Palestine Journal of Mathematics

Vol 13(Special Issue II)(2024) , 136–142

On S-Principal Ideal Domain

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Communicated by Manoj Kumar Patel

Dedicated to Prof. B. M. Pandeya on his 78th birthday

MSC 2020 Classifications: Primary 13A15, 13A02; Secondary 13F10, 13E15.

Keywords and phrases: S-PID, S-prime ideal, S-Noetherian ring, S-free module..

The authors grateful thank the anonymous reviewer for his/her valuable suggestions and comments to improve the paper.

Abstract Let A be a commutative integral domain with identity, S a multiplicatively closed subset of A. An ideal I of A is called S-principal if there exist $s \in S$ and $x \in I$ such that $sI \subseteq < x > \subseteq I$. Also, an S-integral domain A is called S-PID if every ideal I of A is S-principal. Using this concept, we give many examples, properties and S-versions of several known results on principal ideal domains. Also, we characterize S-PID in terms of S-prime ideals. Moreover, we investigate structural properties of modules over S-PID.

1 Introduction

Theory of principal ideal domains played an important role in the developement of structure theory of finitely generated abelian groups. This theory has a history extending over more than hundred years. Recall that a commutative integral domain with identity is called a principal ideal domain, or PID, if its each ideal can be generated by a single element. Prominent examples of principal ideal domains include the set of integers $\mathbb{Z}$, polynomial ring $k[X]$ over a field k and the ring of Gaussian integers $\mathbb{Z}[i]$. Several attempts have been made to generalize the concept of principal ideal domain in order to extend its structural properties. In 1988, Hamann et al. [8] introduced the notion of almost principal ideal domain as a generalization of principal ideal domain. Let D be an integral domain with field of fraction K. An ideal I of $D[x]$ is called almost principal if there exist an $f(x) \in I$ of positive degree and a nonzero $s \in D$ such that $sI \subseteq f(x)D[x]$. A polynomial ring $D[x]$ is called an almost principal ideal domain if all ideals of $D[x]$ with proper extensions to $K[x]$ are almost principal. They introduced this notion to study the following questions due to Ratliff, Houston and Arnold:

(i) When is $(ax - b)K[x] \cap D[x]$ generated by linear polynomials?

(ii) When is $f(x)K[x] \cap D[x]$ divisorial?

(iii) When is an ideal I, which is its own extension-contraction from $D[x]$ to $D[[x]]$ and back, equal to $cl(I)$ in the x-adic topology?

In 2002, Anderson and Dumitrescu [1] abstracted the notion of almost principal ideal domain for any commutative integral domain and called it an S-principal ideal domain (briefly, S-PID). They have transferred several results on PID to S-PID. For example, they proved S-version of Cohen-type theorem i.e., a ring A is an S-PID if and only if every prime ideal of A (disjoint from S) is S-principal. Recently, S-version of many special rings and modules has received much attention; see, [1], [4], [5] and [6], for example.

In this paper, we show that S-PID enjoy analogue of many properties of PID. Cohen's theorem for PID is the classic result which states that an integral domain is a PID if and only if its prime ideals are principal. We prove this result for S-PID (Theorem 2.13). It is well known that a submodule of a finitley generated module over a PID is finitely generated ([9, Theorem 6.2.6, Corollary 6.2.10]). We provide an example to show that it is not true for S-PID (Example 2.15). In Theorem 2.14 and Proposition 2.16, we generalize [9, Theorem 6.2.6, Corollary 6.2.10] for

S-PID. It is well known that a finitely generated torsion free module over a PID is free. Finally, in Theorem 2.17, we prove S-version of [9, Theorem 6.2.14].

Throughout the paper, A will be a commutative ring with identity and S be a multiplicatively closed subset of A unless otherwise stated.

2 Properties of S-PID and Modules over S-PID

Let A be a commutative ring and S a multiplicatively closed subset of A. We say that A is an *S-integral domain* if there exists $s \in S$ such that for any $a, b \in A$ with $ab = 0$, we have either $sa = 0$ or $sb = 0$. It is clear from the definition that an integral domain is an S-integral domain but the converse need not be true in general. For this, consider the following example.

