Palestine Journal of Mathematics

Vol 13(Special Issue II)(2024) , 128–135

COPURE $D2$ MODULES

Kaushal Gupta[1], Sonal Gupta[2] and Ashok Ji Gupta[3]

Communicated by Mohammad Ashraf

Dedicated to Prof. B. M. Pandeya on his 78th birthday

MSC 2020 Classification: Primary 16D10; Secondary 16D40, 16E50, 16E60.

Keywords and phrases: $D2$ module, Copure submodule, Copure $D2$ module.

Acknowledgement: The authors would like to thank the editors and the referees for their valuable comments and suggestions, which improved the quality of the paper. Kaushal Gupta acknowledges Senior Research Fellowship from the UGC-INDIA (JUNE18-413785) to carry out this research work.

Abstract This paper presents and investigate the concept of copure $D2$ modules, which serve as a generalization of $D2$ modules and the dual notion of copure $C2$ modules [8]. A module $\mathcal{N}$ is considered to be copure $D2$ if a quotient module determined by a copure submodule of $\mathcal{N}$, is isomorphic to a summand of $\mathcal{N}$, then that copure submodule is also a summand of $\mathcal{N}$. In our discussion, we explore summands and direct sums of copure $D2$ modules and give new characterizations of semi-simple rings with respect to relative copure $D2$ modules. Also, we find connections among copure $D2$ modules, copure-Rickart modules, and endoregular modules.

1 Introduction

Throughout the paper, we consider an associative ring denoted by $\mathcal{R}$, which is assumed to have an identity element, unless explicitly stated otherwise, and right $\mathcal{R}$-modules will be considered as unital. In 1976, Nicholson [9], an $\mathcal{R}$-module $\mathcal{N}$ is called $D2$ if the quotient module $\mathcal{N}/\mathcal{K}$ is isomorphic to a summand of $\mathcal{N}$, then $\mathcal{K}$ is also a summand of $\mathcal{N}$. $D2$ modules are alternatively referred to as Direct-projective modules. In 1984, Hiremath introduced the concept of copurity [4], as the dual counterpart to the concept of purity by employing co-finitely related modules. In an $\mathcal{R}$-module $\mathcal{N}$, a submodule $\mathcal{L}$ is classified as copure if, for any co-finitely related $\mathcal{R}$-module $\mathcal{K}$, any homomorphism from $\mathcal{L}$ to $\mathcal{K}$ can be extended to a homomorphism from $\mathcal{N}$ to $\mathcal{K}$. In 2012, Copure-projectivity was introduced and its properties are investigated in [2].

In recent years various generalizations of $D2$ modules have been studied like finite-$D2$ modules, semi-projective modules, simple-$D2$ modules, and pure-$D2$ modules, etc. Motivated by these generalizations and work done in [8] on copure $C2$ modules, we present the concept of copure $D2$ modules, which serves as the dual counterpart to copure $C2$ modules and represents a broader extension of $D2$ modules. An $\mathcal{R}$-module $\mathcal{N}$ is considered copure $D2$ if there exists a copure submodule $\mathcal{P}$ of $\mathcal{N}$ such that $\mathcal{M}/\mathcal{P} \cong \mathcal{K} \leq^{\oplus} \mathcal{N}$ for some submodule $\mathcal{K}$ of $\mathcal{N}$, then $\mathcal{P} \leq^{\oplus} \mathcal{N}$.

Suppose $\mathcal{N}$ and $\mathcal{K}$ are copure $D2$ $\mathcal{R}$-modules. We say that $\mathcal{N}$ is relatively copure $D2$ to $\mathcal{K}$ if there exists an $\mathcal{R}$-homomorphism $f \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{N}, \mathcal{K})$ such that $Img(f)$ is a summand of $\mathcal{K}$ and $Ker(f)$ is a copure submodule of $\mathcal{N}$, then $Ker(f)$ is also a summand of $\mathcal{N}$.

To establish a foundation, section 2 provides several definitions and compiles a number of properties concerning copure $D2$ modules and copure short exact sequences.

In section 3, we first define copure $D2$ module and also give some examples of copure $D2$ module and give counterexamples such that the module is copure $D2$ but not $D2$. In remark 3.3 we give the equivalent condition of the copure $D2$ module with respect to the copure exact sequence. In Proposition 3.4, we find an equivalent definition of copure $D2$ module with respect to its endomorphism ring, which is defined as, an $\mathcal{R}$-module $\mathcal{N}$ is copure $D2$ if and only if, for every $f \in \mathrm{End}_{\mathcal{R}}(\mathcal{N})$, if $Img(f)$ is a summand of $\mathcal{N}$ and $Ker(f)$ is copure in $\mathcal{N}$, then $Ker(f)$ is also a summand of $\mathcal{N}$. Next, we use the Morita equivalence property, if an $\mathcal{R}$-module $\mathcal{N}$ is a copure $D2$, then some module which is Morita equivalent to $\mathcal{N}$, will also be copure $D2$.

