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# Ring endomorphisms satisfying the $\mathcal{Z}$-reversible property

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**Abstract** We introduce a notion of $\alpha$-skew $\mathcal{Z}$-reversible rings describing $\mathcal{Z}$-reversibility of rings in terms of their endomorphisms. In support, we give some examples and counter examples. We go through some of its characteristics and provide a number of characterizations with the help of their extension rings. Also, we discuss some sufficient condition over which the notion of $\alpha$-skew $\mathcal{Z}$-reversible rings and $\mathcal{Z}$-reversible rings are same.

## 1 Introduction

All rings in the present study are associative rings having identity and all the modules are unitary. In a ring $R$, $C(R)$ represents the set of all of its central elements. We denote the ring consisting of all $n$-square matrices over a ring $R$ by $M_n(R)$; and $E_{ij}$ represents the matrix having 1 at $(i,j)^{th}$ place and elsewhere 0.

Recall [10], if $E$ is a right ideal of a ring $R$ with the property that $E \cap E' \neq 0$ for each nonzero right ideal $E'$ of $R$, then right ideal $E$ is called as *essential* and symbolized by $E \leq_e R_R$. An element $x \in R$ is termed as *right singular* in a ring $R$ if $ann_r(x) = \{a \in R | xa = 0\} \leq_e R_R$. We denote by $\mathcal{Z}_r(R)$ to the set of all such elements of $R$ which forms an ideal of $R$, i.e., $\mathcal{Z}_r(R) = \{x \in R | ann_r(x) \leq_e R_R\}$. The notion of left singular ideal $\mathcal{Z}(_R R)$ can be given in the similar way.

Recall [7], if $xy = 0$ predicts $yx = 0$ for any $x, y \in R$, then $R$ is called a *reversible* ring. As an extension of reversible rings, Kose et al. [8] gave the idea of central reversible rings. When $xy = 0$ predicts $yx \in C(R)$ for each $x, y \in R$, the ring $R$ is called as *central reversible*. Recently, the idea of central reversible rings has been extended to the concept of $\mathcal{Z}$-reversible rings by us [5]. We called a ring $R$ right (resp., left) *$\mathcal{Z}$-reversible* if $xy = 0$ implies $yx \in \mathcal{Z}_r(R)\ \forall x, y \in R$ (resp., $yx \in \mathcal{Z}_l(R)\ \forall x, y \in R$). We called a ring $\mathcal{Z}$-reversible when it is left as well as right $\mathcal{Z}$-reversible.

In 2009, Baser et al. [3] called an endomorphism $\alpha \in End(R)$ right (resp., left) *skew reversible* if $xy = 0$ gives $y\alpha(x) = 0$ for all $x$ and $y$ in $R$ (resp., $\alpha(y)x = 0$ for all $x$ and $y$ in $R$). Whenever $\alpha$ is a left (right) skew reversible endomorphism of ring $R$, then $R$ is called left (right) $\alpha$-skew reversible; and when $R$ is both right and left $\alpha$-skew reversible, it is called as $\alpha$-skew reversible. Recently, Bhattacharjee et al. [4] presented a new concept of $\alpha$-skew central reversible rings that generalized the class of $\alpha$-skew reversible rings. They called endomorphism $\alpha \in End(R)$ as right (left) *skew central reversible* if $xy = 0$ gives $y\alpha(x) \in C(R)$, for all $x$ and $y$ in ring $R$ (resp., $\alpha(y)x \in C(R)$, for all $x$ and $y$ in ring $R$). The ring $R$ is referred to as being (left) $\alpha$-skew central reversible. When $R$ is both right as well as left $\alpha$-skew central reversible, then the ring $R$ is referred to as being $\alpha$-skew central reversible.

Recently, we introduced the idea of $\alpha$-skew Z-symmetric rings and studied in [6]. Here, we introduce the idea of skew $\mathcal{Z}_r$-reversible rings analogous to $\alpha$-skew Z-symmetric rings. Some results and their proofs are analogous to that of $\alpha$-skew Z-symmetric rings.

**Definition 1.1.** Let $\alpha$ be an endomorphism of a ring $R$.

(i) $\alpha$ is called right (resp., left) skew $\mathcal{Z}_r$-reversible whenever for any $a, b \in R, ab = 0$ implies that $b\alpha(a) \in \mathcal{Z}_r(R)$ (resp., $\alpha(b)a \in \mathcal{Z}_r(R)$).

