Palestine Journal of Mathematics

Vol 13(Special Issue II)(2024) , 114–127 © Palestine Polytechnic University-PPU 2024

Arithmetic Properties of Certain Partition Functions With Parity Restrictions

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Communicated by Manoj Kumar Patel

Dedicated to Prof. B. M. Pandeya on his 78th birthday.

MSC 2020 Classification: Primary 11P83; Secondary 05A17.

Keywords and phrases: integer partition with restricted odd and even parts, congruence, q-identities.

Abstract Andrews (2010) investigated the Rogers-Ramanujan-Gordon partitions of positive integers with some restrictions on even and odd parts, and introduced two partition functions, $W_{r,s}(n)$ and $\overline{W}_{r,s}(n)$, where r and s are positive integers. Sang, Shi and Yee (2020) defined two Rogers-Ramanujan-Gordon type overpartition functions, $U_{r,s}$ and $\overline{U}_{r,s}$, with similar parity restrictions on even and odd parts. In this paper, we give partition-interpretations of $U_{r,s}$ and $\overline{U}_{r,s}$ using the notion of colour partition of integers and prove some congruences for the partition functions $W_{r,s}(n)$, $\overline{W}_{r,s}(n)$, $U_{r,s}(n)$ and $\overline{U}_{r,s}(n)$ for some particular values of r and s.

1 Introduction

For any complex numbers B and q, define

$$(B)_n := (B;q)_n := \prod_{k=0}^{n-1}(1-Bq^k), \text{ for } n \geq 1$$

and

$$(B)_\infty := (B;q)_\infty := \prod_{k=0}^{\infty}(1-Bq^k).$$

For brevity, we write

$$\prod_{i=0}^{k}(B_i;q)_\infty = (B_1, B_2, \ldots, B_k; q)_\infty$$

and $g_t = (q^t;q^t)_\infty$ for any integer $t \geq 1$.

A partition of a positive integer n is a sequence of integers $\delta_1 \geq \delta_2 \geq \delta_3 \geq \cdots \geq \delta_k \geq 1$ such that $\sum_{j=1}^{k} \delta_j = n$. The integers δ_j are called parts or summands of the partition. If $p(n)$ denotes the number of partitions of n, then its generating function satiesfies the identity

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{g_1}.$$

A summand in a partition of n has t colours if there are t copies of each summand available and all of them are viewed as distinct objects. If a, b and t are positive integers, then the coefficient of q^n in the expansion of $(q^a;q^b)^{-t}$ enumerates the number of partitions n where summands are congruent to a modulo b with each summand having t colours.

If the number of partitions of n with distinct even summands is denoted by $ped(n)$, then

$$\sum_{n=0}^{\infty} ped(n)q^n = \frac{g_4}{g_1}. \tag{1.1}$$

Andrews et al. [3] proved that

$$ped\left(3^{2\alpha+1}n+\frac{17\cdot 3^{2\alpha}-1}{8}\right)\equiv 0 \pmod 6 \tag{1.2}$$

$$\text{and}\quad ped\left(3^{2\alpha+2}n+\frac{19\cdot 3^{2\alpha+1}-1}{8}\right)\equiv 0 \pmod 6 \tag{1.3}$$

for all $\alpha \geq 1$ and $n \geq 0$. Next we recall the following theorem of Gordon (see [8]).

Theorem 1.1. *For $r \geq s \geq 1$, the number of partitions of n of the form $\xi_1+\xi_2+\cdots+\xi_k$ such that $\xi_j \geq \xi_{j+1}$, $\xi_j-\xi_{j+r-1} \geq 2$ and part 1 appears at most $s-1$ times is denoted by $B_{r,s}(n)$. Let $A_{r,s}(n)$ represent the number of partitions of n into parts $\not\equiv 0, \pm s \pmod{2r+1}$. Then for any $n \geq 0$, $A_{r,s}(n)=B_{r,s}(n)$.*

For $r, s \geq 1$, the Andrews-Gordon identity (see [1])

$$\sum_{k_1\geq k_2\geq\cdots\geq k_{r-1}\geq 0}\frac{q^{k_1^2+k_2^2+\cdots+k_{r-1}^2+k_s+\cdots+k_{r-1}}}{(q)_{k_1-k_2}\cdots(q)_{k_{r-2}-k_{r-1}}(q)_{k_{r-1}}}=\frac{(q^s,q^{2r+1-s},q^{2r+1};q^{2r+1})_\infty}{(q)_\infty} \tag{1.4}$$

generalizes Theorem 1.1. In [2] Andrews established analogous results for the function $W_{r,s}(n)$ (resp. $\overline{W}_{r,s}(n)$) which counts the partitions enumerated by $B_{r,s}(n)$ where even (resp. odd) parts occur an even number of times.

Theorem 1.2. *([13, p. 39, Entry 24] & [2]) For $r \geq s \geq 1$ with $r \equiv s \pmod 2$,*

$$\sum_{n\geq 0}W_{r,s}(n)q^n=\frac{(-q;q^2)_\infty \mathfrak{f}(-q^s,-q^{2r+2-s})}{(q^2;q^2)_\infty}, \tag{1.5}$$

where $\mathfrak{f}(x,y)$ [5, p. 34, 18.1] is given by

$$\mathfrak{f}(x,y)=\sum_{n=-\infty}^{\infty}x^{n(n+1)/2}y^{n(n-1)/2}. \tag{1.6}$$

If $r \geq s \geq 2$ with r odd and s even, then

$$\sum_{n\geq 0}\overline{W}_{r,s}(n)q^n=\frac{\mathfrak{f}(-q^s,-q^{2r+2-s})}{(-q;q^2)_\infty(q;q)_\infty}, \tag{1.7}$$

An overpartition of a positive integer n is a partition of n such that the first occurence of any part may be overlined. Let $\overline{\lambda}(n)$ denote the number of overpartitions of n, then its generating function satiesfies the identity

$$\sum_{n\geq 0}\overline{\lambda}(n)q^n=\frac{g_2}{g_1^2}.$$

Kursungöz [12] and Kim and Yee [13] studied the partition functions $W_{r,s}(n)$ and $\overline{W}_{r,s}(n)$ by considering different parities of r and s. Andrew [2] posted fifteen open problems, of which the eleventh was related to the overpartition of integers. Chen et al. [6] investigated the eleventh problem of Andrews and derived the overpartition analogies of Theorems 1.1 and (1.4).