Example 2.1. Let $A = \mathbb{Z}[x]/ < 4x >$. Then, clearly A is not an integral domain. Consider multiplicatively closed subset $S = \{2^n + < 4x >: n \in \mathbb{N} \cup 0\}$ of A. If $\overline{f}, \overline{g} \in A$ such that $\overline{f}\overline{g} = 0$. Then $4x | fg$, and so $x | fg$. This implies that either $x|f$ or $x|g$. Put $s = 4+ < 4x > \in S$. Then either $s\overline{f} = 0$ or $s\overline{g} = 0$. Thus, A is an S-integral domain.

Hamed and Malik [7] introduced the concept of S-prime ideals as a generalization of prime ideals as follows: Let A be a ring, $S \subseteq A$ a multiplicatively closed subset and P an ideal of A such that $P \cap S = \emptyset$. Then P is said to be S-prime ideal if there exists an $s \in S$ such that for all $a, b \in A$ with $ab \in P$, we have $sa \in P$ or $sb \in P$. Also, for each multiplicatively closed subset S of A, $S^* = \{r \in A : \frac{r}{1}$ is a unit of $S^{-1}A\}$ is said to be saturation of S. Note that S^* is a multiplicatively closed subset containing S.

In the following result, we include some basic properties of S-integral domains.

Proposition 2.2. *Let A be a ring and S a multiplicatively closed subset of A. Then the following are hold.*

(i) Let S^* be the saturation of S. Then A is an S-integral domain if and only if A is an S^*-integral domain.

(ii) If A is an S-integral domain, then $S^{-1}A$ is an integral domain.

(iii) An ideal P is an S-prime ideal if and only if A/P is an $\overline{S}$-integral domain, where $\overline{S} = \{\overline{s} = s + P : s \in S\}$.

Proof. (i) Suppose A is an S-integral domain. Since $S \subseteq S^*$, so A is an S^*-integral domain. Conversely, suppose A is an S^*-integral domain. Let $a, b \in A$ with $ab = 0$, then there exists $s^* \in S^*$ such that either $s^*a = 0$ or $s^*b = 0$. Since $s^* \in S^*$, $\frac{s^*}{1}$ is a unit in $S^{-1}A$. This implies that there exists $\frac{u}{t_1} \in S^{-1}A$ such that $\frac{s^*}{1}\frac{u}{t_1} = 1$ and so there exists $t \in S$ such that $t(s^*u - t_1) = 0$. Thus, $ts^*u = tt_1$. Now, $s^*a = 0$ or $s^*b = 0$ implies that $tus^*a = 0$ or $tus^*b = 0$. This implies that $tt_1a = 0$ or $tt_1b = 0$. Put $s = tt_1$, then either $sa = 0$ or $sb = 0$. Therefore, A is an S-integral domain.

(ii) Let A be an S-integral domain. Let $\frac{a}{t_1}, \frac{b}{t_2} \in S^{-1}A$ with $\frac{a}{t_1}\frac{b}{t_2} = 0$. Then there exists $s' \in S$ such that $s'ab = 0$. Since A is S-integral domain, there exists $s \in S$ such that $ss'a = 0$ or $sb = 0$. This implies that $\frac{a}{1} = 0$ or $\frac{b}{1} = 0$. Consequently, $\frac{a}{t_1} = 0$ or $\frac{b}{t_2} = 0$. Therefore, $S^{-1}A$ is an integral domain.

(iii) Suppose P is an S-prime ideal of A. Let $a+P, b+P \in A/P$ be such that $(a+P)(b+P) = P$, zero of A/P. Then $ab \in P$. Since P is an S-prime ideal of A, there exists $s \in S$ such that $sa \in P$ or $sb \in P$. This implies that $sa + P = P$ or $sb + P = P$; whence $(s + P)(a + P) = P$ or $(s + P)(b + P) = P$, and so $\overline{s}(a + P) = P$ or $\overline{s}(b + P) = P$. Thus, A/P is an S-integral domain. Conversely, suppose A/P is an S-integral domain. Let $a, b \in A$ with $ab \in P$. Then $ab + P = P$, and so $(a + P)(b + P) = P$. Since A/P is an S-integral domain, there exists $\overline{s} = s + P \in A/P$ such that either $\overline{s}(a + P) = P$ or $\overline{s}(b + P) = P$. This implies that either $sa \in P$ or $sb \in P$. Hence P is an S-prime ideal of A.