In section 4, we discuss the direct sum and summand of copure $D2$ modules. In Proposition 4.1, we prove that the summand of the copure $D2$ module is copure $D2$. Then we infer that the sum of two copure $D2$ modules need not be a copure $D2$, a counterexample is also provided, so we find the condition under which the direct sum of copure $D2$ modules is copure $D2$. In Proposition 4.4, we show that if $\mathcal{N} = \bigoplus_{i\in\mathbb{N}}\mathcal{N}_i$, where each $\mathcal{N}_i$ is relatively copure $D2$ modules with others if and only if $\mathcal{N}$ is a copure $D2$ module. Also, we evaluate under which condition a semi-simple ring is equivalent to a ring over which all modules are relatively copure $D2$ to each other.

In section 5, we characterize some rings with respect to copure $D2$ modules. In Proposition 5.2, we establish the proof that a copure $D2$ module $\mathcal{N}$ is both a copure Rickart as well as dual Rickart module if and only if $\mathcal{N} \oplus \mathcal{N}$ possesses the S.S.P. (strong summand property) and copure intersection properties. Also, we show that $\mathcal{R}$ is copure semi-simple if and only if each 2-generated $\mathcal{R}$ module is copure $D2$ (see Proposition 5.4). In Proposition 5.8, we derive that each copure injective module over a copure hereditary ring is a copure $D2$ module. In proposition 5.10, we compare copure $D2$ modules to pure $D2$ modules as they are offered as one more generalization of $D2$ modules.

Throughout the paper, basic definitions, symbols, and notations are used to refer to [1] and [12].

2 Preliminaries

This section covers fundamental definitions and characteristics of copure submodules, copure short exact sequences.

Definition 2.1. An $\mathcal{R}$-module $\mathcal{N}$ is considered generated co-finitely if its injective envelope $\mathcal{E}(\mathcal{N})$ can be expressed as $\mathcal{E}(\mathcal{N}) = \mathcal{E}(S_1) \oplus \mathcal{E}(S_2) \oplus \ldots \oplus \mathcal{E}(S_n)$, where S_i for $i = 1, 2, \ldots, n$ represents a simple $\mathcal{R}$-module ([7], [10]).

Definition 2.2. A cofree module is a $\mathcal{R}$-module $\mathcal{N}$ which is expressed as the direct product, denoted as $\Pi_{\alpha\in\mathcal{I}}\mathcal{E}(S_\alpha)$, where $\mathcal{I}$ is an index set and S_α represents a simple $\mathcal{R}$-module, then $\mathcal{N}$ is isomorphic to that direct product. ([5])

Definition 2.3. An $\mathcal{R}$-module is considered co-finitely related if there exists an exact sequence $0 \to \mathcal{N} \to \mathcal{K} \to \mathcal{L} \to 0$ of $\mathcal{R}$-modules, where $\mathcal{K}$ generated co-finitely, cofree and $\mathcal{L}$ generated co-finitely ([5]).

Definition 2.4. A short exact sequence of $\mathcal{R}$ modules

$$0 \to \mathcal{N} \to \mathcal{K} \to \mathcal{L} \to 0.$$

is called a copure short exact sequence if every co-finitely related $\mathcal{R}$-module is injective with respect to this sequence. So a submodule $\mathcal{S}$ of a $\mathcal{R}$-module $\mathcal{N}$ is said to be copure in $\mathcal{N}$ if the canonical short exact sequence

$$0 \to \mathcal{S} \to \mathcal{N} \to \mathcal{N}/\mathcal{S} \to 0.$$

is copure.

In the following section, we begin by introducing copure $D2$ modules and providing several examples to illustrate this concept. Additionally, we delve to examine the various properties associated with the copure $D2$ module.

3 Copure $D2$ Modules

Definition 3.1. An $\mathcal{R}$-module $\mathcal{N}$ is considered a copure $D2$ module (or copure-direct-projective module), if for any copure submodule $\mathcal{B}$ of $\mathcal{N}$ such that $\mathcal{N}/\mathcal{B} \cong \mathcal{A}$ and $\mathcal{A} \leq^{\oplus} \mathcal{N}$, it follows that $\mathcal{B}$ is a summand of $\mathcal{N}$.

In general, each summand of an $\mathcal{R}$-module $\mathcal{N}$ can be considered as a copure submodule. Consequently, it follows that a $D2$ module is also a copure $D2$ module. Next, we provide some examples of copure $D2$ modules.