For an overpartition λ and for any integer ℓ, the numbers of occurences of non-overlined and overlined parts of size ℓ in λ are denoted by $M_\ell(\lambda)$ and $M_{\overline{\ell}}(\lambda)$, respectively.

Sang et al. [14] proved the following results on restricted overpartition functions:

Theorem 1.3. *[14] Suppose $r \geq s \geq 1$, ℓ is any integer, and denote by $U_{r,s}(n)$ the number of overpartitions λ of n satisfying*
(i) $M_1(\lambda) \leq s-1+M_{\overline{1}}(\lambda)$;
(ii) $M_{2\ell-1}(\lambda) \geq M_{\overline{2\ell-1}}(\lambda)$;
(iii) $M_{2\ell}(\lambda)+M_{\overline{2\ell}}(\lambda) \equiv 0 \pmod 2$;
(iv) $M_\ell(\lambda)+M_{\overline{\ell}}(\lambda)+M_{\ell+1}(\lambda) \leq r-1+M_{\overline{\ell+1}}(\lambda)$.
If $r \equiv s \pmod 2$, then

$$\sum_{n\geq 0}U_{r,s}(n)q^n=\frac{(-q;q)_\infty \mathfrak{f}(-q^s,-q^{2r-s})}{(q^2;q^2)_\infty}. \tag{1.8}$$

Theorem 1.4. *[14] Suppose $r \geq s \geq 1$, ℓ is any integer, and denote by $\overline{U}_{r,s}(n)$ the number of overpartitions λ of n satisfying*
(i) $M_1(\lambda) \leq s-1+M_{\overline{1}}(\lambda)$;
(ii) $M_{2\ell}(\lambda) \geq M_{\overline{2\ell}}(\lambda)$;
(iii) $M_{2\ell-1}(\lambda)+M_{\overline{2\ell-1}}(\lambda) \equiv 0 \pmod{2}$;
(iv) $M_\ell(\lambda)+M_{\overline{\ell}}(\lambda)+M_{\ell+1}(\lambda) \leq r-1+M_{\overline{\ell+1}}(\lambda)$.
If $r \geq s \geq 2$ and $s = $ even, then

$$\sum_{n\geq 0} \overline{U}_{r,s}(n)q^n = \frac{(-q^2;q^2)_\infty^2 \mathfrak{f}(-q^s,-q^{2r-s})}{(q^2;q^2)_\infty}. \tag{1.9}$$

In this paper, we investigate some arithmetic properties of the partition functions $W_{r,s}(n)$, $\overline{W}_{r,s}(n)$, $U_{r,s}(n)$ and $\overline{U}_{r,s}(n)$. We establish congruences modulo 3, 4, 6 and 12 for $W_{5,3}(n)$ and $\overline{W}_{3,2}(n)$. For example, we prove for all $\alpha \geq 1$ and $n \geq 0$,

$$W_{5,3}\left(9^\alpha n + \frac{7\cdot 3^{2\alpha-1}-1}{4}\right) \equiv 0 \pmod{3},$$
$$\overline{W}_{3,2}\left(2\cdot 3^{2\alpha+1}n + \frac{17\cdot 3^{2\alpha}-1}{4}\right) \equiv 0 \pmod{6},$$
$$\overline{W}_{3,2}\left(2\cdot 3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{4}\right) \equiv 0 \pmod{6}.$$

In Sect. 3, we give colour partition interpretations of $U_{r,s}(n)$ and $\overline{U}_{r,s}(n)$ which are analogues of Theorem 1.1. In Sect. 4, we prove some particular and infinite families of congruences for the partition functions $W_{5,3}(n)$, $\overline{W}_{3,2}(n)$ and $\overline{W}_{5,4}(n)$, and in Sect. 5, we prove congruences for the partition functions $U_{5,5}(n)$, $\overline{U}_{3,2}(n)$, $\overline{U}_{4,2}(n)$, and $\overline{U}_{6,2}(n)$. In order to prove our results, we will employ some q-identities collected in Sect. 2.

2 Preliminaries

Four important special cases of (1.6) considered by Ramanujan satisfy the identities [5, p. 36, Entry 22 (i), (ii), (iii)]

$$\phi(q) := \mathfrak{f}(q,q) = \sum_{t=-\infty}^{\infty} q^{t^2} = \frac{g_2^5}{g_1{}^2 g_4{}^2}, \tag{2.1}$$

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{t=0}^{\infty} q^{t(t+1)/2} = \frac{g_2^2}{g_1}, \tag{2.2}$$

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{t=-\infty}^{\infty} (-1)^t q^{t(3t+1)/2} = g_1 \tag{2.3}$$

and [5, p. 37, Entry 22 (iv)]

$$\chi(q) := (-q;q^2) = \frac{g_2^2}{g_1 g_4}. \tag{2.4}$$

One can use elementary q-operations to show that

$$\phi(-q) = \frac{g_1^2}{g_2}, \quad \chi(-q) = \frac{g_1}{g_2}, \quad \psi(-q) = \frac{g_1 g_4}{g_2}. \tag{2.5}$$

We now collect some identities involving the theta-function $\mathfrak{f}(x,y)$ defined in (1.6).