□

The concept of S-PID was introduced by Anderson and Dumitrescu [1], where the authors assumed that S-PID is an integral domain. Now, we provide more general definition of S-PID as follows:

Definition 2.3. Let A be a ring, S a multiplicatively closed subset of A and I an ideal of A. We say that I is S-principal if there exists $s \in S$, $a \in I$ such that $sI \subseteq < a >$. Also, an S-integral domain A is said to be an S-PID if each ideal I of A is an S-principal ideal.

It is clear from the definition that a PID is always an S-PID but the converse need not be true. For this consider the following examples.

Example 2.4. Let $A = F[x_1, x_2, \ldots]$ be a polynomial ring in infinitely many indeterminats over a field F. Let $S = A \setminus \{0\}$, then S is a multiplicatively closed subset of A. Let I be any nonzero ideal of A, then $I \cap S \neq \emptyset$. Let $s \in I \cap S$, then $sI \subseteq < s > \subseteq I$. Consequently, I is S-principal. Thus, A is an S-PID which is clearly not a PID.

Example 2.5. Let $A = \mathbb{Z}_6$ and $S = \{\overline{1}, \overline{2}, \overline{4}\}$. Then A is an S-integral domain. Also, every ideal of A is principal, so S-principal (take $s = \overline{1}$). Thus, A is S-PID. Clearly, A is not a PID since it is not an integral domain.

The following result provides a connection between S-PID and PID.

Proposition 2.6. *If A is an S-PID, then $S^{-1}A$ is a PID.*

Proof. Suppose A is an S-PID. Let $S^{-1}I$ be a proper ideal of $S^{-1}A$. Then I is an ideal of A. Since A is an S-PID, there exist $s \in S$ and $a \in I$ such that $sI \subseteq < a >$. This implies that $S^{-1}(sI) \subseteq S^{-1} < a >$, and so $\frac{s}{1}S^{-1}I \subseteq S^{-1} < a >$. Therefore, $S^{-1}I \subseteq S^{-1} < a >$ (since $\frac{s}{1}$ is a unit in $S^{-1}A$). On the other hand, $a \in I$ implies that $S^{-1} < a > \subseteq S^{-1}I$. Thus, $S^{-1}I = S^{-1} < a > = < \frac{a}{1} >$. Therefore, $S^{-1}A$ is a PID. □

In [3], Aqalmoun introduced the concept of S-maximal ideal as a generalization of maximal ideal.

Definition 2.7. [3, Definition 3.1] Let A be a commutative ring, $S \subseteq A$ be a multiplicatively closed subset and P an ideal of A such that $P \cap S = \emptyset$. Then P is said to be an S-maximal ideal of A if there exists an $s \in S$ such that whenever $P \subseteq I$ for some ideal I of A then either $sI \subseteq M$ or $I \cap S \neq \emptyset$.

It is well known that in a PID, each non zero prime ideal is maximal. In the following, we generalize this result for S-PID.

Proposition 2.8. *Let A be an S-PID, then every nonzero S-prime ideal of A is an S-maximal ideal.*

Proof. Let P be a nonzero S-prime ideal of A. Then there exists $s \in S$ such that for any $a, b \in A$ with $ab \in P$, we have $sa \in P$ or $sb \in P$. Let $P \subseteq I$ for some ideal I and $I \cap S = \emptyset$. Then $S^{-1}P \subseteq S^{-1}I$. Since P is an S-prime ideal, by [7, Remark 1(3)], $S^{-1}P$ is a prime ideal of $S^{-1}A$. Also, $I \cap S = \emptyset$ implies that $S^{-1}I$ is a proper ideal of $S^{-1}A$. By Proposition 2.6, $S^{-1}A$ is a PID because A is an S-PID. But then $S^{-1}P$ is a maximal ideal of $S^{-1}A$ which is contained in a proper ideal $S^{-1}I$. Thus $S^{-1}P = S^{-1}I$. Now, let $a \in I$, then $\frac{a}{1} \in S^{-1}I = S^{-1}P$. This implies that there exists $t \in S$ such that $ta \in P$ and so $sI \subseteq P$. Therefore, P is S-maximal ideal of A. □

Proposition 2.9. *Let A be a ring. Then following are hold.*

(i) Let $S_1 \subseteq S_2$ be two multiplicatively closed subset of A. If A is an S_1-PID, then A is an S_2-PID.

