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GENERALIZED REGULARIZED GAP FUNCTION AND D-GAP FUNCTION FOR RANDOM GENERALIZED VARIATIONAL-LIKE INEQUALITY PROBLEM WITH APPLICATIONS

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Abstract This study presented a new generalized regularized gap function (short for GRGF) for random generalized variational-like inequality (short for RGVLIP) in fuzzy contexts. A new D-gap function for RGVLIP was also introduced, utilizing the reverse Schwarz inequality and GRGF, which we also established for real Hilbert space in fuzzy settings. Furthermore, we developed error bound outcomes for the new D-gap function and GRGF as an application. Our findings are new and generalize several established findings for variational inequality problems (abbreviated VIPs) and generalized variational inequality problems (abbreviated GVIPs) in fuzzy contexts.

1 Introduction

Fuzzy theory, also known as fuzzy logic or fuzzy sets, originated in 1965 by Zadeh [1]. The concept was developed as a way to deal with uncertainty and imprecision that exist in real-world problems. Unlike classical set theory, which assigns objects to a set or its complement, fuzzy sets allow for gradual membership where an element can belong to multiple sets with varying degrees of membership. Due to its flexibility, it is suitable for modeling and solving complicated systems that contain ambiguity, vagueness, and partial knowledge. Numerous domains, including control systems, decision-making procedures, pattern recognition, artificial intelligence, data analysis, expert systems, and natural language processing, use fuzzy theory.

Similarly, the theory of variational inequality in fuzzy settings has been extensively studied and has captured the interest of numerous authors in recent years. Chang and Zhu initially explored this area [2]; addressing the existence of solutions, algorithms, and convergence results for various generalized versions of the variational inequality problem (referred to as VIP for brevity).

A gap function is a tool or method used to transform an optimization problem from an identical VIP, essential in convex optimization theory. Numerous approaches and efforts have been made to construct suitable gap functions for various VIP types. Examples of these efforts can be found in references [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 27, 28, 29, 30]. Gap functions have proven highly useful in determining error bounds, indicating the separation between the solution set and any given point. In problems involving variational inequalities and solved through iterative algorithms, error bounds play a crucial role in sensitivity and convergence analyses. Hence, investigating error bounds for gap functions associated with different variational inequalities is of great interest. Examples of such investigations can be found in references [5, 6, 7, 8, 9, 10, 12, 14, 15, 19, 22, 27, 28, 29, 30]. In 1992, Fukushima [20] introduced the concept of a regularized gap function, which was further generalized differently by Wu et al. [21]. The application of gap functions and error bounds in fuzzy settings for VIPs was first introduced by Khan et al. [22]. To our knowledge, limited literature on gap functions in fuzzy settings is available, as can be seen in references [19, 22, 27].

VIPs involving random fuzzy mappings need the use of a gap function in order to quantify the violation of the inequality condition while considering the effects of randomization and fuzziness in the mapping. Its use enables academics and practitioners to develop numerical approaches, algorithms, and strategies for successfully addressing these sorts of problems, even in instances when standard deterministic techniques would be insufficient. The reason behind our study is to contribute to the literature, which may aid others in making new findings in this area and related fields.

In this paper, our primary goal is to study the theory of the generalized regularized gap function, the D-gap function, and error bounds for RGVLIP with the assistance of the reverse Schwarz inequality. We have organized this paper as follows: In Section 2, we gather all the basic notions, definitions, and preliminary results that will be useful in the subsequent sections. Section 3 introduces a new generalized regularized gap function (GRGF) for solving random generalized variational-like inequality problems (RGVLIP) in fuzzy settings. Section 4 presents the result related to the reverse Schwarz inequality. Section 5 introduces a new D-gap function for RGVLIP using the reverse Schwarz inequality and the GRGF. Finally, we establish an application section where we derive error bound results.