Lemma 2.1. *[5, p. 35, Entry 19] We have*

$$\mathfrak{f}(x,y) = (-x;xy)_\infty(-y;xy)_\infty(xy;xy)_\infty. \tag{2.6}$$

Lemma 2.2. *[7, Theorem 2.2] Suppose $p \geq 5$ is any prime. Then we have*

$$g_1 = \sum_{\substack{k=-(p-1)/2 \\ k \neq (\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} \mathfrak{f}\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right) + (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} g_{p^2},$$

where

$$\frac{\pm p - 1}{6} := \begin{cases} \dfrac{(p-1)}{6}, & if\ p \equiv 1 \pmod 6 \\ \dfrac{(-p-1)}{6}, & if\ p \equiv -1 \pmod 6. \end{cases}$$

Furthermore, if

$$\frac{-(p-1)}{2} \leq k \leq \frac{(p-1)}{2} \text{ and } k \neq \frac{(\pm p - 1)}{6},$$

then

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p.$$

Lemma 2.3. *[7, Theorem 2.1] Suppose $p \geq 3$ is any prime. Then we have*

$$\psi(q) = \sum_{i=0}^{(p-3)/2} q^{(i^2+i)/2} \mathfrak{f}\left(q^{(p^2+(2i+1)p)/2}, q^{(p^2-(2i+1)p)/2}\right) + q^{(p^2-1)/8} \psi(q^{p^2}).$$

Furthermore, $\dfrac{(i^2+i)}{2} \not\equiv \dfrac{(p^2-1)}{8} \pmod p$, *when* $0 \leq i \leq \dfrac{(p-3)}{2}$.

Lemma 2.4. *[5, p. 49, Cor. (ii)] We have*

$$\psi(q) = \mathfrak{f}(q^3, q^6) + q\psi(q^9). \tag{2.7}$$

Lemma 2.5. *[5, p. 51, Example (v)] We have*

$$\mathfrak{f}(q, q^5) = \psi(-q^3)\chi(q). \tag{2.8}$$

Lemma 2.6. *[11, Eqn. (3.2.7)] We have*

$$\frac{1}{g_1} \equiv \frac{1}{g_5}\Big(A_0B_0 + (A_0B_1 + A_1B_0) + (A_1B_1 + A_2B_0) + A_2B_1\Big) \pmod 5,$$

where

$$A_0 = \sum_{m=-\infty}^{\infty} (-1)^m q^{5(15m^2+m)/2} + \sum_{k=-\infty}^{\infty} (-1)^k q^{5(15k^2+11k+2)/2},$$

$$A_1 = -q \sum_{m=-\infty}^{\infty} (-1)^m q^{25(3m^2+m)/2},$$

$$A_2 = -q^2 \left[\sum_{m=-\infty}^{\infty} (-1)^{m+1} q^{5(15m^2+13m+2)/2} + \sum_{k=-\infty}^{\infty} (-1)^{k+1} q^{5(15k^2+23k+8)/2}\right],$$

$$B_0 = \sum_{m=-\infty}^{\infty} (-1)^m q^{(25m^2-5m)/2},$$

$$B_1 = -3q \sum_{m=-\infty}^{\infty} (-1)^m q^{(25m^2-15m)/2}.$$

Next lemma is a easy consequence of (1) and the binomial theorem.

Lemma 2.7. *Suppose $k \geq 1$, $m \geq 1$ are any integer, and p is any prime. Then we have*

$$g_{pm}^{p^{k-1}} \equiv g_m^{p^k} \pmod{p^k}. \tag{2.9}$$

Lemma 2.8. *We have*

$$\frac{1}{g_1^2} = \frac{g_8^5}{g_2^5 g_{16}^2} + 2q\frac{g_4^2 g_{16}^2}{g_2^5 g_8}, \tag{2.10}$$

$$\frac{g_1^2}{g_2} = \frac{g_9^2}{g_{18}} - 2q\frac{g_3 g_{18}^2}{g_6 g_9}, \tag{2.11}$$

$$\frac{g_2}{g_1^2} = \frac{g_6^4 g_9^6}{g_3^8 g_{18}^3} + 2q\frac{g_6^3 g_9^3}{g_3^7} + 4q^2\frac{g_6^2 g_{18}^3}{g_3^6}, \tag{2.12}$$

$$\frac{g_2^2}{g_1} = \frac{g_6 g_9^2}{g_3 g_{18}} + q\frac{g_{18}^2}{g_9}, \tag{2.13}$$

$$\frac{g_3}{g_1} = \frac{g_4 g_6 g_{16} g_{24}^2}{g_2^2 g_8 g_{12} g_{48}} + q\frac{g_6 g_8^2 g_{48}}{g_2^2 g_{16} g_{24}}, \tag{2.14}$$

$$\frac{g_4}{g_1} = \frac{g_{12} g_{18}^4}{g_3^3 g_{36}^2} + q\frac{g_6^2 g_9^3 g_{36}}{g_3^4 g_{18}^2} + 2q^2\frac{g_6 g_{18} g_{36}}{g_3^3}. \tag{2.15}$$

Proof. Using (2.1) and (2.2) in (1.9.4) of [11], we obtain

$$\frac{g_2^5}{g_1^2 g_4^2} = \frac{g_8^5}{g_4^2 g_{16}^2} + 2q\frac{g_{16}^2}{g_8}. \tag{2.16}$$

Now (2.10) follows from (2.16). (2.11) and (2.13) follow from (14.3.2) and (14.3.3) of [11], respectively. (2.12) is from [9], (2.14) is from [15] and (2.15) is the Lemma 2.6 of [4]. □

3 Colour Partition Interpretations of $U_{r,s}(n)$ and $\overline{U}_{r,s}(n)$

In this section, we give colour partition interpretations of the partition functions $U_{r,s}(n)$ and $\overline{U}_{r,s}(n)$.

Theorem 3.1. *(a) Suppose r and s satisfy $1 \leq s < r$ and $r \equiv s \pmod 2$. Then $U_{r,s}(n)$ is equal to the number of partitions of n containing no summand congruent to 0, s or $2r - s$ modulo $2r$.*

(b) Suppose r and s are positive integers such that s is even and $2 \leq s < r$. Then, $\overline{U}_{r,s}(n)$ is equal to the number of partitions of n into summands congruent to 2 modulo 4 in two colours and even summands in a third color congruent to neither 0, s nor $2r - s$ modulo $2r$.