(ii) Let S^* be the saturation of S. Then A is an S-PID if and only if A is an S^*-PID.

Proof. 1. Straightforward.

2. Suppose A is an S-PID. Since $S \subseteq S^*$, so by Proposition 2.9 (1), A is an S^*-PID. Conversely, suppose A is an S^*-PID. Let I be an ideal of A, then there exist $s^* \in S$ and $a \in I$ such that $s^*I \subseteq < a >$. Since $s^* \in S^*$, $\frac{s^*}{1}$ is a unit in $S^{-1}A$, and so there exists $\frac{u}{t'} \in S^{-1}A$ such that $\frac{s^*}{1}\frac{u}{t'} = 1$. This implies that there exists $t \in S$ such that $t(s^*u - t') = 0$ and so $ts^*u = tt'$. Now, $s^*I \subseteq < a >$ implies that $ts^*uI \subseteq < a >$ so that $tt'I \subseteq < a >$. Put $s = tt'$, then $s \in S$ such that $sI \subseteq < a >$. Therefore, I is an S-principal ideal of A. Hence A is an S-PID. □

In general, homomorphic image of an S-PID need not be an S-PID. For example, let $A = \mathbb{Z}, B = \mathbb{Z}_m$ and $f : A \to B$ be defined by $f(x) = \overline{x}$, for all $x \in A$. Then, f is a surjective map. Let $S = \{1\}$, then A is an S-PID but $\mathbb{Z}_m$ is not $\overline{S} = \{\overline{1}\}$-PID because $\mathbb{Z}_m$ is not $\overline{S}$-integral domain. However, we have the following:

Proposition 2.10. *Let A be an S-PID and $f : A \to B$ be a surjective homomorphism. If B is an S-integral domain, then B is an S-PID.*

Proof. Let I be an ideal of B. Then $f^{-1}(I)$ is an ideal of A. Since A is S-PID, there exists $s \in S$ such that $sf^{-1}(I) \subseteq < a >$, for some $a \in f^{-1}(I)$. Now, $f(sf^{-1}(I)) \subseteq f(< a >)$ implies that $sf(f^{-1}(I)) \subseteq f(< a >)$ and so $sI \subseteq < b >$, where $b = f(a) \in I$. Therefore, I is an S-principal ideal and so B is an S-PID. □

Anderson and Dumitrescu [1] introduced S-Noetherian rings as a generalization of Noetherian rings. Let A be a commutative ring with identity and S a multiplicatively closed subset of A. Then A is called S-Noetherian if for any ideal I of A, there exist an $s \in S$ and a finitely generated ideal J of A such that $sI \subseteq J \subseteq I$.

Proposition 2.11. *Let A be an S-PID, then A is an S-Noetherian ring.*

Proof. Let I be an ideal of A. Then there exist $s \in S$ and $a \in I$ such that $sI \subseteq < a > \subseteq I$. Consequently, I is S-finite, and so A is an S-Noetherian ring. □

Remark 2.12. Converse of the Proposition 2.11 need not be true. For example, let $A = \mathbb{Z}[x]$ and $S = \{1\}$, then A is an S-Noetherian ring but not an S-PID.

Now we prove S-version of Cohen-type theorem which states that an integral domain A is a PID if and only if each prime ideal of A is principal.

Theorem 2.13. *Let A be an S-integral domain. Then A is an S-PID if and only if every S-prime ideal of A is S-principal.*

