(\dot\kappa^{\circledast}, q_{\dot\kappa}))$-AFI of $\tilde{\mathrm{E}}$, but it is not an $(\in, \in \vee(\dot\kappa^{\circledast}, \mathrm{q}_{\dot\kappa}))$-AFBI because $0.2 = \Im(\dot\omega) \not\leq \Im((\dot\omega \circledast \dot\rho) \circledast \dot\theta) \vee \Im(\dot\theta) \vee 0 = \Im(0) = 0$.

## 4 $(\in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}), \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$-Anti-fuzzy b-Ideals

**Definition 4.1.** A FSU $\Im$ of $\tilde{\mathrm{E}}$ is called an $(\in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}), \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$-AFBI of $\tilde{\mathrm{E}}$ if

(1) $\dot{\rho}_{\zeta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im \Rightarrow 0_{\zeta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$

(2) $((\dot{\omega} \divideontimes \dot{\rho}) \divideontimes \dot{\theta})_{\zeta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$ and $\dot{\theta}_{\eta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im \Rightarrow \dot{\omega}_{\zeta\vee\eta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im, \zeta, \eta \in (0, 1]$ and $\forall \dot{\omega}, \dot{\theta}, \dot{\rho} \in \tilde{\mathrm{E}}$.

**Example 4.2.** Take a BCI-algebra $\tilde{\mathrm{E}} = \{0, \dot{\omega}, \dot{\theta}, \dot{\rho}\}$ with $(\divideontimes)$ which is described in Table 5.

Table 5: Cayley table for binary $\divideontimes$.

| $\divideontimes$ | $0$ | $\dot{\omega}$ | $\dot{\theta}$ | $\dot{\rho}$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $\dot{\rho}$ |
| $\dot{\omega}$ | $\dot{\omega}$ | $0$ | $0$ | $\dot{\rho}$ |
| $\dot{\theta}$ | $\dot{\theta}$ | $\dot{\theta}$ | $0$ | $\dot{\rho}$ |
| $\dot{\rho}$ | $\dot{\rho}$ | $\dot{\rho}$ | $\dot{\rho}$ | $0$ |

Define $\Im : \tilde{\mathrm{E}} \to [0, 1]$ by

$$\Im(\dot{\omega}) = \begin{cases} 0.2, & \text{if } \dot{\omega} = 0 \\ 0.8, & \text{if } \dot{\omega} \in \{\dot{\omega}, \dot{\theta}, \dot{\rho}\} \end{cases}$$

Take $\dot{\kappa}^{*} = 0.15$ and $\dot{\kappa} = 0.05$. Then $\Im$ is an $(\in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}), \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$-AFBI of $\tilde{\mathrm{E}}$.

**Lemma 4.3.** *In* $\tilde{\mathrm{E}}$, $(\in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}), \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$*-AFBI is* $(\in, \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$*-AFBI.*

*Proof.* Let $\Im$ be $(\in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}), \in \vee(\dot{\kappa}^{*}, \mathrm{q}))$-AFBI of $\tilde{\mathrm{E}}$. Then, for any $\dot{\rho}_{\zeta} \in \Im$ for $\dot{\rho} \in \tilde{\mathrm{E}}$ and $\zeta \in (0, 1]$. Thus, $\dot{\rho}_{\zeta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$. So, by speculation, $0_{\zeta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$. Let us assume that $((\dot{\omega} \divideontimes \dot{\rho}) \divideontimes \dot{\theta})_{\zeta} \in \Im$ and $\dot{\theta}_{\eta} \in \Im$. Then, $((\dot{\omega} \divideontimes \dot{\rho}) \divideontimes \dot{\theta})_{\zeta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$ and $\dot{\theta}_{\eta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im \Rightarrow \dot{\omega}_{\zeta\wedge\eta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$. Thus, $\Im$ is an $(\in, \in \vee(\dot{\kappa}^{*}, \mathrm{q}))$-AFBI of $\tilde{\mathrm{E}}$. □

Generally, the inverse of Lemma 4.3 is invalid, as examined by the following example.

**Example 4.4.** Consider a BCI-algebra $\tilde{\mathrm{E}} = \{0, \dot{\omega}, \dot{\theta}, \dot{\rho}, \dot{\varpi}\}$ with $(\divideontimes)$ described in Table 6.