Proof. Using (1.8) and (2.6), we obtain

$$\sum_{n\geq 0} U_{r,s}(n)q^n = \frac{(-q;q)_\infty (q^s;q^{2r})_\infty (q^{2r-s};q^{2r})_\infty (q^{2r};q^{2r})_\infty}{(q^2;q^2)_\infty} = \frac{(q^s;q^{2r})_\infty (q^{2r-s};q^{2r})_\infty (q^{2r};q^{2r})_\infty}{(q,q)_\infty}, \tag{3.1}$$

from which our result (a) follows. Similarly, we can prove (b). □

4 Congruences for $W_{r,s}(n)$ and $\overline{W}_{r,s}(n)$

In this section, we prove congruences for the partition functions $W_{5,3}(n)$, $\overline{W}_{3,2}(n)$ and $\overline{W}_{5,4}(n)$.

Theorem 4.1. *Let $p \equiv 3 \pmod 4$ be prime, $1 \leq j \leq p-1$ and α, $\beta \geq 0$. Then*

$$\sum_{n\geq 0} W_{5,3}\left(2\cdot 9^{\alpha+1}p^{2\beta}n + \frac{5\cdot 9^{\alpha+1}p^{2\beta} - 1}{4}\right)q^n \equiv \psi(q)\psi(q^4) \pmod 3, \tag{4.1}$$

and for all $n \geq 0$ *we have*

$$W_{5,3}\left(2\cdot 9^{\alpha+1}p^{2\beta+2}n + 2\cdot 9^{\alpha+1}p^{2\beta+1}j + \frac{5\cdot 9^{\alpha+1}p^{2\beta+2}-1}{4}\right) \equiv 0 \pmod 3. \tag{4.2}$$

Proof. Setting $r=5$ and $s=3$ in (1.5), we obtain

$$\sum_{n\geq 0} W_{5,3}(n)q^n = \frac{(-q;q^2)_\infty \mathfrak{f}(-q^3,-q^9)}{g_2}. \tag{4.3}$$

Simplifying (4.3) using (2.2), (2.4) and (2.5), we obtain

$$\sum_{n\geq 0} W_{5,3}(n)q^n = \frac{g_2g_3g_{12}}{g_1g_4g_6}. \tag{4.4}$$

Utilizing (2.14) in (4.4), we obtain

$$\sum_{n\geq 0} W_{5,3}(n)q^n = \frac{g_{16}g_{24}^2}{g_2g_8g_{48}} + q\frac{g_8^2g_{12}g_{48}}{g_2g_4g_{16}g_{24}}. \tag{4.5}$$

Collecting the terms involving odd powers of q from both sides of (4.5) and simplifying, we obtain

$$\sum_{n\geq 0} W_{5,3}(2n+1)q^n = \frac{g_4^2g_6g_{24}}{g_1g_2g_8g_{12}}. \tag{4.6}$$

Utilizing (2.9) in (4.6) and then applying (2.2), we obtain

$$\sum_{n\geq 0} W_{5,3}(2n+1)q^n \equiv \frac{g_2^2g_8^2}{g_1g_4} \equiv \psi(q)\psi(q^4) \pmod 3. \tag{4.7}$$

Substituting (2.13) in (4.7) and simplifying, we get

$$\sum_{n\geq 0} W_{5,3}(2n+1)q^n \equiv \frac{g_6g_9^2g_{24}g_{36}^2}{g_3g_{12}g_{18}g_{72}} + q\frac{g_{18}^2g_{24}g_{36}^2}{g_9g_{12}g_{72}} + q^4\frac{g_6g_9^2g_{72}^2}{g_3g_{18}g_{36}} + q^5\frac{g_{18}^2g_{72}^2}{g_9g_{36}} \pmod 3. \tag{4.8}$$

Collecting the terms involving powers of q that are congruent to 2 modulo 3 from both sides of (4.8) and simplifying the resulting equality yields

$$\sum_{n\geq 0} W_{5,3}(6n+5)q^n \equiv q\frac{g_6^2g_{24}^2}{g_3g_{12}} \pmod 3. \tag{4.9}$$

Collecting the terms involving powers of q that are congruent to 1 modulo 3 from both sides of (4.9) and simplifying the resulting equality and then applying (2.2) yields

$$\sum_{n\geq 0} W_{5,3}(18n+11)q^n \equiv \frac{g_2^2g_8^2}{g_1g_4} \equiv \psi(q)\psi(q^4) \pmod 3. \tag{4.10}$$

From (4.7) and (4.10), we see that

$$W_{5,3}(18n+11) \equiv W_{5,3}(2n+1) \pmod 3, \tag{4.11}$$

and iterating (4.11) yields

$$W_{5,3}\left(2\cdot 9^{\alpha+1}n + \frac{5\cdot 9^{\alpha+1}-1}{4}\right) \equiv W_{5,3}(2n+1) \pmod 3, \quad \text{for all} \quad \alpha \geq 1. \tag{4.12}$$

By substituting (4.7) in (4.12), we obtain

$$\sum_{n\geq 0} W_{5,3}\left(2\cdot 9^{\alpha+1}n + \frac{5\cdot 9^{\alpha+1}-1}{4}\right)q^n \equiv \psi(q)\psi(q^4) \pmod 3, \tag{4.13}$$

which is the $\beta = 0$ case of (4.1). Now suppose that (4.1) holds for some $\beta \geq 0$. Lemma 2.3 then yields

$$\sum_{n\geq 0} W_{5,3}\left(2\cdot 9^{\alpha+1}p^{2\beta}n+\frac{5\cdot 9^{\alpha+1}p^{2\beta}-1}{4}\right)q^n$$
$$\equiv \left[\sum_{m=0}^{(p-3)/2} q^{(m^2+m)/2}\mathfrak{f}\left(q^{(p^2+(2m+1)p)/2}, q^{(p^2-(2m+1)p)/2}\right)+q^{(p^2-1)/8}\psi(q^{p^2})\right]$$
$$\times\left[\sum_{k=0}^{(p-3)/2} q^{4(k^2+k)/2}\mathfrak{f}\left(q^{2(p^2+(2k+1)p)}, q^{2(p^2-(2k+1)p)}\right)+q^{(p^2-1)/2}\psi(q^{4p^2})\right] \pmod 3. \tag{4.14}$$