(ii) $R$ is said to be right (resp., left) $\alpha$-skew $\mathcal{Z}_r$-reversible if $\alpha$ is right (resp., left) skew $\mathcal{Z}_r$-reversible.

(iii) $R$ is said to be $\alpha$-skew $\mathcal{Z}_r$-reversible if it is left as well as right $\alpha$-skew $\mathcal{Z}_r$-reversible.

**Remark 1.2.** Left-right symmetry is not applicable to the idea of $\alpha$-skew $\mathcal{Z}_r$-reversible rings. For instance, let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$. Then, $\mathcal{Z}_r(R) = 0$. Take into consideration the endomorphism $\alpha$ of $R$ presented by $\alpha\left(\begin{bmatrix} x & \bar{y} \\ 0 & \bar{z} \end{bmatrix}\right) = \begin{bmatrix} x & \bar{0} \\ 0 & \bar{0} \end{bmatrix}$. Let $A_1 = \begin{bmatrix} a_1 & \bar{b}_1 \\ 0 & \bar{c}_1 \end{bmatrix} \in R$ and $A_2 = \begin{bmatrix} a_2 & \bar{b}_2 \\ 0 & \bar{c}_2 \end{bmatrix} \in R$ such that $A_1A_2 = \begin{bmatrix} a_1a_2 & a_1\bar{b}_2 + \bar{b}_1\bar{c}_2 \\ 0 & \bar{c}_1\bar{c}_2 \end{bmatrix} = 0$. Then, $A_2\alpha(A_1) = \begin{bmatrix} a_2 & \bar{b}_2 \\ 0 & \bar{c}_2 \end{bmatrix} \begin{bmatrix} a_1 & \bar{0} \\ 0 & \bar{0} \end{bmatrix} = \begin{bmatrix} a_2a_1 & \bar{0} \\ 0 & \bar{0} \end{bmatrix} = 0 \in \mathcal{Z}_r(R)$. As a result, $R$ is right $\alpha$-skew $\mathcal{Z}_r$-reversible. As for $A_1 = \begin{bmatrix} 0 & \bar{1} \\ 0 & \bar{0} \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & \bar{0} \\ 0 & \bar{0} \end{bmatrix} \in R$, $A_1A_2 = 0$ while $\alpha(A_2)A_1 = A_2A_1 = \begin{bmatrix} 0 & \bar{1} \\ 0 & \bar{0} \end{bmatrix} \notin \mathcal{Z}_r(R)$, $R$ is not left $\alpha$-skew $\mathcal{Z}_r$-reversible.

## 2 Properties and Characterizations

Recall from [1], $\alpha \in End(R)$ is called *compatible* whenever for all $x, y \in R$, $xy = 0$ if and only if $x\alpha(y) = 0$.

**Lemma 2.1.** *If $\alpha \in End(R)$ is compatible, then $\alpha(x) \in \mathcal{Z}_r(R)$ implies that $x \in \mathcal{Z}_r(R)$.*

*Proof.* Suppose $r \in R$ such that $\alpha(r) \in \mathcal{Z}_r(R)$. Then, $\exists I \leq_q R_R$ such that $\alpha(r)I = 0$. Hence, we have, $\alpha(a)x = 0, \forall x \in I$. Since $\alpha$ is compatible, $\alpha(rx) = \alpha(r)\alpha(x) = 0$, $\forall x \in I$ and so $rx = 0$, for all $x \in I$ by [1, Lemma 2.1(ii)]. Therefore, $rI = 0$ which implies that $r \in \mathcal{Z}_r(R)$. □

From Remark 1.2, it is clear that there is no left-right symmetry in the idea of $\alpha$-skew $\mathcal{Z}_r$-reversible rings. However, we offer a sufficient condition for this concept to be left-right symmetric in the next result.

**Proposition 2.2.** *The following statements are equivalent for a ring $R$ with a compatible endomorphism $\alpha$:*

*(i) $R$ is right $\mathcal{Z}$-reversible;*

*(ii) $R$ is right $\alpha$-skew $\mathcal{Z}_r$-reversible;*

*(iii) $R$ is left $\alpha$-skew $\mathcal{Z}_r$-reversible.*

*Proof.* (1) $\Longrightarrow$ (2). Let $xy = 0$ for two elements $x, y \in R$. Since $\alpha$ is compatible, by [1, Lemma 2.1(ii)], $\alpha(x)y = 0$. Now as $R$ is right $\mathcal{Z}$-reversible, it gives $y\alpha(x) \in \mathcal{Z}_r(R)$. This proves (2).

