Palestine Journal of Mathematics

Vol 13(Special Issue II)(2024) , 143–154

Additive and Modulo Properties with Binet Formula of $p-$Tribonacci Numbers

Gautam Hathiwala[1], Minakshi Biswas Hathiwala[1] and Devangi Sojitra[1]

Communicated by Mohammad Ashraf

Dedicated to Prof. B. M. Pandeya on his 78th birthday

MSC 2010 Classification: 11B39; 65Q30.

Keywords and phrases: Fibonacci Number, $p-$ Tribonacci Number, Tribonacci Polynomial.

Acknowledgement: Authors would like to thank the anonymous referees for their valuable comments, which helped to improve the research article.

Abstract We discuss some unknown properties of $p-$Tribonacci numbers i.e. the Tribonacci polynomials, as has been defined and studied by B. Rybołowicz and A. Tereszkiewicz [1]. We also derive the Binet formula for $p-$ Tribonacci numbers [1]. Finally we talk about some interesting properties related to the congruence of $p-$ Tribonacci numbers *mod* p^k, for $k = 1, 2, 3$.

1 Introduction

There is a rich history of Fibonacci numbers, defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}; n > 2$, where $F_1 = F_2 = 1$. Many geometrical as well as algebraic properties of Fibonacci numbers have been studied over the last few centuries [2, 3, 4, 5, 6, 7]. S. Falcon and A. Plaza [8] studied the Fibonacci $k-$numbers which are defined as

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2}\,;\ k \geq 1, \text{ provided } F_{k,0} = 0 \ \&\ F_{k,1} = 1 \tag{1.1}$$

S. Falcon and A. Plaza proved several interesting properties of Fibonacci $k-$ numbers using elementary linear algebra, which have applications in 4˘triangle longest-edge (4TLE) partition [8]. Many other interesting properties of Fibonacci $k-$numbers modulo m and derivatives of Fibonacci $k-$numbers have been discussed in [9, 10].

Analogous to definition 1.1, given $T_{p,0} = 0, T_{p,1} = 1 \,\&\, T_{p,2} = p^2$, the $p-$Tribonacci numbers can be defined as

$$T_{p,n+3} = p^2 T_{p,n+2} + pT_{p,n+1} + T_{p,n}\,;\ n \geq 0. \tag{1.2}$$

Here p is a positive integer throughout the discussion, and if we consider x instead of p, we have the Tribonacci polynomials given as

$$T_{n+3}(x) = x^2 T_{n+2}(x) + xT_{n+1}(x) + T_n(x)\,;\ n \geq 0. \tag{1.3}$$

Here, $T_0(x) = 0, T_1(x) = 1 \,\&\, T_2(x) = x^2$.

B. Rybołowicz & A. Tereszkiewicz [1] discussed Tribonacci polynomials and studied many simple yet interesting properties. They discussed properties involving the Binet formula for Tribonacci polynomials and also a few results related to its derivative.

In section 2 we prove some additive properties of $p-$Tribonacci numbers. While proving them, we use concepts of matrix algebra. In section 3 we discuss on the generating function for the $p-$Tribonacci numbers, using which we derive the Binet formula for the same. Then in section 4 we prove some congruence properties of $p-$Tribonacci numbers modulo p^k, for $k = 1, 2, 3$, using induction. We conclude with section 5 making few remarks on some open problems related to $p-$Tribonacci numbers.

2 The $p-$Tribonacci Numbers & their Properties

We have Tribonacci numbers [11], $T_1 = T_2 = 1$, $T_3 = 2$, defined as

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}\,;\, n \geq 4.$$

Several interesting properties related to this sequence have been discussed in [12, 13]. As given by 1.2, $p-$Tribonacci numbers are defined as $T_{p,n+3} = p^2 T_{p,n+2} + pT_{p,n+1} + T_{p,n}\,;\, n \geq 0$, where $T_{p,0} = 0, T_{p,1} = 1 \,\&\, T_{p,2} = p^2$. Correspondingly, by 1.3, the Tribonacci polynomials [1] are given as $T_{n+3}(x) = x^2 T_{n+2}(x) + xT_{n+1}(x) + T_n(x)\,;\, n \geq 0$, where $T_0(x) = 0, T_1(x) = 1 \,\&\, T_2(x) = x^2$. The first few $p-$Tribonacci numbers are given as

$$\begin{aligned}
T_{p,1} &= 1\\
T_{p,2} &= p^2\\
T_{p,3} &= p^4 + p\\
T_{p,4} &= p^6 + 2p^3 + 1\\
T_{p,5} &= p^8 + 3p^5 + 3p^2\\
T_{p,6} &= p^{10} + 4p^7 + 6p^4 + 2p\\
T_{p,7} &= p^{12} + 5p^9 + 10p^6 + 7p^3 + 1
\end{aligned}$$

Lemma 2.1.

$$\sum_{n=1}^{k} T_{p,n} = \frac{1}{p(1+p)}\Big[T_{p,k+1} + (1+p)T_{p,k} + T_{p,k-1} - 1\Big]$$