Example 3.2. We discuss the following examples of copure $D2$ modules:-

(1) Copure $D2$ module is a strict generalization of the $D2$ module. Given a right $\mathcal{R}$-module $\mathcal{N}$, each summand of $\mathcal{N}$ is also a copure submodule of $\mathcal{N}$. Consequently, each $D2$ module is a copure $D2$ module, but the converse is not necessarily true. As an example, consider the $\mathbb{Z}$ module $\mathbb{Z} \oplus \mathbb{Z}_n$, where $n \in \mathbb{N}$. This module is copure $D2$ but not a $D2$ module since both $\mathbb{Z}$ and $\mathbb{Z}_n$ have no copure submodules over $\mathbb{Z}$.

(2) We can similarily construct another example $\mathbb{Q} \bigoplus \mathbb{Q}/\mathbb{Z}$ over the ring $\mathbb{Z}$, this module is a copure $D2$ but not a $D2$.

(3) If $\mathcal{N}$ is a copure simple module, then it is also a copure $D2$ module. A module $\mathcal{N}$ is classified as copure simple when its only copure submodules are the trivial submodules.

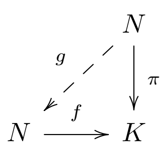
Remark 3.3. An $\mathcal{R}$-module $\mathcal{N}$ is considered as copure $D2$ if the copure exact sequence $0 \to Ker(f) \to \mathcal{N} \to \mathcal{N}/Ker(f) \to 0$ splits for each $f \in \text{End}_{\mathcal{R}}(\mathcal{N})$ such that $Img(f) \leq^{\oplus} \mathcal{N}$.

Proposition 3.4. *The following conditions are equivalent for an $\mathcal{R}$-module $\mathcal{N}$:*

(1) $\mathcal{N}$ is a copure $D2$ module.

(2) If $Img(f)$ is a summand of $\mathcal{N}$ with $Ker(f)$ is a copure submodule of $\mathcal{N}$ for each $f \in End_R(\mathcal{N})$, then $Ker(f)$ is also a summand of $\mathcal{N}$.

(3) Suppose $\mathcal{K}$ be a summand of $\mathcal{N}$ and let $f : \mathcal{N} \to \mathcal{K}$ be an epimorphism such that $Ker(f)$ is copure in $\mathcal{N}$. In this case, there exists $g \in End_{\mathcal{R}}(\mathcal{N})$ such that the following diagram commutes:



equivalently, $f \circ g = \pi$, where π is a projection map.

Proof. (1) $\Rightarrow$ (2) Using the fact, $\mathcal{N}/Ker(f) \cong Img(f)$ (By Fundamental theorem of Module Homomorphism) and $Ker(f)$ is copure submodule of $\mathcal{N}$, hence $Ker(f) \leq^{\oplus} \mathcal{N}$.

(2) $\Rightarrow$ (1) Consider $\mathcal{N}/\mathcal{B} \cong \mathcal{K} \leq^{\oplus} \mathcal{N}$ with $\mathcal{B}$ copure submodule of $\mathcal{N}$ and let $\mathcal{N}/\mathcal{B} \cong e\mathcal{N}$ for some idempotent $e \in End_{\mathcal{R}}(\mathcal{N})$. Let π be a projection map from $\mathcal{N}$ to $\mathcal{K}$ and f be an epimorphism from $\mathcal{N}$ to $\mathcal{K}$. We set some s as $s = e \circ f^{-1} \circ \pi$, then $Ker(s) = \mathcal{B}$ and $Img(s) = e\mathcal{N}$, Since $Ker(s)$ is summand in $\mathcal{N}$ implies $\mathcal{B}$ is summand of $\mathcal{N}$.

(1) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) Let $\mathcal{B}$ be any copure submodule of $\mathcal{N}$ such that $\mathcal{N}/\mathcal{B} \cong K \leq^{\oplus} \mathcal{N}$. Consider $h : \mathcal{N} \to \mathcal{N}/\mathcal{B}$, since $Ker(h) \cong \mathcal{B}$ and copure in $\mathcal{N}$. By hypothesis, we will have some $g \in End_{\mathcal{R}}(\mathcal{N})$ such that $h \circ g = \pi$. Hence g is the required splitting of h, and $Ker(h)$ is a summand of $\mathcal{N}$. □

Next, we discuss the Morita equivalence with respect to copure $D2$ modules. Two rings say $\mathcal{R}$ and S are Morita equivalent if their category of modules say $\mathcal{R}$-module $\mathcal{N}$ and S-module $\mathcal{N}$ are equivalent.