(2) $\Longrightarrow$ (1). Let $xy = 0$ for two elements $x, y \in R$. Since $\alpha$ is compatible, by [1, Lemma 2.1(ii)], $\alpha(x)y = 0$. Now since $R$ is right $\alpha$-skew $\mathcal{Z}$-reversible, $\alpha(yx) = \alpha(y)\alpha(x) \in \mathcal{Z}_r(R)$. Therefore, $yx \in \mathcal{Z}_r(R)$ by Lemma 2.1. This proves (1).

(1) $\Longleftrightarrow$ (3). Similar to above cases. □

**Proposition 2.3.** *A ring $R$ is right $\alpha$-skew $\mathcal{Z}_r$-reversible if for any endomorphism $\alpha$ of $R$ such that $x\alpha(x) = 0 \implies x \in \mathcal{Z}_r(R), \forall x \in R$.*

*Proof.* Let $uv = 0$ for two elements $u, v \in R$. Then, we have $v\alpha(u)\alpha(v\alpha(u)) = v\alpha(uv)\alpha^2(u) = 0$. Hence $v\alpha(u) \in \mathcal{Z}_r(R)$ by the hypothesis. This proves that $R$ is right $\alpha$-skew $\mathcal{Z}_r$-reversible. □

**Proposition 2.4.** *The ring* $S = \left\{ \begin{bmatrix} r & t \\ 0 & r \end{bmatrix} | r, t \in R \right\}$ *is* $\alpha$*-skew* $\mathcal{Z}_r$*-reversible for any right* $\mathcal{Z}$*-reversible ring* $R$*, here endomorphism* $\alpha$ *of* $S$ *is defined by* $\alpha\left( \begin{bmatrix} r & t \\ 0 & r \end{bmatrix} \right) = \begin{bmatrix} r & -t \\ 0 & r \end{bmatrix}$.

*Proof.* By [2, Lemma 2.25], we have $\mathcal{Z}_r(S) = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} | a \in \mathcal{Z}_r(R), b \in R \right\}$. Let $A_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & a_1 \end{bmatrix}, A_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & a_2 \end{bmatrix} \in S$ such that $A_1 A_2 = \begin{bmatrix} a_1 a_2 & a_1 b_2 + b_1 a_2 \\ 0 & a_1 a_2 \end{bmatrix} = 0$. Then $a_1 a_2 = 0$ which gives $a_2 a_1 \in \mathcal{Z}_r(R)$ as $R$ is right $\mathcal{Z}$-reversible. Therefore, $A_2 \alpha(A_1) = \begin{bmatrix} a_2 a_1 & b_2 a_1 - a_2 b_1 \\ 0 & a_2 a_1 \end{bmatrix} \in \mathcal{Z}_r(S)$ and $\alpha(A_2) A_1 = \begin{bmatrix} a_2 a_1 & a_2 b_1 - b_2 a_1 \\ 0 & a_2 a_1 \end{bmatrix} \in \mathcal{Z}_r(S)$. Thus, $S$ is a $\alpha$-skew $\mathcal{Z}_r$-reversible ring. □

**Remark 2.5.** 1. A right $\alpha$-skew $\mathcal{Z}_r$-reversible ring does not necessarily have to be right $\alpha$-skew central reversible. In support, let $R$ represent the ring as in [4, Example 1.4] and $S = \left\{ \begin{bmatrix} r & t \\ 0 & r \end{bmatrix} | r, t \in R \right\}$. Then, by [4, Example 1.4], $R$ is central reversible and $S$ is not right $\alpha$-skew central reversible with respect to endomorphism $\alpha$ of $S$, defined by $\alpha\left( \begin{bmatrix} r & t \\ 0 & r \end{bmatrix} \right) = \begin{bmatrix} r & -t \\ 0 & r \end{bmatrix}$. Since $R$ is central reversible, so by [5, Lemma 1] it follows that $R$ is right $\mathcal{Z}$-reversible. Hence, by Proposition 2.4, $S$ is right $\alpha$-skew $\mathcal{Z}_r$-reversible.