Table 6: Cayley table for binary $(\in, \in \vee(\dot{\kappa}^{*}, \mathrm{q}))$-AFBI.

| $\divideontimes$ | $0$ | $\dot{\omega}$ | $\dot{\theta}$ | $\dot{\rho}$ | $\dot{\varpi}$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\dot{\omega}$ | $\dot{\omega}$ | $0$ | $\dot{\omega}$ | $0$ | $\dot{\omega}$ |
| $\dot{\theta}$ | $\dot{\theta}$ | $\dot{\theta}$ | $0$ | $\dot{\theta}$ | $0$ |
| $\dot{\rho}$ | $\dot{\rho}$ | $\dot{\omega}$ | $\dot{\rho}$ | $0$ | $\dot{\rho}$ |
| $\dot{\varpi}$ | $\dot{\varpi}$ | $\dot{\varpi}$ | $\dot{\theta}$ | $\dot{\varpi}$ | $0$ |

Define $\Im : \tilde{\mathrm{E}} \to [0, 1]$ by

$$\Im(\dot{\omega}) = \begin{cases} 0, & \text{if } \dot{\omega} = 0 \\ 0.5, & \text{if } \dot{\omega} \in \{\dot{\omega}, \dot{\rho}\} \\ 0.3, & \text{if } \dot{\omega} = \{\dot{\theta}, \dot{\varpi}\} \end{cases}$$

Take $\dot{\kappa} = 0$, $\dot{\kappa}^{*} = 0.6$. Then, $\Im$ is an $(\in, \in \vee(\dot{\kappa}^{*}, \mathrm{q}))$-AFBI of $\tilde{\mathrm{E}}$, but it is not an $(\in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}), \in \vee(\dot{\kappa}^{*}, \mathrm{q}))$-AFBI of $\tilde{\mathrm{E}}$ as $\dot{\theta}_{\zeta=0.90} = (\dot{\theta} \divideontimes (0 \divideontimes \dot{\omega}))_{\zeta=0.90} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$ and $0_{\eta=0.5} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$, but $\dot{\theta} = (\dot{\theta} \divideontimes \dot{\omega})_{\zeta\vee\eta=0.5} \overline{\in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})}\Im$.

## 5 Level Subsets of $(\in, \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$-Anti Fuzzy b-Ideals

**Definition 5.1.** Let $\Im$ be a FS of $\tilde{\mathrm{E}}$. Then the level subset is defined as

$$\Im_{\zeta} = \{\dot{\rho} \in \tilde{\mathrm{E}} \mid \Im(\dot{\rho}) \geq \zeta, where\ \zeta \in (0, 1]\}.$$

**Theorem 5.2.** *If $\Im$ be a FSU of $\tilde{\text{E}}$, then, the set $\Im_\zeta \neq \varnothing$ is a b-ideal of $\tilde{\text{E}}, \forall \zeta \in (0, (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2})] \Leftrightarrow \Im$ is an $(\in, \in \vee(\dot\kappa^*, q_{\dot\kappa}))$-AFBI of $\tilde{\text{E}}$.*

*Proof.* Assume that $\Im_\zeta$ is b-ideal of $\tilde{\text{E}}, \forall \zeta \in (0, (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2})]$. Take $\Im(0) > \Im(\dot\rho) \vee (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2}), \exists \dot\rho \in \tilde{\text{E}}$. Then $\exists \zeta \in (0, (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2})]$ be such that $\Im(0) > \zeta \geq \Im(\dot\rho) \vee (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2})$. That's what it follows $\dot\rho \in \Im_\zeta$ but $0 \notin \Im_\zeta$, a contradiction. Thus, $\Im(0) \leq \Im(\dot\theta) \vee (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2})$. Also, let $\Im(\dot\omega) > \Im((\dot\omega \divideontimes \dot\rho) \divideontimes \dot\theta) \vee \Im(\dot\theta) \vee (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2}), \exists \dot\omega, \dot\theta, \dot\rho \in \tilde{\text{E}}$. Then $\zeta \in (0, (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2})]$ be such that

$$\Im(\dot\omega) > \zeta \geq \Im((\dot\omega \divideontimes \dot\rho) \divideontimes \dot\theta) \vee \Im(\dot\theta) \vee \frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2}.$$

This follows that $((\dot\omega \divideontimes \dot\rho) \divideontimes \dot\theta) \in \Im_\zeta$ and $\dot\theta \in \Im_\zeta$ but $\dot\omega \notin \Im_\zeta$, which is again a contradiction. Thus, $\Im(\dot\omega) \leq \Im((\dot\omega \divideontimes \dot\rho) \divideontimes \dot\theta) \vee \Im(\dot\theta) \vee (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2})$. Therefore, $\Im$ is an $(\in, \in \vee(\dot\kappa^*, \text{q}_{\dot\kappa}))$-AFBI of $\tilde{\text{E}}$.

Conversely, let $\zeta \in (0, (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2})]$ with $\Im_\zeta \neq \varnothing$. By Theorem 3.9, we have

$$\Im(0) \leq \Im(\dot\rho) \vee \frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2},$$

with $\dot\rho \in \Im_\zeta \Rightarrow \Im(0) \leq \zeta \vee (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2}) = \zeta$. Therefore, $0 \in \Im_\zeta$.