Next consider the congruence

$$\left(\frac{m^2+m}{2}\right)+4\left(\frac{k^2+k}{2}\right)\equiv 5\left(\frac{p^2-1}{8}\right) \pmod p, \quad \text{for} \quad 0\leq k,m\leq p-1, \tag{4.15}$$

which is equivalent to

$$(2m+1)^2+(4k+2)^2\equiv 0 \pmod p. \tag{4.16}$$

Since $\left(\frac{-1}{p}\right) = -1$, the only solution of (4.16) is $k = m = (p-1)/2$. Therefore, collecting the terms involving powers of q that are congruent to $5(p^2-1)/8$ modulo p from both sides of (4.14) and simplifying the resulting equality yields

$$\sum_{n\geq 0} W_{5,3}\left(2\cdot 9^{\alpha+1}p^{2\beta+1}n+\frac{5\cdot 9^{\alpha+1}p^{2\beta+2}-1}{4}\right)q^n\equiv\psi(q^p)\psi(q^{4p}) \pmod 3. \tag{4.17}$$

Collecting the terms involving powers of q that are congruent to 0 modulo p from both sides of (4.17) and simplifying the resulting equality yields

$$\sum_{n\geq 0} W_{5,3}\left(2\cdot 9^{\alpha+1}p^{2\beta+2}n+\frac{5\cdot 9^{\alpha+1}p^{2\beta+2}-1}{4}\right)q^n\equiv\psi(q)\psi(q^{4}) \pmod 3, \tag{4.18}$$

which is the $\beta+1$ case of (4.1). Finally, collecting the terms involving powers of q that are congruent to j modulo p from both sides of (4.17) yields (4.2). □

Corollary 4.2. *Nothing that the power of q in every term on the right hand side of (4.9) is congruent to 1 modulo 3 immediately yields the following:*

$$W_{5,3}(18n+5)\equiv W_{5,3}(18n+17)\equiv 0 \pmod 3.$$

Theorem 4.3. *For any integer $\alpha \geq 1$, we have*

$$W_{5,3}\left(9^{\alpha}n+\frac{9^{\alpha}-1}{4}\right)\equiv W_{5,3}(n) \pmod 3, \tag{4.19}$$

$$W_{5,3}\left(9^{\alpha}n+\frac{7\cdot 3^{2\alpha-1}-1}{4}\right)\equiv 0 \pmod 3, \tag{4.20}$$

$$W_{5,3}\left(9^{\alpha}n+\frac{11\cdot 3^{2\alpha-1}-1}{4}\right)\equiv 0 \pmod 3. \tag{4.21}$$

Proof. Simplifying (4.4) using (2.9) and employing (2.5), we obtain

$$\sum_{n\geq 0} W_{5,3}(n)q^n\equiv\left(\frac{g_1g_4}{g_2}\right)^2\equiv\psi^2(-q) \pmod 3. \tag{4.22}$$

Then (2.7) yields

$$\sum_{n\geq 0} W_{5,3}(n)q^n \equiv \mathfrak{f}^2(-q^3,q^6) - 2q\,\mathfrak{f}(-q^3,q^6)\psi(-q^9) + q^2\psi^2(-q^9) \pmod 3. \tag{4.23}$$

Collecting the terms involving powers of q that are congruent to 2 modulo 3 from both sides of (4.23) and simplifying the resulting equality yields

$$\sum_{n\geq 0} W_{5,3}(3n+2)q^n \equiv \psi^2(-q^3) \pmod 3. \tag{4.24}$$

Collecting the terms involving powers of q that are congruent to 0 modulo 3 from both sides of (4.24) and simplifying the resulting equality yields

$$\sum_{n\geq 0} W_{5,3}(9n+2)q^n \equiv \psi^2(-q) \pmod 3. \tag{4.25}$$

Combining (4.22) and (4.25), we find

$$W_{5,3}(9n+2) \equiv W_{5,3}(n) \pmod 3, \tag{4.26}$$

and iterating (4.26) yields (4.19). Next note that since the right hand side of (4.24) is a series in q^3, we have

$$W_{5,3}(9n+5) \equiv 0 \pmod 3 \quad \text{and} \quad W_{5,3}(9n+8) \equiv 0 \pmod 3, \tag{4.27}$$

which are the $\alpha = 1$ cases of (4.20) and (4.21). Finally, replacing n by $9n+5$ and $9n+8$ in (4.19) yields (4.20) and (4.21) for $\alpha \geq 2$. □

Theorem 4.4. *For any integers $\alpha \geq 1$ and $n \geq 0$, we have*

$$\overline{W}_{3,2}\left(2\cdot 9^{\alpha}n + \frac{9^{\alpha}-1}{4}\right) \equiv \overline{W}_{3,2}(2n) \pmod 4, \tag{4.28}$$

$$\overline{W}_{3,2}\left(2\cdot 3^{2\alpha+1}n + \frac{17\cdot 3^{2\alpha}-1}{4}\right) \equiv 0 \pmod 6, \tag{4.29}$$

$$\overline{W}_{3,2}\left(2\cdot 3^{2\alpha+2}n + \frac{19\cdot 3^{2\alpha+1}-1}{4}\right) \equiv 0 \pmod 6. \tag{4.30}$$

Proof. Setting $r = 3$ and $s = 2$ in (1.7) and simplifying using (2.2) and (2.4) and then applying (2.5), we obtain

$$\sum_{n\geq 0} \overline{W}_{3,2}(n)q^n = \frac{g_8}{g_2}. \tag{4.31}$$

Since the right hand side of (4.31) is a series in q^2, it follows that

$$\sum_{n\geq 0} \overline{W}_{3,2}(2n)q^n = \frac{g_4}{g_1}. \tag{4.32}$$

Next, using (2.15), we obtain

$$\sum_{n=0}^{\infty} \overline{W}_{3,2}(2n)q^n = \frac{g_{12}g_{18}^4}{g_3^3 g_{36}^2} + q\frac{g_6^2 g_9^3 g_{36}}{g_3^4 g_{18}^2} + 2q^2\frac{g_6 g_{18} g_{36}}{g_3^3}. \tag{4.33}$$