2. A right $\alpha$-skew central reversible ring does not necessarily have to be right $\alpha$-skew $\mathcal{Z}_r$-reversible. In support, every commutative nonsingular ring is $\alpha$-skew central reversible but may or may not be right $\alpha$-skew $\mathcal{Z}_r$-reversible. In support, let $S = \mathbb{R} \times \mathbb{R}$ and consider the ring endomorphism $\alpha$ of $S$ presented by $\alpha(r, t) = (t, r)$. Then, $S$ is right $\alpha$-skew central reversible as $S$ is commutative but it is not right $\alpha$-skew $\mathcal{Z}_r$-reversible as $(1, 0)(0, 1) = (0, 0)$ but $(0, 1)\alpha(1, 0) = (0, 1)(0, 1) = (0, 1) \notin \mathcal{Z}_r(S) = \{(0, 0)\}$.

3. Whenever a ring is right $\alpha$-skew reversible, it is right $\alpha$-skew $\mathcal{Z}_r$-reversible. But it isn't guaranteed to be so in the other direction. For example, ring $S$ in (1) is right $\alpha$-skew $\mathcal{Z}_r$-reversible but not $\alpha$-skew reversible otherwise it will become $\alpha$-skew central reversible, a contradiction.

**Proposition 2.6.** *A ring* $R$ *is right* $\alpha$*-skew* $\mathcal{Z}_r$*-reversible for a compatible endomorphism* $\alpha \in End(R)$ *if* $R$ *is right* $\alpha$*-skew central reversible.*

*Proof.* Let $rt = 0$ for two elements $r, t \in R$. Then, $t\alpha(r) \in C(R)$. Since $\alpha$ is a compatible endomorphism and $rt = 0$, we have $\alpha(r)t = 0$ and so $(t\alpha(r))^2 = 0$. Hence, $t\alpha(r) \in \mathcal{Z}_r(R)$ by [10, Lemma 7.11]. □

**Proposition 2.7.** *Let* $\sigma : R \to S$ *be a ring isomorphism and* $\alpha \in End(R)$*. Then,* $R$ *is right* $\alpha$*-skew* $\mathcal{Z}_r$*-reversible if and only if* $S$ *is right* $\bar{\alpha}$*-skew* $\mathcal{Z}_r$*-reversible, where* $\bar{\alpha} = \sigma\alpha\sigma^{-1} \in End(S)$.

*Proof.* Suppose that $R$ is right $\alpha$-skew $\mathcal{Z}_r$-reversible. Let $xy = 0$ for two elements $x, y \in R$. Then $\sigma(u) = x, \sigma(v) = y$ for some $u, v \in R$ as $\sigma$ is surjective. Now since $\sigma(uv) = \sigma(u)\sigma(v) = xy = 0$, we have $uv = 0$ as $\sigma$ is injective. It follows that $v\alpha(u) \in \mathcal{Z}_r(R)$. Therefore, $y\bar{\alpha}(x) = \sigma(v)\sigma\alpha\sigma^{-1}(\sigma(u)) = \sigma(v)\sigma(\alpha(u)) = \sigma(v\alpha(u)) \in \sigma(\mathcal{Z}_r(R)) \subseteq \mathcal{Z}_r(S)$. Thus, $S$ is right $\bar{\alpha}$-skew $\mathcal{Z}_r$-reversible.

In the opposite direction, suppose that $S$ is right $\bar{\alpha}$-skew $\mathcal{Z}_r$-reversible. Let $uv = 0$ for two elements $u, v \in R$. Then $\sigma(u)\sigma(v) = \sigma(uv) = 0$. Hence, $\sigma(v\alpha(u)) = \sigma(v)\sigma(\alpha(u)) = \sigma(v)\sigma\alpha\sigma^{-1}(\sigma(u)) = \sigma(v)\bar{\alpha}(\sigma(u)) \in \mathcal{Z}_r(S)$. Since $\sigma$ is an isomorphism, $v\alpha(u) \in \mathcal{Z}_r(R)$. This proves that $R$ is right $\alpha$-skew $\mathcal{Z}_r$-reversible. □

**Theorem 2.8.** *The following are equivalent for $\alpha \in End(R)$:*

*(i) $R$ is right $\alpha$-skew $\mathcal{Z}_r$-reversible;*

*(ii) $S = \{(a,b) \in R \times R | a - b \in \mathcal{Z}_r(R)\}$ with usual addition and multiplication is a right $\bar{\alpha}$-skew $\mathcal{Z}_r$-reversible ring, where $\bar{\alpha} \in End(S)$ is given by $\bar{\alpha}(r,t) = (\alpha(r), \alpha(t))$.*