Proof: We have

$$\begin{aligned}
T_{p,1} &= 1 = T_{p,1}\\
T_{p,2} &= p^2 = p^2 T_{p,1}\\
T_{p,3} &= p^2 T_{p,2} + pT_{p,1} + T_{p,0}\\
T_{p,4} &= p^2 T_{p,3} + pT_{p,2} + T_{p,1}\\
T_{p,5} &= p^2 T_{p,4} + pT_{p,3} + T_{p,2}\\
&\vdots\\
T_{p,k-1} &= p^2 T_{p,k-2} + pT_{p,k-3} + T_{p,k-4}\\
T_{p,k} &= p^2 T_{p,k-1} + pT_{p,k-2} + T_{p,k-3}
\end{aligned}$$

Adding the above equations, we have

$$\sum_{n=1}^{k} T_{p,n} = 1 + p^2 \sum_{n=1}^{k-1} T_{p,n} + p \sum_{n=1}^{k-2} T_{p,n} + \sum_{n=0}^{k-3} T_{p,n}$$

$$\therefore \sum_{n=1}^{k} T_{p,n} = 1 + p^2 \sum_{n=1}^{k} T_{p,n} - p^2 T_{p,k} + p \sum_{n=1}^{k} T_{p,n} - pT_{p,k-1} - pT_{p,k} + \sum_{n=1}^{k} T_{p,n} - T_{p,k-2} - T_{p,k-1} - T_{p,k}.$$

By simple evaluations we get

$$-p(1+p) \sum_{n=1}^{k} T_{p,n} = 1 - [T_{p,k+1} + (1+p)T_{p,k} + T_{p,k-1}]$$

$$\therefore \sum_{n=1}^{k} T_{p,n} = \frac{1}{p(1+p)}[T_{p,k+1} + (1+p)T_{p,k} + T_{p,k-1} - 1]. \ \square$$

We can verify the above result by considering $k = 9$.

$$\sum_{n=1}^{9} T_{p,n} = p^{16} + p^{14} + 7p^{13} + p^{12} + 6p^{11} + 22p^{10} + 5p^{9} + 16p^{8} + 34p^{7} + 11p^{6} + 19p^{5} + 26p^{4} + 9p^{3} + 10p^{2} + 6p + 3.$$

$$\frac{1}{p(1+p)}[T_{p,10} + (1+p)T_{p,9} + T_{p,8} - 1] = \frac{1}{p(1+p)}[p^{18} + p^{17} + p^{16} + 8p^{15} + 8p^{14} + 7p^{13} + 28p^{12} + 27p^{11} + 21p^{10} + 50p^{9} + 45p^{8} + 30p^{7} + 45p^{6} + 35p^{5} + 19p^{4} + 16p^{3} + 9p^{2} + 3p].$$

On simplification, it is observed that

$$\frac{1}{p(1+p)}\left[T_{p,10} + (1+p)T_{p,9} + T_{p,8} - 1\right] = \sum_{n=1}^{9} T_{p,n}.$$

Lemma 2.2.

$$(i) \sum_{n=1}^{k} T_{p,2n} = \frac{1}{p(1+p)\left(p^2-p+2\right)}\left[\left(1+p^2\right)T_{p,2k+1} + (1+p)T_{p,2k} + (1-p)T_{p,2k-1}\right] - \frac{\left(1+p^2\right)}{p(1+p)\left(p^2-p+2\right)}.$$

$$(ii) \sum_{n=1}^{k} T_{p,2n-1} = \frac{1}{p(1+p)\left(p^2-p+2\right)}\left[(1-p)T_{p,2k+1} + \left(1+p^3\right)T_{p,2k} + \left(1+p^2\right)T_{p,2k-1}\right] - \frac{(1-p)}{p\left(1+p\right)\left(p^2-p+2\right)}.$$

Proof:
(i)

$$\begin{aligned}
T_{p,2} &= p^2 T_{p,1} \\
T_{p,4} &= p^2 T_{p,3} + pT_{p,2} + T_{p,1} \\
T_{p,6} &= p^2 T_{p,5} + pT_{p,4} + T_{p,3} \\
T_{p,8} &= p^2 T_{p,7} + pT_{p,6} + T_{p,5} \\
&\vdots \\
T_{p,2k-2} &= p^2 T_{p,2k-3} + pT_{p,2k-4} + T_{p,2k-5} \\
T_{p,2k} &= p^2 T_{p,2k-1} + pT_{p,2k-2} + T_{p,2k-3}
\end{aligned}$$

Adding all the above equalities, we have

$$\sum_{n=1}^{k} T_{p,2n} = p^2 \sum_{n=1}^{k} T_{p,2n-1} + p\sum_{n=1}^{k-1} T_{p,2n} + \sum_{n=1}^{k-1} T_{p,2n-1}$$

$$\therefore (1-p)\sum_{n=1}^{k} T_{p,2n} = \left(1+p^2\right)\sum_{n=1}^{k} T_{p,2n-1} - T_{p,2k-1} - pT_{p,2k}$$

$$\therefore \left(2-p+p^2\right)\sum_{n=1}^{k} T_{p,2n} = \left(1+p^2\right)\sum_{n=1}^{2k} T_{p,n} - T_{p,2k-1} - pT_{p,2k}$$

By lemma 2.1, we have

$$\left(2-p+p^2\right)\sum_{n=1}^{k} T_{p,2n} = \left(1+p^2\right)\left[\frac{1}{p(1+p)}[T_{p,2k+1}+(1+p)T_{p,2k}+T_{p,2k-1}-1]\right]-T_{p,2k-1}$$
$$-pT_{p,2k}.$$