Proposition 3.5. *Consider two Morita equivalent rings, denoted as $\mathcal{R}$ and S. Let ϕ : Mod-$R \to$ Mod-S be their Morita equivalence. In this context, if a module $\mathcal{N}$ is copure $D2$, then its image under ϕ, denoted as $\phi(\mathcal{N})$, is also copure $D2$, and vice versa.*

Proof. Let $\mathcal{K} \leq \mathcal{N}$ be a copure submodule then $\phi(\mathcal{K})$ is copure submodule of $\phi(\mathcal{N})$. Since copurity is a Morita equivalent property. Hence, the proof is clear. □

Proposition 3.6. *Assume that $\mathcal{N}$ is a copure $D2$ $\mathcal{R}$-module. If we can express $\mathcal{N}$ as the direct sum $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$, where $f : \mathcal{N}_1 \to \mathcal{N}_2$ is an $\mathcal{R}$-homomorphism satisfying the conditions that the $Img(f)$ is a summand of $\mathcal{N}_2$ and the $Ker(f)$ is a copure submodule of $\mathcal{N}_1$, then $Ker(f)$ is also a summand of $\mathcal{N}_1$.*

Proof. Consider the module homomorphism $f : \mathcal{N}_1 \to \mathcal{N}_2$. Let $\pi : \mathcal{N} \to \mathcal{N}_1$ be the canonical projection. We have $f \circ \pi : \mathcal{N} \to \mathcal{N}_2$. Moreover, $Img(f \circ \pi) = Img(f)$. Therefore, $\mathcal{N}/Ker(f \circ \pi) \cong Img(f \circ \pi) = Img(f)$. Since $Img(f) \leq^{\oplus} \mathcal{N}_2$ and it is a summand of the module $\mathcal{N}$, by the definition of a copure $D2$ module, we have that $Ker(f \circ \pi) = \mathcal{N}_2 \oplus Ker(f)$ is copure in $\mathcal{N}_2 \oplus \mathcal{N}_1$. Since $Ker(f)$ is copure in $\mathcal{N}_1$, it follows that $Ker(f \circ \pi)$ is copure in $\mathcal{N}$. Therefore, we have $Ker(f \circ \pi) \leq^{\oplus} \mathcal{N}$. Since $Ker(f)$ is a submodule of $\mathcal{N}_1$, we can conclude that $Ker(f) \leq^{\oplus} \mathcal{N}_1$. □

Corollary 3.7. *Consider a copure $D2$ $\mathcal{R}$-module $\mathcal{N}$. If we can write $\mathcal{N}$ as the direct sum $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ with an $\mathcal{R}$-epimorphism $f : \mathcal{N}_1 \to \mathcal{N}_2$ such that $Ker(f)$ is copure in $\mathcal{N}_1$, then $Ker(f)$ is a summand of $\mathcal{N}_1$.*

Proof. Follows from Proposition 3.6. □

Definition 3.8. A module $\mathcal{N}$ is considered as a copure dual Rickart module if for each $f \in End_{\mathcal{R}}(\mathcal{N})$, $Img(f)$ is a copure submodule of the module $\mathcal{N}$.

Proposition 3.9. *Suppose that the $\mathcal{R}$-module $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ is a copure $D2$ module, where $\mathcal{N}_1$ is copure simple and copure dual Rickart. Then, either $Hom_{\mathcal{R}}(\mathcal{N}_1, \mathcal{N}_2) = 0$ or for each nonzero homomorphism $f : \mathcal{N}_1 \to \mathcal{N}_2$, if $Ker(f)$ is a copure submodule of $\mathcal{N}$, then f is a monomorphism.*

Proof. We consider $Hom_R(\mathcal{N}_1, \mathcal{N}_2) \neq 0$ as other case is trivial. If a non-zero $\mathcal{R}$-homomorphism $f : \mathcal{N}_1 \to \mathcal{N}_2$ and $\mathcal{N}_1$ is copure simple, this implies $Ker(f) \neq 0$ is not copure submodule, implies $Ker(f) \neq 0$ is not summand of $\mathcal{N}_1$. Hence, $Ker(f)$ is trivial and $f \in Hom_{\mathcal{R}}(\mathcal{N}_1, \mathcal{N}_2)$ is a monomorphism. □

Proposition 3.10. *Let $\mathcal{N}$ be an $\mathcal{R}$-module such that for each $f \in End_{\mathcal{R}}(\mathcal{N})$, the image $Img(f)$ is a summand of $\mathcal{N}$ and $\mathcal{N}/Ker(f)$ is a copure injective submodule of $\mathcal{N}$. Then, $\mathcal{N}$ is a copure $D2$ module if and only if $\mathcal{N}$ is a $D2$ module.*

Proof. Since $\mathcal{N}/Ker(f)$ is copure injective, we can conclude that the exact sequence is copure and $0 \to \mathcal{N}/Ker(f) \to \mathcal{N}$ splits. Moreover, since $\mathcal{N}/Ker(f) \cong Img(f)$, which is a summand of $\mathcal{N}$. Therefore, we got our assertion. □

4 Summands and Direct Sums of Copure $D2$ modules

In this section, our focus revolves around exploring the properties of summands and direct sums of copure $D2$ modules. The subsequent proposition verifies that a summand of a copure $D2$ module is itself a copure $D2$ module. Additionally, we delve into the outcomes concerning the direct sum of copure $D2$ modules.