*Proof.* (1) $\Longrightarrow$ (2). Let $(a_1,b_1),(a_2,b_2) \in S$ such that $(a_1,b_1)(a_2,b_2) = (0,0)$. Then, $a_1a_2 = 0 = b_1b_2$. Since $R$ is right $\alpha$-skew $\mathcal{Z}_r$-reversible, $a_2\alpha(a_1), b_2\alpha(b_1) \in \mathcal{Z}_r(R)$. Therefore, $(a_2,b_2)\bar{\alpha}(a_1,b_1) = (a_2\alpha(a_1), b_2\alpha(b_1)) \in \mathcal{Z}_r(R) \times \mathcal{Z}_r(R) \subseteq \mathcal{Z}_r(S)$. Thus, $S$ is right $\bar{\alpha}$-skew $\mathcal{Z}_r$-reversible.

(2) $\Longrightarrow$ (1). For two elements $a, b \in R$, let $ab = 0$. Then for $(a,a),(b,b) \in S$, we have $(a,a)(b,b) = (0,0)$. So, $(b,b)\bar{\alpha}(a,a) = (b\alpha(a), b\alpha(a)) \in \mathcal{Z}_r(S)$ as $S$ is right $\bar{\alpha}$-skew $\mathcal{Z}_r$-reversible. Hence, $ann_r(b\alpha(a), b\alpha(a)) \leq_e S$. We want to show that $ann_r(b\alpha(a)) \leq_e R$ so that $b\alpha(a) \in \mathcal{Z}_r(R)$. Let $0 \neq x \in R$. Then, $(0,0) \neq (x,x) \in S$. Since $ann_r(b\alpha(a), b\alpha(a)) \leq_e S$, there exists $(0,0) = (u,v) \in S$ such that $(0,0) \neq (x,x)(u,v) = (xu, xv) \in ann_r(b\alpha(a), b\alpha(a))$. This implies that $xu, xv \in ann_r(b\alpha(a))$ and at least one of $xu, xv$ must be nonzero. Thus, $ann_r(b\alpha(a)) \leq_e R$ and so $ann_r(b\alpha(a)) \in \mathcal{Z}_r(R)$. □

**Proposition 2.9.** *Let $\alpha_i \in End(R_i)$ for each $i \in I$. Then $R = \prod_{i\in I} R_i$ is right $\alpha$-skew $\mathcal{Z}_r$-reversible if and only if each $R_i$ is right $\alpha_i$-skew $\mathcal{Z}_r$-reversible, where $\alpha = (\alpha_i)_{i\in I}$.*

*Proof.* This implied by the fact that $\mathcal{Z}_r(R) = \prod_{i\in I} \mathcal{Z}_r(R_i)$. □

Recall [11], a ring $R$ is called an *Armendariz* ring if for any $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x], f(x)g(x) = 0$ implies that $a_ib_j = 0$ for all $i, j$.

**Proposition 2.10.** *With an endomorphism of $\alpha$, let $R$ be an Armendariz ring. Then $R$ is right $\alpha$-skew $\mathcal{Z}_r$-reversible if and only if $R[x]$ is right $\bar{\alpha}$-skew $\mathcal{Z}_r$-reversible, where $\bar{\alpha}$ is the endomorphism of $R[x]$ given by $\bar{\alpha}(\sum t_i x^i) = \sum \alpha(t_i)x^i$.*

*Proof. Sufficiency part:-* Let $A = \sum_{i=0}^{m} r_i x^i, B = \sum_{j=0}^{n} t_j x^j \in R[x]$ such that $AB = 0$. Then we have $r_it_j = 0$ for any pair $(i,j)$ as because $R$ is an Armendariz ring. So $t_j\alpha(r_i) \in \mathcal{Z}_r(R)$ for any pair $(i,j)$ as $R$ is right $\alpha$-skew $\mathcal{Z}_r$-reversible. This implies that $B\bar{\alpha}(A) = (\sum_{j=0}^{n} t_j x^j)(\sum_{i=0}^{m} \alpha(r_i)x^i) \in \mathcal{Z}_r(R)[x] = Z_r(S)$ (see [9, Exercise 7.35]). This proves sufficiency part.