2 Preliminaries

We adopt the following notations: $\mathcal{H}$ is a real Hilbert space in which $\langle ., . \rangle$ denotes the inner product and $\|.\|$ is induced norm. Let Σ be a σ-algebra of subsets of a set τ such that (τ, Σ) forms a measurable space. Also consider $2^{\mathcal{H}}$ as the family of non-empty subsets of $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ as the family of Borel σ-fields in $\mathcal{H}$ and $\mathcal{CB}(\mathcal{H})$ as the family of all non-empty closed and bounded subsets of $\mathcal{H}$. Let $\mathbb{F}$ denotes the collection of all fuzzy sets over $\mathcal{H}$. A mapping $\Gamma : \mathcal{H} \longrightarrow \mathbb{F}(\mathcal{H})$ is known as a fuzzy map on $\mathcal{H}$. Also $\Gamma(a)$ or Γ_a is a fuzzy set if the map Γ is a fuzzy map on $\mathcal{H}$ and $\Gamma_a(b)$ is the characteristic function or the membership function of b in Γ_a. Suppose $A \in \mathbb{F}(\mathcal{H})$ and $\alpha \in [0, 1]$ such that the set $(A)_\alpha = \{a \in \mathcal{H} : A(a) \geq \alpha\}$ is known as the α-cut set of $\mathcal{H}$.

Definition 2.1. (i) A map $\theta_a : \tau \longrightarrow \mathcal{H}$ is called measurable if $\{s \in \tau : \theta_a(s) \in B\}, \forall\, B \in \mathcal{B}(\mathcal{H})$ and a set-valued map $\Gamma : \tau \longrightarrow 2^{\mathcal{H}}$ is called measurable if $\Gamma^{-1} = \{s \in \tau : \Gamma(s) \cap B \neq \psi\} \in \Sigma, \forall\, B \in \mathcal{B}(\mathcal{H})$.

(ii) A map $\phi : \tau \times \mathcal{H} \longrightarrow \mathcal{H}$ is called random operator if $\phi(s, \theta_a(s)) = \theta_a(s)$ is measurable, $\forall$ $\theta_a(s) \in \mathcal{H}$ and a random operator ϕ is continuous if $\phi(s, .) : \mathcal{H} \longrightarrow \mathcal{H}$ is continuous map, $\forall\, s \in \tau$.

(iii) A map $\Gamma : \tau \times \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ is known to be a random set-valued map if $\Gamma(., \theta_a(.))$ is a measurable, $\forall\, \theta_a(.) \in \mathcal{H}$ and a random set-valued map $\Gamma : \tau \times \mathcal{H} \longrightarrow \mathcal{BC}(\mathcal{H})$ is called ϕ-continuous if $\Gamma(s, .)$ is continuous map in Hausdorff metric, $\forall\, s \in \tau$.

(iv) A fuzzy map $\Gamma : \tau \longrightarrow \mathbb{F}(\mathcal{H})$ is known to be measurable if $(\Gamma(.))_\alpha : \tau \longrightarrow 2^{\mathcal{H}}$ is measurable set-valued map, $\forall\, \alpha \in (0, 1]$

(v) A fuzzy map $\Gamma : \tau \times \mathcal{H} \longrightarrow \mathbb{F}(\mathcal{H})$ is known to be random if a fuzzy map $\Gamma(., \theta_a(.)) : \tau \longrightarrow \mathbb{F}(\mathcal{H})$ is measurable, $\forall\, \theta_a(.) \in \mathcal{H}$.

As obvious from the above observation, fuzzy maps, random set-valued maps and set-valued maps are special cases for random fuzzy maps.

Consider a random fuzzy map $\Gamma^* : \tau \times \mathcal{H} \longrightarrow \mathbb{F}(\mathcal{H})$ have following property:

(C_1): $\exists$ a map $e : \mathcal{H} \longrightarrow [0, 1]$ such that $(\Gamma^*_{s,\theta_a})_{e(\theta_a)} \in \mathcal{BC}(\mathcal{H}), \forall\ (s, \theta_a) \in \tau \times \mathcal{H}$. By definition of random fuzzy map Γ^*, define random set-valued, $\Gamma : \tau \times \mathcal{H} \longrightarrow \mathcal{BC}(\mathcal{H}), (s, \theta_a) \in (\Gamma^*_{s,a})_{e(a)}; \forall\, (s, \theta_a) \in \tau \times \mathcal{H}$. Γ is known to be random set-valued induced by Γ^*.