On simplification we have

$$\sum_{n=1}^{k} T_{p,2n} = \frac{1}{p(1+p)\left(p^2-p+2\right)}\left[\left(1+p^2\right)T_{p,2k+1}+(1+p)T_{p,2k}+(1-p)T_{p,2k-1}\right]$$
$$-\frac{\left(1+p^2\right)}{p(1+p)\left(p^2-p+2\right)}.$$

(ii) We write

$$\begin{aligned}
T_{p,1} &= 1\\
T_{p,3} &= p^2T_{p,2}+pT_{p,1}+T_{p,0}\\
T_{p,5} &= p^2T_{p,4}+pT_{p,3}+T_{p,2}\\
T_{p,7} &= p^2T_{p,6}+pT_{p,5}+T_{p,4}\\
&\vdots\\
T_{p,2k-3} &= p^2T_{p,2k-4}+pT_{p,2k-5}+T_{p,2k-6}\\
T_{p,2k-1} &= p^2T_{p,2k-2}+pT_{p,2k-3}+T_{p,2k-4}
\end{aligned}$$

Therefore, adding we get

$$\sum_{n=1}^{k} T_{p,2n-1} = 1+p^2\sum_{n=1}^{k-1} T_{p,2n}+p\sum_{n=1}^{k-1} T_{p,2n-1}+\sum_{n=0}^{k-2} T_{p,2n}$$

$$\therefore (1-p)\sum_{n=1}^{k} T_{p,2n-1} = 1-p^2T_{p,2k}-T_{p,2k-2}-T_{p,2k}-pT_{p,2k-1}+\left(1+p^2\right)\sum_{n=1}^{k} T_{p,2n}.$$

$$\therefore (1-p)\sum_{n=1}^{k} T_{p,2n-1} = 1-p^2T_{p,2k}-T_{p,2k-2}-T_{p,2k}-pT_{p,2k-1}+\left(1+p^2\right)$$
$$\left[\frac{1}{p(1+p)\left(p^2-p+2\right)}\left[\left(1+p^2\right)T_{p,2k+1}+(1+p)T_{p,2k}+(1-p)T_{p,2k-1}\right]-\right.$$
$$\left.\frac{\left(1+p^2\right)}{p(1+p)\left(p^2-p+2\right)}\right].$$

$$\therefore (1-p)\sum_{n=1}^{k} T_{p,2n-1} = 1-T_{p,2k+1}-T_{p,2k}+\frac{(1+p^2)^2}{p(1+p)(p^2-p+2)}T_{p,2k+1}$$
$$+\frac{(1+p)(1+p^2)}{p(1+p)(p^2-p+2)}T_{p,2k}+\frac{(1-p)(1+p^2)}{p(1+p^2)(p^2-p+2)}T_{p,2k-1}-\frac{(1+p^2)^2}{p(1+p)(p^2-p+2)}$$

$$\therefore (1-p)\sum_{n=1}^{k} T_{p,2n-1} = \left[\frac{(1+p^2)^2-p(1+p)(p^2-p+2)}{p(1+p)(p^2-p+2)}\right]T_{p,2k+1}$$
$$+\left[\frac{(1+p)(1+p^2)-p(1+p)(p^2-p+2)}{p(1+p)(p^2-p+2)}\right]T_{p,2k}$$
$$+\frac{(1-p)(1+p^2)}{p(1+p)(p^2-p+2)}T_{p,2k-1}+\frac{p(1+p)(p^2-p+2)-(1+p^2)^2}{p(1+p)(p^2-p+2)}$$

On simplification, we get the required result. □

Lemma 2.3.

$(i) \sum_{n=1}^{k} T_{p,3n} = \frac{1}{p(1+p^3)} \left[p^2 T_{p,3k+1} + T_{p,3k-1}\right] - \frac{p}{(1+p^3)}.$

$(ii) \sum_{n=1}^{k} T_{p,3n-1} = \frac{1}{p(1+p^3)} \left[T_{p,3k+1} - pT_{p,3k-1}\right] - \frac{1}{p(1+p^3)}.$

$(iii) \sum_{n=1}^{k} T_{p,3n-2} = \frac{1}{p(1+p^3)} \left[-pT_{p,3k+1} + \left(1+p^3\right) T_{p,3k} + p^2 T_{p,3k-1}\right] + \frac{1}{1+p^3}.$

Proof: In order to prove these results, we make use of some simple matrix algebra.
Now, we write

$$
\begin{aligned}
T_{p,3} &= p^2 T_{p,2} + pT_{p,1} + T_{p,0} \\
T_{p,6} &= p^2 T_{p,5} + pT_{p,4} + T_{p,3} \\
T_{p,9} &= p^2 T_{p,8} + pT_{p,7} + T_{p,6} \\
T_{p,12} &= p^2 T_{p,11} + pT_{p,10} + T_{p,9} \\
&\vdots \\
T_{p,3k-3} &= p^2 T_{p,3k-4} + pT_{p,3k-5} + T_{p,3k-6} \\
T_{p,3k} &= p^2 T_{p,3k-1} + pT_{p,3k-2} + T_{p,3k-3}
\end{aligned}
$$