*Necessity part:-* Let $uv = 0$ for two elements $u, v \in R$. Then, $v\bar{\alpha}(u) = v\alpha(u) \in Z_r(S_S) = \mathcal{Z}_r(R)[x]$. This implies that $v\alpha(u) \in Z_r(R)$. This proves necessity part. □

Recall [5], if $S^{-1}R$ is the localization ring of a ring $R$ over a multiplicatively closed subset $S$ of $R$ consisting of the central regular elements of $R$, then $\mathcal{Z}_r(S^{-1}R) = S^{-1}\mathcal{Z}_r(R)$.

**Proposition 2.11.** *If $S^{-1}R$ is the localization ring of a ring $R$ over a multiplicatively closed subset $S$ of $R$ consisting of the central regular elements of $R$ and $\alpha \in End(R)$ be such that $\alpha(S) \subseteq S$, then*

*(i) $\bar{\alpha} : S^{-1}R \to S^{-1}R$ given by $\bar{\alpha}(rs^{-1}) = \alpha(r)(\alpha(s))^{-1}$ is a ring homomorphism.*

*(ii) $R$ is right $\alpha$-skew $\mathcal{Z}_r$-reversible if and only if $S^{-1}R$ is right $\bar{\alpha}$-skew $\mathcal{Z}_r$-reversible.*

*Proof.* (1). Clear.

(2). *Sufficiency part:-* Let $r_1s_1^{-1}, r_2s_2^{-1} \in S^{-1}R$ such that $(r_1s_1^{-1})(r_2s_2^{-1}) = 0$. Then, we have $r_1r_2 = 0$ and so $r_2\alpha(r_1) \in \mathcal{Z}_r(R)$. Hence, $(r_2s_2^{-1})\bar{\alpha}(r_1s_1^{-1}) = (r_2s_2^{-1})(\alpha(r_1)\alpha(s_1)^{-1}) = (r_2\alpha(r_1))(s_2\alpha(s_1))^{-1} \in S^{-1}\mathcal{Z}_r(R) = \mathcal{Z}_r(S^{-1}R)$. This proves sufficiency part.

*Necessity part:-* Let $r_1, r_2 \in R$ such that $r_1r_2 = 0$. Then for any $s \in S$, $(r_1s^{-1})(r_2s^{-1}) = 0$ and so $(r_2s^{-1})\bar{\alpha}(r_1s^{-1}) = (r_2s^{-1})(\alpha(r_1)\alpha(s)^{-1}) = (r_2\alpha(r_1))\ (s\alpha(s))^{-1} \in \mathcal{Z}_r(S^{-1}R) = S^{-1}\mathcal{Z}_r(R)$. This implies that $(r_2\alpha(r_1) \in \mathcal{Z}_r(R)$. This proves necessity part. □

Recall from [4], the map $\bar{\alpha} : R[x, x^{-1}] \to R[x, x^{-1}]$ given by $\bar{\alpha}(\sum_{i=k}^{n} a_i x^i) = \sum_{i=k}^{n} \alpha(a_i)x^i$ is a ring homomorphism of $R[x, x^{-1}]$ and its restriction $\bar{\alpha}|_{R[x]}$ lies in $End(R[x])$ for any $\alpha \in End(R)$.

**Corollary 2.12.** *For any endomorphism $\alpha$, sending identity on itself, of a ring $R$, $R[x]$ is right $\bar{\alpha}$-skew $\mathcal{Z}_r$-reversible if and only if $R[x, x^{-1}]$ is right $\bar{\alpha}$-skew $\mathcal{Z}_r$-reversible.*

*Proof.* Since $R[x, x^{-1}] = S^{-1}(R[x])$ is the localization ring of $R$ over the multiplicatively closed subset $S = \{1, x, x^2, x^3 \cdots\}$ of $R$ consisting of the central regular elements of $R$ and $\bar{\alpha}(S) \subseteq S$, result follows from Proposition 2.11. □

**Remark 2.13.** Let $R$ be a ring that has an endomorphism $\alpha$ that has the property $\alpha(1) = 1$. Then for any $n \geq 2$, $M_n(R)$ and $T_n(R)$ are not right $\bar{\alpha}$-skew $\mathcal{Z}_r$-reversible, where $\bar{\alpha}$ is given by $\bar{\alpha}(a_{ij}) = (\alpha(a_{ij}))$.

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