Collecting the terms involving powers of q that are congruent to 1 modulo 3 from both sides of (4.33) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty} \overline{W}_{3,2}(6n+2)q^n = \frac{g_2^2 g_3^3 g_{12}}{g_1^4 g_6^2} = \left(\frac{g_2}{g_1^2}\right)^2 \frac{g_3^3 g_{12}}{g_6^2}. \tag{4.34}$$

Employing (2.12) then yields

$$\sum_{n=0}^{\infty}\overline{W}_{3,2}(6n+2)q^n = \frac{g_6^6g_9^{12}g_{12}}{g_3^{13}g_{18}^6}+4q\frac{g_6^5g_9^9g_{12}}{g_3^{12}g_{18}^3}+12q^2\frac{g_6^4g_9^6g_{12}}{g_3^{11}}+16q^3\frac{g_6^3g_9^3g_{12}g_{18}^3}{g_3^{10}}+16q^4\frac{g_6^2g_{12}g_{18}^6}{g_3^9}. \tag{4.35}$$

Collecting the terms involving powers of q that are congruent to 0 modulo 3 from both sides of (4.35) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty}\overline{W}_{3,2}(18n+2)q^n \equiv \frac{g_2^6g_3^{12}g_4}{g_1^{13}g_6^6} \pmod{16}. \tag{4.36}$$

Using (2.9), we obtain

$$\sum_{n=0}^{\infty}\overline{W}_{3,2}(18n+2)q^n \equiv \frac{g_4}{g_1} \pmod{4}, \tag{4.37}$$

which by (4.32) yields

$$\overline{W}_{3,2}(18n+2) \equiv \overline{W}_{3,2}(2n) \pmod{4}. \tag{4.38}$$

Iterating (4.38), we acquire (4.28).

Combining (1.1) and (4.32), we obtain

$$\overline{W}_{3,2}(2n) = ped(n). \tag{4.39}$$

Employing (1.2) and (1.3) in (4.39), we arrive at (4.29) and (4.30), respectively. □

Corollary 4.5. *For any integer $n \geq 0$, we have*

$$\overline{W}_{3,2}(18n+8) \equiv 0 \pmod{4}, \tag{4.40}$$

$$\overline{W}_{3,2}(18n+14) \equiv 0 \pmod{12}. \tag{4.41}$$

Proof. Collecting the terms involving powers of q that are congruent to 1 modulo 3 and congruent to 2 modulo 3 from both sides of (4.35), we complete the proof of (4.40) and (4.41), respectively. □

Theorem 4.6. *Suppose p is an odd prime such that $\left(\frac{-3}{p}\right) = -1$, $1 \leq j \leq p-1$ and $\alpha \geq 0$. Then*

$$\sum_{n\geq 0}\overline{W}_{5,4}\left(4p^{2\alpha}n+\frac{13p^{2\alpha}-1}{6}\right)q^n \equiv 2(-1)^{\alpha(\pm p-1)/6}g_1\psi(q^4) \pmod{8}, \tag{4.42}$$

and for all $n \geq 0$ we have

$$\overline{W}_{5,4}\left(4p^{2\alpha+2}n+4p^{2\alpha+1}j+\frac{13p^{2\alpha+2}-1}{6}\right) \equiv 0 \pmod{8}. \tag{4.43}$$

Proof. Setting $r=5$ and $s=4$ in (1.7) and employing (2.3) and (2.4), we obtain

$$\sum_{n\geq 0}\overline{W}_{5,4}(n)q^n = \frac{\mathfrak{f}(-q^4,-q^8)}{(-q;q^2)_\infty(q;q)_\infty} = \frac{g_4^2}{g_2^2}. \tag{4.44}$$

Since the right hand side of (4.44) is a series in q^2, it follows that

$$\sum_{n\geq 0}\overline{W}_{5,4}(n)q^n = \frac{g_2^2}{g_1^2}. \tag{4.45}$$

Employing (2.10) then yields

$$\sum_{n\geq 0}\overline{W}_{5,4}(2n)q^n = \frac{g_8^5}{g_2^3g_{16}^2}+2q\frac{g_4^2g_{16}^2}{g_2^3g_8}. \tag{4.46}$$

Collecting the terms involving odd powers of q from both sides of (4.46) and simplifying, we obtain

$$\sum_{n\geq 0}\overline{W}_{5,4}(4n+2)q^n = 2\frac{g_2^2g_8^2}{g_1^3g_4}. \tag{4.47}$$

Employing (2.9) in (4.47) and then applying (2.2), we obtain

$$\sum_{n\geq 0}\overline{W}_{5,4}(4n+2)q^n \equiv 2g_1\psi(q^4) \pmod 8, \tag{4.48}$$

which is the $\alpha = 0$ case of (4.42). Now suppose that (4.42) holds for some $\alpha \geq 0$. Lemmas 2.2 and 2.3 then yields

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{W}_{5,4}\left(4p^{2\alpha}n+\frac{13p^{2\alpha}-1}{6}\right)q^n \\
&\equiv 2(-1)^{\alpha(\pm p-1)/6}\Bigg[\sum_{\substack{k=-(p-1)/2\\ k\neq(\pm p-1)/6}}^{(p-1)/2}(-1)^k q^{(3k^2+k)/2}\mathfrak{f}\left(-q^{(3p^2+(6k+1)p)/2},-q^{(3p^2-(6k+1)p)/2}\right) \\
&\qquad + (-1)^{(\pm p-1)/6}q^{(p^2-1)/24}g_{p^2}\Bigg] \\
&\quad\times\Bigg[\sum_{m=0}^{(p-3)/2}q^{4(m^2+m)/2}\mathfrak{f}\left(q^{2(p^2+(2m+1)p)},q^{2(p^2-(2m+1)p)}\right) \\
&\qquad + q^{(p^2-1)/2}\psi(q^{4p^2})\Bigg] \pmod 8.
\end{aligned} \tag{4.49}$$