Therefore, adding them all, we get

$$\sum_{n=1}^{k} T_{p,3n} = p^2 \sum_{n=1}^{k} T_{p,3n-1} + p \sum_{n=1}^{k} T_{p,3n-2} + \sum_{n=0}^{k-1} T_{p,3n}$$

$$\therefore p \sum_{n=1}^{k} T_{p,3n-2} + p^2 \sum_{n=1}^{k} T_{p,3n-1} + 0. \sum_{n=1}^{k} T_{p,3n} = T_{p,3k} \tag{2.1}$$

Again,

$$
\begin{aligned}
T_{p,2} &= p^2 = p^2 T_{p,1} \\
T_{p,5} &= p^2 T_{p,4} + pT_{p,3} + T_{p,2} \\
T_{p,8} &= p^2 T_{p,7} + pT_{p,6} + T_{p,5} \\
T_{p,11} &= p^2 T_{p,10} + pT_{p,9} + T_{p,8} \\
&\vdots \\
T_{p,3k-4} &= p^2 T_{p,3k-5} + pT_{p,3k-6} + T_{p,3k-7} \\
T_{p,3k-1} &= p^2 T_{p,3k-2} + pT_{p,3k-3} + T_{p,3k-4}
\end{aligned}
$$

Therefore, addition gives

$$\sum_{n=1}^{k} T_{p,3n-1} = p^2 \sum_{n=1}^{k} T_{p,3n-2} + p \sum_{n=1}^{k-1} T_{p,3n} + \sum_{n=1}^{k-1} T_{p,3n-1}$$

$$\therefore \sum_{n=1}^{k} T_{p,3n-1} = p^2 \sum_{n=1}^{k} T_{p,3n-2} + p \sum_{n=1}^{k} T_{p,3n} - pT_{p,3k} + \sum_{n=1}^{k} T_{p,3n-1} - T_{p,3k-1}$$

$$\therefore p^2 \sum_{n=1}^{k} T_{p,3n-2} + 0. \sum_{n=1}^{k} T_{p,3n-1} + p \sum_{n=1}^{k} T_{p,3n} = T_{p,3k-1} + pT_{p,3k} \tag{2.2}$$

Lastly,

$$T_{p,1} = 1$$
$$T_{p,4} = p^2 T_{p,3} + pT_{p,2} + T_{p,1}$$
$$T_{p,7} = p^2 T_{p,6} + pT_{p,5} + T_{p,4}$$
$$T_{p,10} = p^2 T_{p,9} + pT_{p,8} + T_{p,7}$$
$$\vdots$$
$$T_{p,3k-5} = p^2 T_{p,3k-6} + pT_{p,3k-7} + T_{p,3k-8}$$
$$T_{p,3k-2} = p^2 T_{p,3k-3} + pT_{p,3k-4} + T_{p,3k-5}$$

Therefore, adding we have

$$\sum_{n=1}^{k} T_{p,3n-2} = 1 + p^2 \sum_{n=1}^{k-1} T_{p,3n} + p \sum_{n=1}^{k-1} T_{p,3n-1} + \sum_{n=1}^{k-1} T_{p,3n-2}$$
$$\therefore \sum_{n=1}^{k} T_{p,3n-2} = 1 + p^2 \sum_{n=1}^{k} T_{p,3n} + p \sum_{n=1}^{k} T_{p,3n-1} + \sum_{n=1}^{k} T_{p,3n-2} - p^2 T_{p,3k} - pT_{p,3k-1} - T_{p,3k-2}$$
$$\therefore 0. \sum_{n=1}^{k} T_{p,3n-2} + p \sum_{n=1}^{k} T_{p,3n-1} + p^2 \sum_{n=1}^{k} T_{p,3n} = T_{p,3k+1} - 1 \tag{2.3}$$

Thus, we can consider equations 2.1, 2.2, 2.3 as linear equations in unknowns $\sum_{n=1}^{k} T_{p,3n-2}$, $\sum_{n=1}^{k} T_{p,3n-1}$ & $\sum_{n=1}^{k} T_{p,3n}$.

Writing this linear system of equations in matrix form, we have

$$\begin{bmatrix} p & p^2 & 0 \\ p^2 & 0 & p \\ 0 & p & p^2 \end{bmatrix} \begin{bmatrix} \sum_{n=1}^{k} T_{p,3n-2} \\ \sum_{n=1}^{k} T_{p,3n-1} \\ \sum_{n=1}^{k} T_{p,3n} \end{bmatrix} = \begin{bmatrix} T_{p,3k} \\ T_{p,3k-1} + pT_{p,3k} \\ T_{p,3k+1} - 1 \end{bmatrix}$$

Therefore, we solve the augmented matrix $\left[\begin{array}{ccc|c} p & p^2 & 0 & T_{p,3k} \\ p^2 & 0 & p & T_{p,3k-1} + pT_{p,3k} \\ 0 & p & p^2 & T_{p,3k+1} - 1 \end{array}\right]$.
Applying elementary row operations and converting above matrix system into row-echelon form, we get

$$\left[\begin{array}{ccc|c} p & p^2 & 0 & T_{p,3k} \\ 0 & p & p^2 & T_{p,3k+1} - 1 \\ 0 & 0 & p + p^4 & p^2 T_{p,3k+1} + T_{p,3k-1} - p^2 \end{array}\right].$$

$$\therefore \left(p + p^4\right) \sum_{n=1}^{k} T_{p,3n} = p^2 T_{p,3k+1} + T_{p,3k-1} - p^2$$
$$\therefore \sum_{n=1}^{k} T_{p,3n} = \frac{1}{p\left(1+p^3\right)} \left[p^2 T_{p,3k+1} + T_{p,3k-1}\right] - \frac{p}{\left(1+p^3\right)}.$$

By back substitution and simplification, we get the required identities. □

The above results can also be proved or verified using the principle of mathematical induction.