Next, assume that $((\dot\omega \divideontimes \dot\rho) \divideontimes \dot\theta) \in \Im_\zeta$ and $\dot\theta \in \Im_\zeta$. Then, $\Im((\dot\omega \divideontimes \dot\rho) \divideontimes \dot\theta) \leq \zeta$ and $\Im(\dot\theta) \leq \zeta$. Again, by Theorem 3.9, we have

$$\Im(\dot\omega) \leq \Im((\dot\omega \divideontimes \dot\rho) \divideontimes \dot\theta) \vee \Im(\dot\theta) \vee \frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2} \leq \zeta \vee \frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2} = \zeta.$$

Thus, $\dot\omega \in \Im_\zeta$. Therefore, $\Im_\zeta$ is a b-ideal of $\tilde{\text{E}}$.

□

**Definition 5.3.** Let $\Im$ be a FSU of $\tilde{\text{E}}$. Then the $(\in \vee(\dot\kappa^*, q_{\dot\kappa}))$-*level subset* of $\Im$ is defined as

$$\overline{[\Im]}_\zeta = \{\dot\rho \in \tilde{\text{E}} \mid \dot\rho_\zeta \in \vee(\dot\kappa^*, q_{\dot\kappa})\Im, where\ \zeta \in (0, 1]\}.$$

**Theorem 5.4.** *Let $\Im$ be a FSU of $\tilde{\text{E}}$. Then, the $(\in \vee(\dot\kappa^*, q_{\dot\kappa}))$-level subset $\overline{[\Im]}_\zeta$ of $\Im$ is a b-ideal of $\tilde{\text{E}}, \forall \zeta \in (0, 1] \Leftrightarrow \Im$ is an $(\in, \in \vee(\dot\kappa^*, q_{\dot\kappa}))$-AFBI of $\tilde{\text{E}}$.*

*Proof.* Let $\overline{[\Im]}_\zeta$ be a b-ideal of $\tilde{\text{E}}, \forall \zeta \in (0, 1]$. On the contrary, let

$$\Im(\dot\rho) \vee \frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2} > \Im(0),$$

for some $\dot\rho \in \tilde{\text{E}}$. Then, $\exists \zeta \in (0, 1]$ be such that $\Im(\dot\rho) \vee (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2}) \geq \zeta > \Im(0)$. It follows that $\dot\rho \in \overline{[\Im]}_\zeta$, but $0 \notin \overline{[\Im]}_\zeta$, which is not possible. Therefore,

$$\Im(\dot\rho) \vee \tfrac{\dot\kappa^*}{2} - \tfrac{\dot\kappa}{2} \leq \Im(0).$$

Also, if $\Im((\dot\omega \divideontimes \dot\rho) \divideontimes \dot\theta) \vee \Im(\dot\theta) \vee (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2}) < \Im(\dot\omega)$ for some, $\dot\omega \in \tilde{\text{E}}$, then $\zeta \in (0, 1]$ be such that

$$\Im((\dot\omega \divideontimes \dot\rho) \divideontimes \dot\theta) \vee \Im(\dot\theta) \vee \frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2} \geq \zeta > \Im(\dot\omega).$$

Thus, $((\dot\omega \divideontimes \dot\rho) \divideontimes \dot\theta) \in \overline{[\Im]}_\zeta$ and $\dot\theta \in \overline{[\Im]}_\zeta$, but $\dot\omega \notin \overline{[\Im]}_\zeta$, which is again a contradiction. Hence, $\Im((\dot\omega \divideontimes \dot\rho) \divideontimes \dot\theta) \vee \Im(\dot\theta) \vee (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2}) \geq \Im(\dot\omega)$. Therefore, $\Im$ is an $(\in, \in \vee(\dot\kappa^*, \text{q}))$-AFBI of $\tilde{\text{E}}$.

Conversely, suppose $\Im$ is an $(\in, \in \vee(\dot\kappa^*, \text{q}))$-AFBI of $\tilde{\text{E}}$. Then for any $\dot\rho \in \overline{[\Im]}_\zeta$. So, $\dot\rho_\zeta \in \vee(\dot\kappa^*, q_{\dot\kappa})\Im$. Thus, $\zeta \geq \Im(\dot\rho)$ or $\dot\kappa^* - \dot\kappa > \Im(\dot\rho) + \zeta$. Now, by using Theorem 3.9, we get $\Im(\dot\rho) \vee (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2}) \geq \Im(0)$. Therefore, $\zeta \vee (\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2}) \geq \Im(0)$ when $\zeta \geq \Im(\dot\rho)$. Let $(\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2}) > \zeta$. Then $(\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2}) \geq \Im(0) \Rightarrow 0 \in \overline{[\Im]}_\zeta$. Also, let $(\frac{\dot\kappa^*}{2} - \frac{\dot\kappa}{2}) \leq \zeta$. Then $\zeta \geq \Im(0) \Rightarrow 0 \in \overline{[\Im]}_\zeta$. Similarly, $0 \in \overline{[\Im]}_\zeta$ when $\dot\kappa^* - \dot\kappa > \Im(\dot\rho) + \zeta$.