Proof. Suppose A is an S-PID. Then every ideal of A is S-principal, in particular, every S-prime ideal of A is S-principal. Conversely, suppose every S-prime ideal of A is S-principal. On contrary, suppose A is not an S-PID. Let X be the set of all non S-principal ideals of A. Then $X \neq \emptyset$. Let $\{I_i\}$ be a chain in X and $I = \cup_i I_i$. If $I \notin X$, then I is an S-principal ideal of A. This implies that there exist an $s \in S$ and $\alpha \in I$ such that $sI \subseteq < \alpha > \subseteq I$. Since $\alpha \in I$, so $\alpha \in I_i$, for some i. This implies that $sI_i \subseteq sI \subseteq < \alpha > \subseteq I_i$. Therefore, I_i is S-principal, which is a contradiction because $I_i \in X$. Thus, $I \in X$ and so every chain in X has an upper bound in X. Then by Zorn's lemma, there exists a maximal element P in X with respect to inclusion. Notice that $P \cap S = \emptyset$. For, if $P \cap S \neq \emptyset$, then there exists $s \in S$ such that $s \in P$. Consequently, $sP \subseteq < s > \subseteq P$, i.e., P is S-principal, which is not possible because $P \in X$. Now, we claim that P is S-prime. If possible, suppose P is not S-prime. Then there exist $a, b \in A$ such that $ab \in P$ but neither $sa \in P$ nor $sb \in P$, for all $s \in S$. Consider the ideal $J = P + < a >$. Then J contains P properly and so $J \notin X$. This implies that J is S-principal and so there exist $s_1 \in S$ and $c \in J$ such that $s_1J \subseteq < c > \subseteq J$. Now since $ab \in P$ and $b \notin P$, $b \in (P : a)$ and $b \notin P$, and so $(P : a)$ contains P properly. Consequently, $(P : a) \notin X$. This implies that $(P : a)$ is S-principal, and so there exist $s_2 \in S$ and $d \in (P : a)$ such that $s_2(P : a) \subseteq < d > \subseteq (P : a)$. Now, let $e \in P$, then $s_1e = uc$, for some $u \in A$. This implies that $u < c > \subseteq P$, and so $s_1uJ \subseteq u < c > \subseteq P$. Consequently, $s_1ua \in P$. This implies that $s_1u \in (P : a)$ so that $s_1s_2u \in s_2(P : a) \subseteq < d >$. Write $s_1s_2u = vd$, for some $v \in A$; whence $s_2s_1^2e = s_1s_2uc = vcd \in < cd >$. Consequently, $s_2s_1^2P \subseteq < cd >$. On the other hand, $d \in (P : a)$ implies that $da \in P$, and so $dJ \subseteq P$; whence

$cd \in P$. Thus, we have $s_2 s_1^2 P \subseteq < cd > \subseteq P$. This implies that P is S-principal, a contradiction since $P \in X$. Consequently, P is S-prime, and so by hypothesis P is an S-principal ideal of A, a contradiction because $P \in X$. Hence, A is an S-PID. □

Now we define the concept of S-free module as a generalization of free modules in order to extend the structure of free modules over S-PID. Let A be a ring, S a multiplicatively closed subset of A and M an A-module. We say that M is *S-free* if there exist $s \in S$ and a free submodule F of M such that $sM \subseteq F$. A basis of F is called an S-basis of M.

The following result is an S-version of [9, Theorem 6.2.6].

Theorem 2.14. *Let A be an S-PID and M be an S-free A-module with S-basis consisting of n elements. Then every nonzero submodule of M is S-free with S-basis containing atmost n elements.*