3 The Binet Formula for $p-$Tribonacci Numbers

B. Rybołowicz & A. Tereszkiewicz [1] have discussed the Binet formula for the generalized Tribonacci polynomials. They have expressed it in terms of the roots of the characteristic equation for generalized Tribonacci polynomials. Here we particularly discuss the Binet formula for $p-$Tribonacci numbers defined by 1.2 for the given initial terms. In fact we obtain the Binet formula for $p-$Tribonacci numbers by detailed discussion regarding the characteristic equation for the same.
Now, let $P(x)$ be the generating function for the sequence $\{T_{p,n}\}_{n=0}^{\infty}$.
Therefore, we can write

$$P(x) = \sum_{n=0}^{\infty} T_{p,n}x^n = T_{p,0} + T_{p,1}x + T_{p,2}x^2 + T_{p,3}x^3 + \ldots + T_{p,k}x^k + \ldots$$

$$p^2xP(x) = p^2\sum_{n=0}^{\infty} T_{p,n}x^{n+1} = p^2T_{p,0}x + p^2T_{p,1}x^2 + p^2T_{p,2}x^3 + p^2T_{p,3}x^4 + \ldots + p^2T_{p,k-1}x^k + p^2T_{p,k}x^{k+1}\ldots$$

$$px^2P(x) = p\sum_{n=0}^{\infty} T_{p,n}x^{n+2} = pT_{p,0}x^2 + pT_{p,1}x^3 + pT_{p,2}x^4 + pT_{p,3}x^5 + \ldots + pT_{p,k-2}x^k + pT_{p,k-1}x^{k+1} + pT_{p,k}x^{k+2} + \ldots$$

$$x^3P(x) = \sum_{n=0}^{\infty} T_{p,n}x^{n+3} = T_{p,0}x^3 + T_{p,1}x^4 + T_{p,2}x^5 + T_{p,3}x^6 + \ldots + T_{p,k-3}x^k + T_{p,k-2}x^{k+1} + T_{p,k-1}x^{k+2} + T_{p,k}x^{k+3} + \ldots$$

$$\begin{aligned}\therefore P(x) - x^3P(x) - px^2P(x) - p^2xP(x) = (&T_{p,0} + T_{p,1}x + T_{p,2}x^2 + T_{p,3}x^3 + \ldots \\ &+ T_{p,k}x^k + \ldots) - (T_{p,0}x^3 + T_{p,1}x^4 + T_{p,2}x^5 + T_{p,3}x^6 + \ldots + T_{p,k-3}x^k \\ &+ T_{p,k-2}x^{k+1} + T_{p,k-1}x^{k+2} + T_{p,k}x^{k+3} + \ldots) - (pT_{p,0}x^2 + pT_{p,1}x^3 + pT_{p,2}x^4 \\ &+ pT_{p,3}x^5 + \ldots + pT_{p,k-2}x^k + pT_{p,k-1}x^{k+1} + pT_{p,k}x^{k+2} + \ldots) \\ &- (p^2T_{p,0}x + p^2T_{p,1}x^2 + p^2T_{p,2}x^3 + p^2T_{p,3}x^4 + \ldots + p^2T_{p,k-1}x^k + p^2T_{p,k}x^{k+1}\ldots).\end{aligned}$$

$$\begin{aligned}\therefore (1 - p^2x - px^2 - x^3)P(x) = 0 + x + p^2x^2 - p^2x^2 + (T_{p,3} - T_{p,0} - pT_{p,1} - p^2T_{p,2})x^3 \\ + (T_{p,4} - T_{p,1} - pT_{p,2} - p^2T_{p,3})x^4 + \ldots + (T_{p,k} - T_{p,k-3} - pT_{p,k-2} \\ - p^2T_{p,k-1})x^k + \ldots\end{aligned}$$

$$\therefore P(x) = \frac{x}{(1 - p^2x - px^2 - x^3)}.$$

Lemma 3.1. *The generating function for $p-$Tribonacci numbers is*

$$P(x) = \frac{x}{(1 - p^2x - px^2 - x^3)}.$$

Mehta D.A. [11], discussed extensively the roots of the characteristic equation for Tribonacci numbers defined in section 2. Here we obtain the roots of the characteristic equation for $p-$Tribonacci numbers and hence derive the Binet formula for the same in terms of these roots.