Proposition 4.1. *If $\mathcal{N}$ is a copure $D2$ module with $\mathcal{K} \leq^{\oplus} \mathcal{N}$, then $\mathcal{K}$ is also a copure $D2$ module.*

Proof. Let $\mathcal{K}$ be a direct summand of $\mathcal{N}$ and $\mathcal{A}$ be a copure submodule of $\mathcal{K}$ such that $\mathcal{K}/\mathcal{A} \cong \mathcal{B} \leq^{\oplus} \mathcal{K}$ for some submodule $\mathcal{B}$ of $\mathcal{K}$, we are required to prove $\mathcal{A} \leq^{\oplus} \mathcal{K}$. Since $\mathcal{N}$ is copure-direct-projective, so there exists some $\mathcal{T}$, a direct summand of $\mathcal{N}$ such that $\mathcal{N}/\mathcal{A} \cong \mathcal{T} \leq^{\oplus} \mathcal{N}$. Since $\mathcal{A}$ is copure in $\mathcal{K}$ this implies $\mathcal{A}$ is copure in $\mathcal{N}$, therefore $\mathcal{A}$ will be a direct summand of $\mathcal{N}$ and $\mathcal{A}$ is a copure submodule of $\mathcal{K}$. Hence $\mathcal{A} \leq^{\oplus} \mathcal{K}$. □

Now, it is worth noting that the direct sum of two copure $D2$ modules may not necessarily be a copure $D2$ module. Consider $\mathbb{Z}$-modules $\mathcal{N}_1 = \prod_{n=1}^{\infty} \mathbb{Z}_2$ and $\mathcal{N}_2 = \prod_{n=1}^{\infty} \mathbb{Z}_2 / \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$ which are both copure $D2$ but $\mathcal{N}_1 \bigoplus \mathcal{N}_2$ is not a copure $D2$ module because $f \in Hom_{\mathcal{R}}(\mathcal{N}_1, \mathcal{N}_2)$ and $Ker(f)$ is copure submodule of $\mathcal{N}_1$ but not a summand of $\mathcal{N}_1$.

Definition 4.2. Let $\mathcal{N}$ and $\mathcal{K}$ be copure $D2$ $\mathcal{R}$-modules, where $\mathcal{N}$ is called relatively copure $D2$ with respect to $\mathcal{K}$, if there exists an $\mathcal{R}$-homomorphism $f \in Hom_{\mathcal{R}},(\mathcal{N},\mathcal{K})$ whose $Img(f)$ is a summand of $\mathcal{K}$ and if its $Ker(f)$ is a copure submodule $\mathcal{N}$, then its $Ker(f)$ is a summand of $\mathcal{N}$.

Next, in our discussion, we will examine the direct sum of a copure $D2$ module in relation to the copure intersection property. A module is said to possess the copure intersection property when the intersection of two copure submodules remains copure. Suppose $\mathcal{N}$ be an $\mathcal{R}$-module such that it satisfies the copure intersection property. For any decomposition of module $\mathcal{N}$ into the direct sums of module $\mathcal{N} = \mathcal{K} \oplus \mathcal{T}$, and for each $f \in \text{Hom}_{\mathcal{R}}(\mathcal{T},\mathcal{K})$, it follows that its $Ker(f)$ is a copure submodule of $\mathcal{T}$.

Proposition 4.3. *Suppose $\mathcal{N} = \mathcal{K} \bigoplus \mathcal{T}$ is a copure $D2$ module with the copure intersection property and $\mathcal{T}$ is a dual Rickart module, then $\mathcal{K}$ is relatively copure $D2$ with respect to $\mathcal{T}$.*

Proof. Clear. □

In particular, each direct sum of copure $D2$ modules is copure $D2$ if modules are relatively copure $D2$ to each other.