Next, let us take any $((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta}) \in \overline{[\Im]}_{\zeta}$ and $\dot{\theta} \in \overline{[\Im]}_{\zeta}$. Then, $((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta}) \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$ and $\dot{\theta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$, i.e., either $\zeta \geq \Im((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta})$ or $\dot{\kappa}^{*} - \dot{\kappa} > \Im((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta}) - \zeta$ and either $\zeta \geq \Im(\dot{\theta})$ or $\dot{\kappa}^{*} - \dot{\kappa} > \Im(\dot{\theta}) - \zeta$. By assumption, $\Im(\dot{\omega} \circledast (\dot{\theta} \circledast \dot{\omega})) \vee \Im(\dot{\theta}) \vee (\frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2}) \geq \Im(\dot{\omega})$. Thus, the following cases arise.

*Case 1*. Let $\zeta \geq \Im((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta})$ and $\zeta \geq \Im(\dot{\theta})$. If $(\frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2}) > \zeta$, then

$$\Im(\dot{\omega}) \leq \Im((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta}) \vee \Im(\dot{\theta}) \vee \frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2} \leq \zeta \vee \frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2} = \frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2},$$

and so, $\dot{\omega}_{\zeta} \in (\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$. If $\zeta \geq (\frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2})$ , then

$$\Im(\dot{\omega}) \leq \Im((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta}) \vee \Im(\dot{\theta}) \vee \frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2} \leq \zeta \vee \frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2} = \zeta.$$

So, $\dot{\omega}_{\zeta} \in \Im$. Thus, $\dot{\omega}_{\zeta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$.

*Case 2*. Let $\zeta \geq \Im((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta})$ and $\dot{\kappa}^{*} - \dot{\kappa} \geq \Im(\dot{\theta}) - \zeta$. If $(\frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2}) > \zeta$, then

$$\Im(\dot{\omega}) \leq \Im((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta}) \vee \Im(\dot{\theta}), \frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2} \leq \zeta \vee \dot{\kappa}^{*} - \dot{\kappa} - \zeta \vee \frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2} = \dot{\kappa}^{*} - \dot{\kappa} - \zeta,$$

i.e., $\dot{\kappa}^{*} - \dot{\kappa} > \Im(\dot{\omega}) - \zeta$, and thus, $\dot{\omega}_{\zeta} \in (\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$. If $(\frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2}) \geq \zeta$, then

$$\Im(\dot{\omega}) \leq \Im((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta}) \vee \Im(\dot{\theta}) \vee \frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2} \leq \zeta \vee \dot{\kappa}^{*} - \dot{\kappa} - \zeta \vee \frac{\dot{\kappa}^{*}}{2} - \frac{\dot{\kappa}}{2} = \zeta.$$

Therefore, $\dot{\omega}_{\zeta} \in \Im$. Hence, $\dot{\omega}_{\zeta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$.

Similarly, for other cases, i.e., when $\dot{\kappa}^{*} - \dot{\kappa} > \Im((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta}) + \zeta,\ \zeta \geq \Im(\dot{\theta}),\ \dot{\kappa}^{*} - \dot{\kappa} > \Im((\dot{\omega} \circledast \dot{\rho}) \circledast \dot{\theta}) + \zeta$, and $\dot{\kappa}^{*} - \dot{\kappa} > \Im(\dot{\theta}) + \zeta$, we have $\dot{\omega}_{\zeta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$. Therefore, for each case, $\dot{\omega}_{\zeta} \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}})\Im$, and thus, $\dot{\omega} \in \overline{[\Im]}_{\zeta}$.

□

## 6 Conclusions

In this paper, we introduced the concept of $(\in, \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$-AFBI in BCI-algebra and explores its properties. This concept extends the notion of $(\in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}), \in \vee(\dot{\kappa}^{*}, \mathrm{q}_{\dot{\kappa}}))$-AFBIs are then presented and related properties are explored. We discussed the relations between $(\in, \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$-AFBI and $(\in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}), \in \vee(\dot{\kappa}^{*}, \mathrm{q}_{\dot{\kappa}}))$-AFBI. It examined how these two concepts are related and identifies their similarities and differences. Furthermore, the concept of $(\in, \in \vee(\dot{\kappa}^{*}, q_{\dot{\kappa}}))$-level subset were applied to anti-fuzzy b-ideal. In future review, these ideas might be reached out to $BCH$-algebras, $BG$-algebras, b-algebras, and EQ-algebras.

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