Suppose $\left(1 - p^2x - px^2 - x^3\right) = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)$, where $\alpha,\ \beta,\ \gamma$ are in terms of the parameter p. Clearly $\frac{1}{\alpha},\ \frac{1}{\beta}\ \& \frac{1}{\gamma}$ are the roots of (say) $q(x) = 1 - p^2x - px^2 - x^3$. Therefore, $\alpha,\ \beta,\ \&\ \gamma$ are the roots of $q\left(\frac{1}{x}\right) = 1 - p^2\frac{1}{x} - p\frac{1}{x^2} - \frac{1}{x^3}$. Therefore, equating to zero, we can say that $\alpha,\ \beta,\ \&\ \gamma$ are the roots of

$$x^3 - p^2x^2 - px - 1 = 0. \tag{3.1}$$

Equation 3.1 is the characteristic equation for $p-$Tribonacci numbers. It should be observed that 3.1 is an equation in one variable 'x' with 'p' as the parameter.
The solutions of 3.1 are obtained using WolframAlpha [14] and are given as

$$\alpha = \frac{p^2}{3} + \frac{\sqrt[3]{2p^6+9p^3+3\sqrt{3}\sqrt{3p^6+14p^3+27}+27}}{3\sqrt[3]{2}} + \frac{\sqrt[3]{2}p\left(p^3+3\right)}{3\sqrt[3]{2p^6+9p^3+3\sqrt{3}\sqrt{3p^6+14p^3+27}+27}}$$

$$\beta = \frac{p^2}{3} - \frac{\left(1-\sqrt{3}i\right)\sqrt[3]{2p^6+9p^3+3\sqrt{3}\sqrt{3p^6+14p^3+27}+27}}{6\sqrt[3]{2}} - \frac{\left(1+\sqrt{3}i\right)p\left(p^3+3\right)}{3\times 2^{\frac{2}{3}}\sqrt[3]{2p^6+9p^3+3\sqrt{3}\sqrt{3p^6+14p^3+27}+27}}$$

$$\gamma = \frac{p^2}{3} - \frac{\left(1+\sqrt{3}i\right)\sqrt[3]{2p^6+9p^3+3\sqrt{3}\sqrt{3p^6+14p^3+27}+27}}{6\sqrt[3]{2}} - \frac{\left(1-\sqrt{3}i\right)p\left(p^3+3\right)}{3\times 2^{\frac{2}{3}}\sqrt[3]{2p^6+9p^3+3\sqrt{3}\sqrt{3p^6+14p^3+27}+27}}$$

If we assume $\omega = \sqrt[3]{2p^6+9p^3+3\sqrt{3}\sqrt{3p^6+14p^3+27}+27}$, then above roots can be written as

$$\left.\begin{aligned} \alpha &= \frac{p^2}{3} + \frac{\omega}{3\sqrt[3]{2}} + \frac{\sqrt[3]{2}p\left(p^3+3\right)}{3\omega} \\ \beta &= \frac{p^2}{3} - \frac{\left(1-\sqrt{3}i\right)\omega}{6\sqrt[3]{2}} - \frac{\left(1+\sqrt{3}i\right)p\left(p^3+3\right)}{3\times 2^{\frac{2}{3}}\omega} \\ \gamma &= \frac{p^2}{3} - \frac{\left(1+\sqrt{3}i\right)\omega}{6\sqrt[3]{2}} - \frac{\left(1-\sqrt{3}i\right)p\left(p^3+3\right)}{3\times 2^{\frac{2}{3}}\omega} \end{aligned}\right\} \tag{3.2}$$

It should be observed that $\alpha+\beta+\gamma = p^2$ & $\alpha\beta\gamma = 1$.
Coming to generating function $P(x) = \frac{x}{(1-p^2x-px^2-x^3)}$, we can write

$$\begin{aligned} P(x) &= \frac{x}{(1-p^2x-px^2-x^3)} = \frac{x}{(1-\alpha x)(1-\beta x)(1-\gamma x)} \\ &= \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} + \frac{C}{1-\gamma x} \end{aligned}$$

$$\therefore P(x) = A(1-\beta x)(1-\gamma x) + B(1-\alpha x)(1-\gamma x) + C(1-\alpha x)(1-\beta x)$$

Therefore by taking $x = \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$, we get the values $A = \frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}$, $B = \frac{\beta}{(\beta-\alpha)(\beta-\gamma)}$, $C = \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}$, respectively.

$$\begin{aligned}\therefore P(x) =& \frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}.\frac{1}{1-\alpha x}+\frac{\beta}{(\beta-\alpha)(\beta-\gamma)}.\frac{1}{1-\beta x}\\ &+\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}.\frac{1}{1-\gamma x}\\ =&\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}\sum_{n=0}^{\infty}\alpha^n x^n+\frac{\beta}{(\beta-\alpha)(\beta-\gamma)}\sum_{n=0}^{\infty}\beta^n x^n\\ &+\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\sum_{n=0}^{\infty}\gamma^n x^n\\ =&\sum_{n=0}^{\infty}\left(\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}\right)x^n\end{aligned}$$

But $P(x)=\sum_{n=0}^{\infty}T_{p,n}x^n$.
Therefore, on comparison we have

$$T_{p,n}=\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus we have the following result.

Theorem 3.2. *The Binet formula for the $p-$ Tribonacci numbers is given as*

$$T_{p,n}=\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)};$$

where $\alpha,\ \beta,\ \gamma$ are given by 3.2.