Proof. Let $B = \{x_1, x_2, \dots, x_n\}$ be an S-basis of M. Then there exists $s_1 \in S$ such that $s_1 M \subseteq F = < x_1, x_2, \dots, x_n > \subseteq M$. We use induction on n to prove this result. Suppose $n = 1$. Then $B = \{x_1\}$ and $s_1 M \subseteq < x_1 > \subseteq M$. Let N be a submodule of M. Define a map $f : A \to Ax_1$ by $f(a) = ax_1$, for all $a \in A$. Then f is an isomorphism. Since $s_1 N$ is a submodule of Ax_1, $f^{-1}(s_1 N)$ is an ideal of A. Consequently, $f^{-1}(s_1 N)$ is an S-principal ideal of A. Since A is an S-PID, there exist $s_2 \in S$ and $a \in A$ such that $s_2 f^{-1}(s_1 N) \subseteq < a > \subseteq f^{-1}(s_1 N)$. This implies that $f(s_2 f^{-1}(s_1 N)) \subseteq < f(a) > \subseteq f(f^{-1}(s_1 N))$; whence $s_2 f f^{-1}(s_1 N) \subseteq < f(a) > \subseteq s_1 N \subseteq N$, and so $s_2 s_1 N \subseteq < f(a) > \subseteq s_1 N \subseteq N$. This implies that N is S-free with S-basis $\{f(a)\}$. Thus, the statement is true for $n = 1$.
Now, suppose the statement of the theorem is true for all S-free modules with S-basis containing m elements, where $m < n$. We need to prove the result for n. Consider the submodule $L = < x_1 > = Ax_1$ and quotient homomorphism $\pi : M \to M/L$ defined by $\pi(x) = x + L$, for all $x \in M$. We claim that $X = \pi(B) = \{\pi(x_2), \dots, \pi(x_n)\}$ is an S-basis of M/L. For this, let $\alpha_2, \alpha_3, \dots, \alpha_n \in A$ such that $\alpha_2\pi(x_2) + \alpha_3\pi(x_3) + \cdots + \alpha_n\pi(x_n) = L$, zero of M/L. This implies that $\pi(\alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_n x_n) = L$; whence $\alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_n x_n + L = L = Ax_1$ which implies that $\alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_n x_n = \alpha_1 x_1$, for some $\alpha_1 \in A$. Since $B = \{x_1, x_2, \dots, x_n\}$ is basis of M, $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Also, since $< B > \subseteq M$, so $< \pi(B) > \subseteq M/L$. Since, $s_1 M \subseteq < x_1, x_2, \dots, x_n >$, so $s_1(M/L) \subseteq < \pi(x_2), \pi(x_3), \dots, \pi(x_n) > \subseteq M/L$. Consequently, M/L is an S-free A-module with S-basis containing $n - 1$ elements. Let N be a nonzero submodule of M. Then $\pi(N)$ is a submodule of M/L. If $\pi(N) = \{L\}$, zero submodule of M/L. Then $\pi(N) = (N + L)/L = \{L\}$, and so $N + L = L$; whence $N \subseteq L = Ax_1$. Then by the case $n = 1$, N is S-free with S-basis containing one element. Suppose $\pi(N) \neq \{L\}$, i.e., $\pi(N)$ is a nonzero submodule of M/L. Then, by second isomorphism theorem $\pi(N) = (N + L)/L \cong N/(L \cap N)$. Since M/L is S-free with S-basis containing $n - 1$ elements, so by induction hypothesis $N/(L \cap N)$ is S-free with S-basis containing atmost $n - 1$ elements. Let $T = \{y_1 + L \cap N, y_2 + L \cap N, \dots, y_r + L \cap N\}$ be an S-basis for $N/(L \cap N)$. Then there exists $s_3 \in S$ such that $s_3(N/(L \cap N)) \subseteq < y_1 + L \cap N, y_2 + L \cap N, \dots, y_r + L \cap N > \subseteq N/(L \cap N)$, where $r \leq n - 1$. Let $\beta_1, \beta_2, \dots, \beta_r \in A$ such that $\beta_1 y_1 + \beta_2 y_2 + \cdots + \beta_r y_r = 0$. This implies that $\beta_1(y_1 + L \cap N) + \beta_2(y_2 + L \cap N) + \cdots + \beta_r(y_r + L \cap N) = L \cap N$, the zero of $N/(L \cap N)$. Since T is linearly independent, so $\beta_i = 0$, for all i. Thus, $T' = \{y_1, y_2, \dots, y_r\}$ is a linearly independent set over A. This implies that $K = < y_1, y_2, \dots, y_r >$ is free with T' as a basis and the map $y_i \to y_i + N \cap L$ is an isomorphism from K to $N/N \cap L$. Let $x \in N$. Then $s_3(x + L \cap N) = a_1(y_1 + L \cap N) + a_2(y_2 + L \cap N) + \cdots + a_r(y_r + L \cap N)$, for some $a_1, a_2, \dots, a_r \in A$. But then $s_3 x - (a_1 y_1 + a_2 y_2 + \cdots + a_r y_r) \in L \cap N$. This implies that $s_3 x = a_1 y_1 + a_2 y_2 + \cdots + a_r y_r + z$, for some $z \in L \cap N$. Consequently, $s_3 x \in K + (L \cap N)$, and so $s_3 N \subseteq K + (L \cap N)$. Now, we show that $K \cap (L \cap N) = \{0\}$. For this, let $w \in K \cap (L \cap N)$. Then, $w + N \cap L = N \cap L$, as $w \in L \cap N$. Then $w = 0$, since the map $w \to w + N \cap L$ is an isomorphism from K to $N/(N \cap L)$. Thus, $K \cap (L \cap N) = \{0\}$. Hence, $s_3 N \subseteq K \oplus (L \cap N) \cong N/(L \cap N) \oplus L \cap N$. Now, since $N/(L \cap N)$ is S-free with S-basis containing atmost $n - 1$ elements and $L \cap N$ is S-free with S-basis containing atmost one element. So, N is S-free with S-basis containing atmost n element, as desired. □