Now for given map $e : \mathcal{H} \longrightarrow [0,1]$, a random fuzzy map $\Gamma^* : \tau \times \mathcal{H} \longrightarrow \mathbb{F}(\mathcal{H})$ have the property (C_1), and random operator $\phi : \tau \times \mathcal{H} \longrightarrow \mathcal{H}$ with Img($\mathcal{H}$)$\cap$domain($\partial\psi$)$\neq\emptyset$, consider RGVLIP:

To find measurable maps $\theta_a, \rho : \tau \longrightarrow \mathcal{H}$ such that $\forall s \in \tau, \theta_b(s) \in \mathcal{H}$,

$$\Gamma^*_{s,\theta_a}(\rho(s)) \geq e(\theta_a(s)), \langle \rho(s), \eta(\theta_b(s), \theta_a(s)) \rangle + \psi(\theta_b(s)) - \psi(\theta_a(s)) \geq 0,$$

or

$$\langle \rho(s), \eta(\theta_b(s), \phi(s, \theta_a(s))) \rangle + \psi(\theta_b(s)) - \psi(\phi(s, \theta_a(s)) \geq 0, \tag{2.1}$$

where $\psi : \mathcal{H} \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuos function with its effective domain is being closed, $\eta : \mathcal{H}^2 = \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ is a map and $\partial\psi$ is sub-differential of function ψ. The sub-differential of ψ at point $a \in \mathcal{H}$ that is $\partial\psi$ is given as $\partial\psi(a) = \{r \in \mathcal{H} : \psi(b) \geq \psi(a) + \langle r, \eta(b,a) \rangle, \forall\, b \in \mathcal{H}\}$ and point $r \in \partial\psi$ is known as sub-gradient of ψ at a.

The pair of measurable maps (θ_a, ρ) known as random solution for RGVLIP(2.1).

Definition 2.2. A random set-valued map $\Gamma : \tau \times \mathcal{H} \longrightarrow \mathcal{H}$ is called

(i) strongly monotone if $\exists$ a measurable map $\alpha : \tau \longrightarrow (0, +\infty)$ such that $\forall s \in \tau, \theta_{a_i}(s) \in \mathcal{H}$, and $\forall \rho_i(s) \in \Gamma(s, \theta_{a_i}(s)),\ (i = 1, 2.)$,

$$\langle \rho_1(s) - \rho_2(s), \eta(\theta_{a_1}(s), \theta_{a_2}(s)) \rangle \geq \alpha(s) \|\theta_{a_1}(s) - \theta_{a_2}(s)\|^2.$$

(ii) strongly ϕ-monotone if $\exists$ a measurable map $\lambda^* : \tau \longrightarrow (0, +\infty)$ such that $\forall s \in \tau, \theta_{a_i}(s) \in \mathcal{H}$, and $\forall \rho_i(s) \in \Gamma(s, \theta_{a_i}(s)),\ (i = 1, 2.)$,

$$\langle \rho_1(s) - \rho_2(s), \eta(\phi(s, \theta_{a_1}(s)), \phi(s, \theta_{a_2}(s))) \rangle \geq \lambda^*(s) \|\theta_{a_1}(s) - \theta_{a_2}(s)\|^2.$$

(iii) Lipschitz continuos if $\exists L : \tau \longrightarrow (0, +\infty)$, a measurable function such that

$$\|\phi(s, \theta_{a_1}(s)) - \phi(s, \theta_{a_2}(s))\| \leq L(s) \|\theta_{a_1}(s) - \theta_{a_2}(s)\|,$$

$\forall \theta_{a_i}(s) \in \mathcal{H}$,$\forall s \in \tau$ (i=1,2).

Definition 2.3. A map $\eta : \mathcal{H}^2 \longrightarrow \mathcal{H}$ is called

(i) Lipschitz continuous if $\exists \mu : \tau \longrightarrow (0, +\infty)$, a measurable function such that $\forall \theta_a(s), \theta_b(s) \in \mathcal{H}; s \in \tau$,

$$\|\eta(\theta_a(s), \theta_b(s))\| \leq \mu(s) \|\theta_a(s) - \theta_b(s)\|.$$