Proposition 4.4. *Suppose $\mathcal{N} = \bigoplus_{i\in\mathbb{N}}\mathcal{N}_i$, where each $\mathcal{N}_i$ is relatively copure $D2$ module with others if and only if $\mathcal{N}$ is a copure $D2$ module.*

Proof. If $\mathcal{N}$ is copure $D2$ module then $\bigoplus_{i\in\mathbb{N}}\mathcal{N}_i$ is copure $D2$, follows from proposition 3.6 and definition 4.2. For the converse part, let for some copure submodule $\mathcal{P}$ of $\mathcal{N}$ such that $\mathcal{N}/\mathcal{P} \cong \mathcal{K} \leq^{\oplus} \mathcal{N}$, then $\mathcal{N}$ will be a summand of some $\mathcal{N}_k$. Moreover, $\mathcal{P}$ is a copure submodule of some $\mathcal{N}_t$, since $\mathcal{N}_t$ is relatively copure $D2$ to $\mathcal{N}_k$. Hence P is a summand of module $\mathcal{N}$. As $k, t \in \mathbb{N}$ are arbitrary, $\mathcal{N}$ is a copure $D2$ module. □

Proposition 4.5. *Suppose we have a ring $\mathcal{R}$ in which each $\mathcal{R}$-module, $\mathcal{N}$ is a fully copure module. Then the following statements are equivalent:*

(1) $\mathcal{R}$ is a semi-simple ring.

(2) All $\mathcal{R}$-modules are relatively copure $D2$ to any $\mathcal{R}$-module.

Proof. $(1) \Rightarrow (2)$ Clear.

$(2) \Rightarrow (1)$ We have to show that $\mathcal{I}$ is a summand of $\mathcal{R}$. Now, suppose I is an ideal of a ring $\mathcal{R}$. From (2), if each $\mathcal{R}$-module is relatively copure $D2$ to any $\mathcal{R}$-module, then the $\mathcal{R}$-module $\mathcal{R}$ is relatively copure $D2$ to $\mathcal{R}/\mathcal{I}$ as an $\mathcal{R}$-module. Then by definition for $\mathcal{R}$-homomorphism $f : \mathcal{R} \to \mathcal{R}/\mathcal{I}$ and $Ker(f) = \mathcal{I}$ is copure in $\mathcal{R}$, $Ker(f)$ is a summand of $\mathcal{R}$. Since $\mathcal{I}$ is an arbitrary ideal of $\mathcal{R}$. Therefore, $\mathcal{R}$ is a semi-simple ring. □

5 Characterization of copure $D2$ modules over rings

This section deals with the new characterizations of copure $D2$ modules over some rings. At first, we will consider its endomorphism ring, and develop equivalent conditions for copure $D2$ modules and endoregular modules. Since each endoregular module is copure $D2$, the converse of this statement may not hold. For example, assume $\mathbb{Z}_{p^n}$ as $\mathbb{Z}$ module where p is any prime number and $n \in \mathbb{N}$, which is copure $D2$ module but not an endoregular module.

Definition 5.1. A module $\mathcal{N}$ is referred to as a copure Rickart module if, for each $f \in \text{End}_{\mathcal{R}}(\mathcal{N})$, the kernel $Ker(f)$ is a copure submodule of module $\mathcal{N}$ and summand of $\mathcal{N}$.

Proposition 5.2. *For copure $D2$ $\mathcal{R}$-module $\mathcal{N}$, the following statements are equivalent :*

(1) Module $\mathcal{N}$ is copure Rickart and dual Rickart ;

(2) $\mathcal{N} \bigoplus \mathcal{N}$ has SSP and copure intersection property;

(3) $\mathcal{N}$ is the Endoregular module.

Proof. (1) $\Rightarrow$ (3) To prove $\mathcal{N}$ is endoregular module then $Ker(f)$ and $Img(f)$ are summands for all $f \in End_{\mathcal{R}}(\mathcal{N})$. $Img(f)$ is a summand of $\mathcal{N}$ since $\mathcal{N}$ is dual Rickart, also $\mathcal{N}$ is a copure Rickart module, hence $Ker(f)$ is copure for all $f \in End_{\mathcal{R}}(\mathcal{N})$. So we get $Ker(f)$ is a summand of $\mathcal{N}$. Hence $\mathcal{N}$ is the Endoregular module.

(2) $\Rightarrow$ (3) $Img(g) \leq^{\oplus} \mathcal{N}$ since $\mathcal{N} \bigoplus \mathcal{N}$ has SSP for each $g \in End_{\mathcal{R}}(\mathcal{N})$ and due to copure intersection property, $Ker(f)$ is copure for each $f \in End_{\mathcal{R}}(\mathcal{N})$. This implies $Ker(f)$ is a summand of $\mathcal{N}$, as $\mathcal{N}$ is a copure $D2$ module.

(3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) It is clear from the definition of Endoregular module. □

In the following proposition, we find when a copure Rickart module is a copure $D2$.