Next, consider the congruence

$$\left(\frac{3k^2+k}{2}\right)+4\left(\frac{m^2+m}{2}\right)\equiv 13\left(\frac{p^2-1}{24}\right) \pmod p, \quad \text{for} \quad 0\leq k,m\leq p-1, \tag{4.50}$$

which is equivalent to

$$(6k+1)^2+3(4m+2)^2\equiv 0 \pmod p. \tag{4.51}$$

Since $\left(\frac{-3}{p}\right) = -1$, the only solution of (4.51) is $k = (\pm p-1)/6$ and $m = (p-1)/2$. Therefore, collecting the terms involving powers of q that are congruent to $13(p^2-1)/24$ modulo p from both sides of (4.49) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty}\overline{W}_{5,4}\left(4p^{2\alpha+1}n+\frac{13p^{2\alpha+2}-1}{6}\right)q^n \equiv 2(-1)^{(\alpha+1)(\pm p-1)/6}g_p\psi(q^{4p}) \pmod 8. \tag{4.52}$$

Collecting the terms involving powers of q that are congruent to 0 modulo p from both sides of (4.52) and simplifying the resulting equality yields

$$\sum_{n=0}^{\infty}\overline{W}_{5,4}\left(4p^{2\alpha+2}n+\frac{13p^{2\alpha+2}-1}{6}\right)q^n \equiv 2(-1)^{(\alpha+1)(\pm p-1)/6}g_1\psi(q^4) \pmod 8, \tag{4.53}$$

which is the $\alpha+1$ case of (4.42). Finally, collecting the terms involving powers of q that are congruent to j modulo p from both sides of (4.52) yields (4.43). □

5 Congruences for $U_{r,s}(n)$ and $\overline{U}_{r,s}(n)$

Theorem 5.1. *For any integer $n \geq 0$, we have*

$$U_{5,5}(5n+4)\equiv 0 \pmod 5.$$

Proof. Setting $r = s = 5$ in (1.8) and using (2.1) and Lemma 2.6, we obtain

$$\sum_{n\geq 0} U_{5,5}(n)q^n \equiv \frac{\phi(-q^5)}{g_5}\Big(A_0B_0+(A_0B_1+A_1B_0)+(A_1B_1+A_2B_0)+A_2B_1\Big) \pmod 5. \quad (5.1)$$

Our result follows by observing that the series on the right hand side of (5.1) has no term whose exponent is congruent to 4 modulo 5. □

Remark 5.2. Setting $r = 3$ and $s = 2$ in (1.9) and simplifying (2.3), we obtain

$$\sum_{n\geq 0} \overline{U}_{3,2}(n)q^n = \frac{(-q^2;q^2)_\infty^2 \mathfrak{f}(-q^2,-q^4)}{(q^2;q^2)_\infty} = \frac{1}{(q^2;q^4)_\infty^2} = \frac{g_4^2}{g_2^2}, \quad (5.2)$$

where we also used the well-known result,

$$(q;q^2)_\infty^{-1} = (-q;q)_\infty. \quad (5.3)$$

Combining (4.44) and (5.2), we find

$$\overline{U}_{3,2}(n) = \overline{W}_{5,4}(n). \quad (5.4)$$

As a consequence, $\overline{U}_{3,2}(n)$ satisfies the congruences given in Theorem 4.6.

Theorem 5.3. *Suppose p is an odd prime such that $\left(\frac{-6}{p}\right) = -1$, $1 \leq j \leq p-1$ and $\alpha \geq 0$. Then*

$$\sum_{n\geq 0} \overline{U}_{4,2}\left(8p^{2\alpha}n + \frac{7p^{2\alpha}-1}{3}\right) q^n \equiv 2(-1)^{\alpha(\pm p-1)/6} g_1\psi(q^2) \pmod 8, \quad (5.5)$$

and for all $n \geq 0$ we have

$$\overline{U}_{4,2}\left(8p^{2\alpha+2}n + 8p^{2\alpha+1}j + \frac{7p^{2\alpha+2}-1}{3}\right) \equiv 0 \pmod 8. \quad (5.6)$$

Proof. Setting $r = 4$ and $s = 2$ in (1.9), we have

$$\sum_{n\geq 0} \overline{U}_{4,2}(n)q^n = \frac{(-q^2;q^2)_\infty^2 \mathfrak{f}(-q^2,-q^6)}{(q^2;q^2)_\infty} = \frac{(-q^2;q^2)_\infty^2 \mathfrak{f}(-q^2,-q^6)}{g_2}. \quad (5.7)$$

Simplifying (5.7) using (5.3), (2.2) and (2.5), we obtain

$$\sum_{n\geq 0} \overline{U}_{4,2}(n)q^n = \frac{\psi(-q^2)}{(q^2;q^4)^2 g_2} = \frac{g_4 g_8}{g_2^2}. \quad (5.8)$$

Collecting the terms involving even powers of q from (5.8) and then applying (2.10) in the resulting equation, we obtain

$$\sum_{n\geq 0} \overline{U}_{4,2}(2n)q^n = \frac{g_4 g_8^5}{g_2^4 g_{16}^2} + 2q\frac{g_4^3 g_{16}^2}{g_2^4 g_8}. \quad (5.9)$$

Collecting the terms involving odd powers of q from both sides of (5.9) and simplifying, we obtain

$$\sum_{n\geq 0} \overline{U}_{4,2}(4n+2)q^n = 2\frac{g_2^3 g_8^2}{g_1^4 g_4}. \quad (5.10)$$

Using (2.9) in (5.10), we obtain

$$\sum_{n\geq 0} \overline{U}_{4,2}(4n+2)q^n \equiv 2\frac{g_2 g_8^2}{g_4} \pmod 8. \quad (5.11)$$

Collecting the terms involving powers of q that are congruent to 0 modulo 2 from both sides of (5.11) and simplifying the resulting equality and then applying (2.2) yields

$$\sum_{n\geq 0}\overline{U}_{4,2}(8n+2)q^n \equiv 2g_1\psi(q^2) \pmod{8}, \tag{5.12}$$

which is the $\alpha = 0$ case of (5.5). As one can now proceed via the same argument used in our proof of Theorem 4.6, we omit the remaining details. □