4 $p-$Tribonacci Numbers Modulo p^k, for $k=1,2,3.$

B. Rybołowicz & A. Tereszkiewicz [1] have discussed about the degree of generalized Tribonacci polynomials. We can say that for $p-$Tribonacci numbers, which can be considered as polynomials in parameter 'p', we have deg $(T_{p,n})=2(n-1)$, $n>0$. This result is easily established by the principle of mathematical induction. We prove some basic and important results for $p-$Tribonacci numbers modulo p, using congruence relation [15].

Lemma 4.1.

$$\begin{aligned}T_{p,3n-2}&\equiv 1\ \ (mod\,p)\\ T_{p,3n-1}&\equiv 0\ \ (mod\,p)\\ T_{p,3n}&\equiv 0\ \ (mod\,p)\end{aligned}$$

Proof: These results can be proved by mathematical induction.
We prove $T_{p,3n}\equiv 0\ \ (mod\,p)$.
Let $n=1$. Therefore, $T_{p,3(1)}=T_{p,3}=p^2T_{p,2}+pT_{p,1}+T_{p,0}=p^4+p\equiv 0\ \ (mod\,p)$.
Hence the result is true for $n=1$. Assume this is true for $n=k$ i.e., we have $T_{p,3k}\equiv 0\ \ (mod\,p)$. Let $n=k+1$. Therefore, $T_{p,3(k+1)}=T_{p,3k+3}=p^2T_{p,3k+2}+pT_{p,3k+1}+T_{p,3k}$. This gives, $T_{p,3k+3}\equiv T_{p,3k}\ \ (mod\,p)$. By induction hypothesis, we have $T_{p,3k}\equiv 0\ \ (mod\,p)$. Using this we get $T_{p,3(k+1)}\equiv 0\ \ (mod\,p)$. Hence result is true for all $n\geq 1$.
The remaining results can be proved in a similar manner. □

Corollary 4.2.

$$\begin{aligned}\sum_{n=1}^{3k}T_{p,n}&\equiv k\ \ (mod\,p)\\ \sum_{n=1}^{3p}T_{p,n}&\equiv 0\ \ (mod\,p)\end{aligned}$$

Proof: Adding the three results of lemma 4.1, we have

$$T_{p,3n-2} + T_{p,3n-1} + T_{p,3n} \equiv 1\,(mod\, p)$$

Taking summation from 1 to k on both sides, we get by properties of modulo function and on simplification, $\sum_{n=1}^{3k} T_{p,n} \equiv k\,(mod\, p)$.
Moreover, considering $k = p$ in the above result, we get

$$\sum_{n=1}^{3p} T_{p,n} \equiv p \equiv 0\,(mod\, p)$$

Hence proved. □

Lemma 4.3.

$$T_{p,3n-2} \equiv 1 \ \left(mod\, p^2\right)$$
$$T_{p,3n-1} \equiv 0 \ \left(mod\, p^2\right)$$
$$T_{p,3n} \equiv np \ \left(mod\, p^2\right)$$

Proof: We prove $T_{p,3n-2} \equiv 1 \ \left(mod\, p^2\right)$ by induction on n.
For $n = 1$, we have $T_{p,3(1)-2} = T_{p,1} = 1 \equiv 1 \ \left(mod\, p^2\right)$. Hence the result is true for $n = 1$. Assume that it is true for $n = k$ i.e., we have $T_{p,3k-2} \equiv 1 \ \left(mod\, p^2\right)$. Now let $n = k + 1$. This gives
$T_{p,3(k+1)-2} = T_{p,3k+1} = p^2 T_{p,3k} + pT_{p,3k-1} + T_{p,3k-2} \equiv pT_{p,3k-1} + T_{p,3k-2} \ \left(mod\, p^2\right)$. By lemma 4.1, we have $T_{p,3k-1} \equiv 0 \ (mod\, p) \Rightarrow pT_{p,3k-1} \equiv 0 \ \left(mod\, p^2\right)$. Using this, we have

$$T_{p,3(k+1)-2} = T_{p,3k+1} \equiv T_{p,3k-2} \ \left(mod\, p^2\right)$$

By induction hypothesis, we have $T_{p,3k-2} \equiv 1 \ \left(mod\, p^2\right) \Rightarrow T_{p,3(k+1)-2} = T_{p,3k+1} \equiv 1 \ \left(mod\, p^2\right)$.
Therefore the result is true for $n = k + 1$, and hence for all $n \geq 1$.
The remaining results are proved in a similar manner. □

Corollary 4.4.

$$\sum_{n=1}^{3k} T_{p,n} \equiv k(1 + np) \ \left(mod\, p^2\right)$$
$$\sum_{n=1}^{3p} T_{p,n} \equiv p(1 + np) \equiv p \ \left(mod\, p^2\right)$$

Lemma 4.5.

$$T_{p,3n-2} \equiv 1 \ \left(mod\, p^3\right)$$
$$T_{p,3n-1} \equiv \frac{n(n+1)}{2} p^2 \ \left(mod\, p^3\right)$$
$$T_{p,3n} \equiv np \ \left(mod\, p^3\right)$$