Proposition 5.3. *Every copure Rickart module is copure $D2$ module over copure semi-simple ring.*

Proof. Suppose $\mathcal{N}$ is a copure Rickart module over a copure semi-simple ring $\mathcal{R}$. For a $\mathcal{P} \leq \mathcal{N}$ satisfy $\mathcal{N}/\mathcal{P} \cong \mathcal{N} \leq^{\oplus} \mathcal{N}$, $\mathcal{P}$ is copure submodule of $\mathcal{N}$. As $\mathcal{R}$ is a copure semi-simple ring, $\mathcal{P}$ is a summand of $\mathcal{N}$ this implies $\mathcal{N}$ is a copure $D2$ module. □

Next, we will discuss copure $D2$ modules with respect to copure semisimple rings. If each copure submodule of an $\mathcal{R}$-module $\mathcal{N}$ is a summand of $\mathcal{N}$ then a ring $\mathcal{R}$ is said to be copure semisimple.

Proposition 5.4. *The following statements are equivalent :-*

(1) $\mathcal{R}$ is a copure semisimple ring;

(2) Every generated finitely $\mathcal{R}$-module is a copure $D2$;

(3) Every 2-generated module is a copure $D2$.

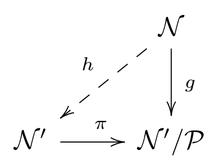
Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1) We are required to prove that each copure ideal, say I of $\mathcal{R}$ is a summand of $\mathcal{R}$. Consider $\mathcal{R} \bigoplus \mathcal{R}/\mathcal{I}$ which is copure $D2$. So from Proposition 3.4, there exists $f : \mathcal{R} \to \mathcal{R}/\mathcal{I}$ then $Ker(f) \cong \mathcal{I}$, which is copure in $\mathcal{R}$, hence $\mathcal{I}$ is summand of $\mathcal{R}$. Therefore $\mathcal{R}$ is a copure semisimple ring.

□

Next, we characterize the two modules when one module is copure $D2$ with respect to another module and characterize this condition with copure semisimple modules.

Definition 5.5. Let $\mathcal{N}$ and $\mathcal{N}'$ be $\mathcal{R}$-modules, then $\mathcal{N}$ is $\mathcal{N}'$ copure $D2$, if for a module homomorphism $g : \mathcal{N} \to \mathcal{N}'/P$ where P is a copure submodule of $\mathcal{N}'$ and $\mathcal{N}'/\mathcal{P}$ is isomorphic to a summand of $\mathcal{N}$, $\exists$ a $h \in Hom_{\mathcal{R}}(\mathcal{N}, \mathcal{N}')$ such that $\pi \circ h = g$, where π is natural epimorphism from $\mathcal{N}'$ to $\mathcal{N}'/\mathcal{P}$.



This implies the above diagram commutes.

Proposition 5.6. *For each copure submodule P of $\mathcal{N}$, if $\mathcal{N}/\mathcal{P}$ isomorphic to a summand of $\mathcal{R}$-module $\mathcal{N}$ and $\mathcal{N}/\mathcal{P}$ is $\mathcal{N}$-copure $D2$, then $\mathcal{R}$-module $\mathcal{N}$ is copure semisimple.*

Proof:- Since $\mathcal{N}/\mathcal{P}$ is $\mathcal{N}$ copure $D2$ implies identity mapping from $\mathcal{N}/\mathcal{P}$ to $\mathcal{N}/\mathcal{P}$ has a lifting $g : \mathcal{N}/\mathcal{P} \to \mathcal{N}$. Hence short exact sequence $0 \to \mathcal{P} \to \mathcal{N} \to \mathcal{N}/\mathcal{P}$ splits, implies $\mathcal{P} \leq^{\oplus} \mathcal{N}$.

Proposition 5.7. *Let $\mathcal{N} = \mathcal{N}' \bigoplus \mathcal{N}''$ be a copure $D2$ and $\mathcal{N}'$ is a copure simple module. Then $\mathcal{N}'$ is $\mathcal{N}''$-copure $D2$ module.*

Proof. Follows from the above definition. □

Recall If every factor module of a copure injective module is copure injective, a ring is considered as copure hereditary.