Theorem 5.4. *Suppose p is an odd prime such that $\left(\frac{-2}{p}\right) = -1$, $1 \leq j \leq p-1$ and $\alpha \geq 0$. Then*

$$\sum_{n\geq 0}\overline{U}_{6,2}\left(54p^{2\alpha}n + \frac{99p^{2\alpha}-3}{4}\right)q^n \equiv 2g_2\psi(q^3) \pmod{4}, \tag{5.13}$$

$$\sum_{n\geq 0}\overline{U}_{6,2}\left(6p^{2\alpha}n + \frac{11p^{2\alpha}-3}{4}\right)q^n \equiv 2(-1)^{\alpha(\pm p-1)/6}g_2\psi(q^3) \pmod{8}, \tag{5.14}$$

and for all $n \geq 0$, we have

$$\overline{U}_{6,2}\left(54p^{2\alpha+2}n + 54p^{2\alpha+1}j + \frac{99p^{2\alpha+2}-3}{4}\right) \equiv 0 \pmod{4}, \tag{5.15}$$

$$\overline{U}_{6,2}\left(6p^{2\alpha+2}n + 6p^{2\alpha+1}j + \frac{11p^{2\alpha+2}-3}{4}\right) \equiv 0 \pmod{8}. \tag{5.16}$$

Proof. Setting $r = 6$, $s = 2$ in (1.9), we obtain

$$\sum_{n\geq 0}\overline{U}_{6,2}(n)q^n = \frac{(-q^2;q^2)_\infty^2 \mathfrak{f}(-q^2,-q^{10})}{g_2}. \tag{5.17}$$

Simplifying (5.17) using (5.3) and (2.8), we obtain

$$\sum_{n\geq 0}\overline{U}_{6,2}(n)q^n = \frac{\psi(q^6)\chi(-q^2)}{(q^2;q^4)_\infty^2 \; g_2}. \tag{5.18}$$

Collecting the terms involving powers of q that are congruent to 0 modulo 2 from both sides of (5.18) and simplifying the resulting equality and then applying (2.2) and (2.5) yields

$$\sum_{n\geq 0}\overline{U}_{6,2}(2n)q^n = \frac{\psi(q^3)\chi(-q)}{(q;q^2)_\infty^2 \; g_1} = \frac{g_2 g_6^2}{g_1^2 g_3}. \tag{5.19}$$

Employing (2.12) then yields

$$\sum_{n\geq 0}\overline{U}_{6,2}(2n)q^n = \frac{g_6^6 g_9^6}{g_3^9 g_{18}^3} + 2q\frac{g_6^5 g_9^3}{g_3^8} + 4q^2\frac{g_6^4 g_{18}^3}{g_3^7}. \tag{5.20}$$

Collecting the terms involving powers of q that are congruent to 0 modulo 3 from both sides of (5.20) and simplifying the resulting equality yields

$$\sum_{n\geq 0}\overline{U}_{6,2}(6n)q^n = \frac{g_2^6 g_3^6}{g_1^9 g_6^3}. \tag{5.21}$$

Employing (2.9) in (5.21), we obtain

$$\sum_{n\geq 0}\overline{U}_{6,2}(6n)q^n \equiv \frac{g_2^2 g_3^6}{g_1 g_6^3} \pmod{8}. \tag{5.22}$$

Next, using (2.13) we obtain

$$\sum_{n\geq 0}\overline{U}_{6,2}(6n)q^n \equiv \frac{g_3^5 g_9^2}{g_6^2 g_{18}} + q\frac{g_3^6 g_{18}^2}{g_6^3 g_9} \pmod 8. \tag{5.23}$$

Collecting the terms involving powers of q that are congruent to 1 modulo 3 from both sides of (5.23) and simplifying the resulting equality yields

$$\sum_{n\geq 0}\overline{U}_{6,2}(18n+6)q^n \equiv \frac{g_1^6 g_6^2}{g_2^3 g_3} \pmod 8. \tag{5.24}$$

Using (2.9) in (5.24), we obtain

$$\sum_{n\geq 0}\overline{U}_{6,2}(18n+6)q^n \equiv \frac{g_1^2 g_6^2}{g_2 g_3} \pmod 4. \tag{5.25}$$

Substituting (2.11) in (5.25), we obtain

$$\sum_{n\geq 0}\overline{U}_{6,2}(18n+6)q^n \equiv \frac{g_6^2 g_9^2}{g_3 g_{18}} - 2q\frac{g_6 g_{18}^2}{g_9} \pmod 4. \tag{5.26}$$

Collecting the terms involving powers of q that are congruent to 1 modulo 3 from both sides of (5.26) and simplifying the resulting equality and then applying (2.2) yields

$$\sum_{n\geq 0}\overline{U}_{6,2}(54n+24)q^n \equiv 2g_2\psi(q^3) \pmod 4, \tag{5.27}$$

which is the $\alpha = 0$ case of (5.13). As one can now proceed via the same argument used in our proof of Theorem 4.6, we omit the remaining details. Collecting the terms involving powers of q that are congruent to 1 modulo 3 from both sides of (5.20) and simplifying the resulting equality yields

$$\sum_{n\geq 0}\overline{U}_{6,2}(6n+2)q^n = 2\frac{g_2^5 g_3^3}{g_1^8} = 2\frac{g_2 g_2^4 g_3^4}{g_1^8 g_3}. \tag{5.28}$$

Employing (2.9) in (5.28) and then applying (2.2), we obtain

$$\sum_{n\geq 0}\overline{U}_{6,2}(6n+2)q^n \equiv 2g_2\psi(q^3) \pmod 8, \tag{5.29}$$

which is the $\alpha = 0$ case of (5.14). As one can now proceed via the same argument used in our proof of Theorem 4.6, we omit the remaining details. □

Corollary 5.5. *For any integer $n \geq 0$,*

$$\overline{U}_{6,2}(54n+24) \equiv \overline{U}_{6,2}(6n+2) \pmod 4, \tag{5.30}$$

$$\overline{U}_{6,2}(18n+12) \equiv 0 \pmod 8. \tag{5.31}$$

Proof. Combining (5.27) and (5.29), we arrive at (5.30). Collecting the terms involving powers of q that are congruent to 2 modulo 3 from both sides of (5.23), we complete the proof of (5.31). □

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