(ii) η-strongly montone if, $\exists \zeta : \tau \longrightarrow (0, +\infty)$, a measurable function such that $\forall \theta_a(s), \theta_b(s) \in \mathcal{H}; s \in \tau$,

$$\langle \theta_a(s) - \theta_b(s), \eta(\theta_a(s), \theta_b(s)) \rangle \geq \zeta(s) \|\theta_a(s) - \theta_b(s)\|^2.$$

(iii) skew if $\forall \theta_a(s), \theta_b(s) \in \mathcal{H}; s \in \tau$,

$$\eta(\theta_a(s), \theta_b(s)) + \eta(\theta_b(s), \theta_a(s)) = 0$$

Lemma 2.4. *A function $G : \mathcal{H} \longrightarrow \mathbb{R}$ is said to be a gap function for a RGVLIP(2.1) if,*

(i) $G(\theta_a) \geq 0, \forall\, \theta_a \in \mathcal{H}$;

(ii) $G(\theta_b) = 0$, *if and only if $\theta_b \in \mathcal{H}$ is the solution of RGVLIP(2.1).*

3 Generalized Regularized Gap Function

Let us define the proximal function in $\mathcal{H}$ for fuzzy map $R^{\theta_a(s)}_{\nu(s)} : \tau \times \mathcal{H} \longrightarrow$dom($\psi$) such that for measurable function $\nu : \tau \longrightarrow (0, \infty)$,

$$R^{\theta_a(s)}_{\nu(s)}(s, \theta_a(s)) = \text{arg min}_{\theta_b(s)\in\mathcal{H}}\{\psi(\theta_b(s)) + \frac{1}{2\nu(s)}||\theta_a(s) - \theta_b(s)||^2\}.$$

Motivating from Fukushima[20] and Wu et al. [21]; we construct a new generalized regularized gap function (GRGF) as follows:

For all measurable function $\lambda : \tau \longrightarrow (0, \infty)$ and $\forall s \in \tau$, $\theta_a(s), \theta_b(s) \in \mathcal{H}$ such that

$$G_{\nu(s)}(\theta_a(s)) = max_{\theta_{b(s)\in\mathcal{H}}}\{\langle\rho(s), \eta(\theta_a(s), \theta_b(s))\rangle + \psi(\theta_a(s)) - \psi(\theta_b(s)) - \frac{1}{\nu(s)}f(\theta_a(s), \theta_b(s)) - \frac{\lambda(s)}{2\nu(s)}||\theta_a(s) - \theta_b(s)||^2\},$$

which can be given as

$$G_{\nu(s)}(\theta_a(s)) = \langle\rho(s), \eta(\phi(s, \theta_a(s)), R^{\theta_a(s)}_{\nu(s)}(\phi(s, \theta_a(s)) - \nu(s)\rho(s)))\rangle + \psi(\phi(s, \theta_a(s))) - \psi(R^{\theta_a(s)}_{\nu(s)}(\phi(s, \theta_a(s)) - \nu(s)\rho(s))) - \frac{1}{\nu(s)}f(\phi(s, \theta_a(s)), R^{\theta_a(s)}_{\nu(s)}(\phi(s, \theta_a(s)) - \nu(s)\rho(s))) - \frac{\lambda(s)}{2\nu(s)}||(\phi(s, \theta_a(s)) - R^{\theta_a(s)}_{\nu(s)}(\phi(s, \theta_a(s)) - \nu(s)\rho(s)))||^2. \quad (3.1)$$

Through out the paper, we consider the function $f : \mathcal{H}^2 \longrightarrow \mathbb{R}$ have the properties as follows:

(A1) $f(.,.) \geq 0$ on $\mathcal{H}^2$;
(A2) $f(.,.)$ is continously differentiable on $\mathcal{H}^2$;
(A3) $\forall s \in \tau, \exists$ a measurable function $\xi : \tau \longrightarrow (0, \infty)$ such that $\theta_a(s) \in \mathcal{H}$ and $\forall\theta_{b_1}(s), \theta_{b_2}(s) \in \mathcal{H}$;

$$f(\theta_a(s), \theta_{b_1}(s)) - f(\theta_a(s), \theta_{b_2}(s)) \geq \langle\nabla_2 f(\theta_a(s), \theta_{b_2}(s)), \eta(\theta_{b_1}(s), \theta_{b_2}(s))\rangle + \xi(s)||\theta_{b_1}(s) - \theta_{b_2}(s)||^2,$$