It is well known that every submodule of a finitely generated module over a PID is finitely generated. However, this result is not true for S-PID. For this, we have the following:

Example 2.15. Let $A = F[x_1, x_2, \dots, x_n, \dots]$ be a polynomial ring in infinitely many indeterminates over a field F. Consider a multiplicatively closed subset $S = A \setminus \{0\}$. Then by Example 2.4, A is an S-PID. Consider A as A-module. Then A is a finitely generated module over an S-PID A, but its A-submodule $< x_1, x_2, \dots >$ is not finitely generated.

However for an S-PID, we have the following result:

Proposition 2.16. *Each submodule of a finitely generated module over an S-PID is S-finite.*

Proof. Let A be an S-PID and M be a finitely generated A-module. Then $M = Ax_1 + Ax_2 + \cdots + Ax_n$, for some $x_1, x_2, \dots, x_n \in M$. Consider the map $f : A^n \to M$ defined by $f(a_1, a_2, \dots, a_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$, for all $(a_1, a_2, \dots, a_n) \in A^n$. Then f is a surjective A-module homomorphism. Let N be a submodule of M, then $f^{-1}(N)$ is a submodule of A^n. Since A^n is a free A-module, so A^n is S-free. By Theorem 2.14, $f^{-1}(N)$ is S-free with S-basis containing atmost n elements. Therefore, there exist $s \in S$ and linearly independent elements $y_1, y_2, \dots, y_r \in A^n$ such that $sf^{-1}(N) \subseteq < y_1, y_2, \dots, y_r > \subseteq f^{-1}(N)$, where $r \leq n$. This implies that $sN \subseteq < f(y_1), f(y_2), \dots, f(y_r) > \subseteq N$. Thus, N is S-finite. □

Finally, we generalize [9, Theorem 6.2.14] for S-PID.

Theorem 2.17. *A finitely generated torsion free module over an S-PID is S-free.*

Proof. Let M be a finitely generated torsion free module over an S-PID A. Let $X = \{x_1, \dots, x_n\}$ be a generating set for M and Y be a maximal linearly independent subset of X. Without loss of generality, we may assume $Y = \{x_1, x_2, \dots, x_r\}$, where $r \leq n$ and $x_i \neq 0$, for all $i = 1, 2, \dots, r$. Since M is torsion free, $\{x_i\}$ is linearly independent. This implies that $Y \neq \emptyset$. If $Y = X$, then X is a basis of M and so M is free. Suppose $Y \neq X$. Since Y is maximal linearly independent set, the set $\{x_1, x_2, \dots, x_r, x_{r+i}\}$ is linearly dependent, for $1 \leq i \leq n - r$. This implies that there exist $\alpha_1, \alpha_2, \dots, \alpha_r, \alpha_{r+i} \in A$ such that $\alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_rx_r + \alpha_{r+i}x_{r+i} = 0$. If $\alpha_{r+i} = 0$, then $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$, which is not possible. So $\alpha_{r+i} \neq 0$, for all $i(1 \leq i \leq n - r)$. Put $\alpha = \alpha_{r+1}\alpha_{r+2} \dots \alpha_n$. Then $\alpha \neq 0 \in A$ because M is torsion free. Now, consider the submodule $N = < x_1, x_2, \dots, x_r >$. Then $\alpha_{r+i}x_{r_i} = -(\alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_rx_r) \in N$. This implies that $\alpha x_{r+i} \in N$, for all $i(1 \leq i \leq n - r)$. But, already $\alpha x_i \in N$, for all $i(1 \leq i \leq r)$. Thus, $\alpha x_i \in N$, for all $i(1 \leq i \leq n)$. Consequently, for all $x \in M, \alpha x \in N$. This induces a map $f : M \to N$ defined by $f(x) = \alpha x$, for all $x \in M$. Since M is torsion free, so f is injective. By fundamental theorem of homomorphism, $M \cong f(M)$. Now, since $f(M)$ is a submodule of N and N is S-free so by Theorem 2.14, $f(M)$ is S-free with S-basis containing atmost r elements. Therefore, M is S-free, as desired. □

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