Proof: We prove $T_{p,3n-1} \equiv \frac{n(n+1)}{2} p^2 \ \left(mod\, p^3\right)$ by using induction on n. The remaining results follow similarly.
For $n = 1$, $T_{p,3(1)-1} = T_{p,2} = p^2 \equiv \frac{1(1+1)}{2} p^2 \ \left(mod\, p^3\right)$. Hence the result is true for $n = 1$. Let it be true for $n = k$. Thus, we have $T_{p,3k-1} \equiv \frac{k(k+1)}{2} p^2 \ \left(mod\, p^3\right)$. Let $n = k + 1$. This gives $T_{p,3(k+1)-1} = T_{p,3k+2} = p^2 T_{p,3k+1} + pT_{p,3k} + T_{p,3k-1}$.
Now, by lemma 4.1, we have $T_{p,3k+1} \equiv 1 \ (mod\, p) \Rightarrow p^2 T_{p,3k+1} \equiv p^2 \ \left(mod\, p^3\right)$.
Also, by lemma 4.3, we have $T_{p,3k} \equiv kp \ \left(mod\, p^2\right) \Rightarrow pT_{p,3k} \equiv kp^2 \ \left(mod\, p^3\right)$.

Therefore, $T_{p,3(k+1)-1} = T_{p,3k+2} \equiv p^2 + kp^2 + T_{p,3k-1} \pmod{p^3}$. Moreover, by induction hypothesis, we get

$$\begin{aligned} T_{p,3(k+1)-1} = T_{p,3k+2} &\equiv p^2 + kp^2 + \frac{k(k+1)}{2}p^2 \pmod{p^3} \\ &\equiv \left(1 + k + \frac{k(k+1)}{2}\right) p^2 (mod\, p^3) \\ &\equiv \frac{(k+1)\left((k+1)+1\right)}{2} (mod\, p^3). \end{aligned}$$

Thus the result is true for $n = k+1$, and hence for all $n \geq 1$. □

Corollary 4.6.

$$\sum_{n=1}^{3k} T_{p,n} \equiv k + \frac{kn(n+1)}{2}p^2 + knp \pmod{p^3}$$

$$\sum_{n=1}^{3p} T_{p,n} \equiv p(1+np) \pmod{p^3}$$

5 Conclusion

We have derived many interesting relations related to the finite addition of $p-$ Tribonacci numbers. We also derived congruence properties for $p-$ Tribonacci numbers modulo p^r and talked about an important identity called the Binet formula. Many more interesting results can be derived using the Binet Formula, like, the identity for $(n-1)^{th}$ or $(n+1)^{th}$ term in the $p-$ Tribonacci sequence, provided we already know the n^{th} term. Also, we can obtain results pertaining to the congruence of $p-$ Tribonacci numbers *mod* p^r, for $r > 3$.

References

[1] B. Rybołowicz and A. Tereszkiewicz, Generalized tricobsthal and generalized tribonacci polynomials, Applied Mathematics and Computation, vol. 325, pp. 297–308, 2018.

[2] P. Singh, The So-called Fibonacci Numbers in Ancient and Medieval India, Historia Mathematica, vol. 12, pp. 229–244, 1985.

[3] W. R. Spickerman and R. N. Joyner, “Binet’s formula for the recursive sequence of order k,” Fibonacci Quarterly, vol. 22, pp. 327–331, 1984.

[4] H. Kim and J. Neggers, Fibonacci means and golden section mean, International Journal Computers & Mathematics with Applications, vol. 56, pp. 228–232, 2008.

[5] G. P. Dresden and Z. Du, A simplified Binet formula for k- generalized Fibonacci numbers, Journal of Integer Sequences, vol. 17, 2014.

[6] D. Garth, D. Mills, and P. Mitchell, Polynomials generated by the Fibonacci sequence, Journal of Integer Sequences, vol. 10, p. 3, 2007.

[7] D. D. Wall, Fibonacci Series Modulo m, The American Mathematical Monthly, vol. 67, pp. 525–532, 1960.

[8] S. Falcon and A. Plaza, On the Fibonacci k – numbers, Chaos, Solitons and Fractals, vol. 32, pp. 1615–1624, 2007.

[9] S. Falcon and A. Plaza, k – Fibonacci sequences modulo m, Chaos, Solitons and Fractals, vol. 41, pp. 497–504, 2009.

[10] S. Falcon and A. Plaza, The k – Fibonacci sequence and the Pascal 2- triangle, Chaos, Solitons and Fractals, vol. 33, pp. 38–49, 2007.

[11] D. A. Mehta, Properties of the Sequences of Tribonacci Numbers and Generalized Cut – off Numbers, PhD Thesis, 2009.

[12] B. Tana and Z.-Y. Wenb, Some properties of the Tribonacci sequence, European Journal of Combinatorics, vol. 28, pp. 1703–1719, 2007.

[13] S. Pethe, Some identities for Tribonacci sequences, Fibonacci Quarterly, vol. 26, pp. 144–151, 1988.

[14] https://www.wolframalpha.com.

[15] D. M. Burton, Elementary Number Theory. Tata McGraw – Hill Pub. Co. Ltd., New Delhi, 2007.

Author information

Gautam Hathiwala[1], Minakshi Biswas Hathiwala[1] and Devangi Sojitra[1], [1]Department of Mathematics, Marwadi University, Rajkot-360003, Gujarat, India.
E-mail: gautamhathiwala1717@gmail.com