Proposition 5.8. *Let a module $\mathcal{N}$ be a copure injective module over a copure hereditary ring $\mathcal{R}$. Then $\mathcal{R}$-module $\mathcal{N}$ is a copure $D2$ module.*

Proof. Since $\mathcal{R}$-module $\mathcal{N}$ is copure injective over the copure hereditary ring, so each quotient module of copure injective is copure injective. For copure submodule $\mathcal{P}$ of $\mathcal{N}$ then $\mathcal{N}/\mathcal{P}$ is copure injective hence $\mathcal{P} \leq^{\oplus} \mathcal{N}$. Therefore $\mathcal{R}$-module $\mathcal{N}$ is copure $D2$. □

Lemma 5.9. *Let $\mathcal{T}$ be a projective module and $\mathcal{T} \bigoplus \mathcal{K}$ be a copure $D2$ module where $\mathcal{K}$ is an $\mathcal{R}$-module. If there exists an epimorphism $f : \mathcal{T} \to \mathcal{K}$ such that $Ker(f)$ is copure, then $\mathcal{K}$ is a projective module.*

Proof. Since f is an epimorphism with $Ker(f)$ as copure in T. From Corollary 3.7 $Ker(f)$ will be a summand of T. Hence epimorphism f splits and $\mathcal{K}$ is projective. □

In the next proposition, we characterize von-Neumann regular rings in terms of copure $D2$ modules.

In [3], The introduction of pure $D2$ modules as a generalization of $D2$ modules. If any pure submodule $\mathcal{B}$ of $\mathcal{N}$ satisfies $\mathcal{N}/\mathcal{B} \cong \mathcal{A}$ and $\mathcal{A} \leq^{\oplus} \mathcal{N}$, then B is a summand of $\mathcal{N}$, is considered as pure $D2$ module. In this section, copure $D2$ modules are introduced as another generalization of $D2$ modules and since these two ideas are comparable, we analyze pure and copure $D2$ modules.

Proposition 5.10. *An $\mathcal{R}$-module $\mathcal{N}$ is pure $D2$ if and only if it is copure $D2$ over a Dedekind domain $\mathcal{R}$.*

Proof. The concepts of copurity and purity coincide over the dedekind domain (Remark 22, [4]). So, copure $D2$ is same as pure $D2$ over dedekind domain. □

Recall a ring $\mathcal{R}$ is considered to be classical if it is commutative and each generated co-finitely $\mathcal{R}$-module is linearly compact.([11])

Proposition 5.11. *For a classical ring $\mathcal{R}$, the following statements are equivalent:*

(1) $\mathcal{R}$ is a copure semi-simple ring.

(2) $\mathcal{R}$ is a PDS ring.

(3) Each $\mathcal{R}$-module is copure $D2$.

(4) Each $\mathcal{R}$-module is pure $D2$.

Proof. $(1) \Leftrightarrow (2)$ Directly follows from [[6], Proposition 19(ii)].

$(2) \Leftrightarrow (4)$ $\mathcal{R}$ is a PDS ring if and only if each pure submodule is a summand of an $\mathcal{R}$ module if and only if each $\mathcal{R}$-module is pure $D2$.

$(1) \Leftrightarrow (3)$ $\mathcal{R}$ is a copure semi-simple ring if and only if each copure submodule is a summand if and only if each $\mathcal{R}$-module is copure $D2$.

□

References

[1] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Graduate Texts in Mathematics, Vol. 13(2), Springer-Verlage, New York, 1992.

[2] X. Fu, H. Zhu and N. Ding, On copure projective modules and copure projective dimensions, Comm. Algebra, 40(1), 343-359, 2012.

[3] S. Gupta and A. J. Gupta, Pure-direct-projective modules, Palestine Journal of Mathematics, 2023.

[4] V. A. Hiremath, Copure submodules, Acta Math. Hungar., 44(1-2), 3-12, 1984.

[5] V. A. Hiremath, Generated co-finitely and co-finitely related modules, Acta Math. Hungar., 39(1-3), 1-9, 1982.

[6] V. A. Hiremath, Copure-injective modules, Indian J. Pure Appl. Math., 20(3), 250-259, 1989.

[7] J.P. Jans, On co-Noetherian rings, J. London Math. Soc., 2(1), 588-590, 1969.

[8] S. K. Maurya and S. E. Toksoy, Copure direct injective modules, J. Algebra Appl., 21(9), 2250187, 2022.

[9] W. K. Nicholson, Semiregular modules and rings, Canad. J. Math., 28(5), 1105-1120, 1976.

[10] P. Vamos, The dual of the notion of generated finitely, J. London Math. Soc., 1(1), 643-646, 1968.

[11] P. Vámos, Classical rings, J. Algebra, 34(1), 114-129, 1975.

[12] R. Wisbauer, Foundations of Module and Ring Theory: A handbook for study and research, Gordon and Breach Science Publishers, 1991.

Author information

Kaushal Gupta[1], Sonal Gupta[2] and Ashok Ji Gupta[3], [1,2,3] Department of Mathematical Sciences, IIT(BHU), Varanasi–221005;, India.
E-mail: `kaushal.gupta.rs.mat18@itbhu.ac.in, sonalgupta.rs.mat17@itbhu.ac.in, agupta.apm@itbhu.ac.in`