($\nabla_2 F$ denoted the derivative of f with respect to 2^{nd} slot);
(A4) $f(.,.)$ is zero if and only if both slots are equal;
(A5) $\exists\epsilon : \tau \longrightarrow (0, \infty)$ such that $\forall s \in \tau$, $\theta_a(s) \in \mathcal{H}$ and $\forall\theta_{b_1}(s), \theta_{b_2}(s) \in \mathcal{H}$

$$||\nabla_2 f(\theta_a(s), \theta_{b_1}(s)) - \nabla_2 f(\theta_a(s), \theta_{b_2}(s))|| \leq \epsilon(s)||\theta_{b_1}(s) - \theta_{b_2}(s)||;$$

(A6) Let measurable functions $M', m' : \tau \longrightarrow (0, \infty)$ such that $\forall s \in \tau$, $\forall\theta_a(s), \theta_b(s), z(s) \in \mathcal{H}$,

$$\langle M'(s)\eta(\theta_a(s), \theta_b(s)) - \nabla_2 f(z(s), \theta_b(s)), m'(s)\nabla_2 f(z(s), \theta_b(s)) - \eta(\theta_a(s), \theta_b(s))\rangle > 0.$$

The Property **(A5)** is known as uniformly Lipschitz continuity.

Lemma 3.1. *[22] Let $f(.,.)$ satisfies properties **(A1-A4)**; then $\nabla_2 f(\theta_a(s), \theta_b(s)) = 0 \forall s \in \tau$ if and only if $\theta_a(s) = \theta_b(s)$.*

Lemma 3.2. *Let $f(.,.)$ satisfies **(A3)** and $\eta(.,.)$ be skew in $\mathcal{H}^2$ then $\forall s \in \tau$, $\theta_a(s) \in \mathcal{H}$ and $\theta_{b_1}(s), \theta_{b_2}(s) \in \mathcal{H}$; such that*

$$\langle\nabla_2 f(\theta_a(s), \theta_{b_1}(s)) - \nabla_2 f(\theta_a(s), \theta_{b_2}(s)), \eta(\theta_{b_1}(s), \theta_{b_2}(s))\rangle \geq 2\xi(s)||\theta_{b_1}(s) - \theta_{b_2}(s)||^2.$$

Proof. Since $f(.,.)$ has property **(A3)**; then $\forall s \in \tau$, $\theta_a(s), \theta_{b_1}(s), \theta_{b_2}(s) \in \mathcal{H}$, we have

$$f(\theta_a(s), \theta_{b_1}(s)) - f(\theta_a(s), \theta_{b_2}(s)) \geq \langle\nabla_2 f(\theta_a(s), \theta_{b_2}(s)), \eta(\theta_{b_1}(s), \theta_{b_2}(s))\rangle + \xi(s)||\theta_{b_1}(s) - \theta_{b_2}(s)||^2, \quad (3.2)$$

interchanging $\theta_{b_1}(s)$ and $\theta_{b_2}(s)$, we get

$$f(\theta_a(s),\theta_{b_2}(s)) - f(\theta_a(s),\theta_{b_1}(s)) \geq \langle \nabla_2 f(\theta_a(s),\theta_{b_1}(s)), \eta(\theta_{b_2}(s),\theta_{b_1}(s))\rangle + \xi(s)||\theta_{b_1}(s) - \theta_{b_2}(s)||^2, \tag{3.3}$$

now adding (3.2) and (3.3) and also using skew property of $\eta(.,.)$ then the Lemma follows.

□

Lemma 3.3. *Let a skew $\eta(.,.)$ map be Lipschitz continuous and satisfies properties **(A1-A5)** with measurable functions $\xi, \epsilon : \tau \longleftarrow (0,\infty)$ such that $\forall s \in \tau$, $\theta_a(s), \theta_b(s) \in \mathcal{H}$,*

$$f(\theta_a(s),\theta_b(s)) \leq (\epsilon(s)\mu(s) - \xi(s))||\theta_a(s) - \theta_b(s)||^2.$$

Proof. Since for all $s \in \tau$, $\theta_a(s) \in \mathcal{H}$, **(A5)** given as

$$||\nabla_2 f(\theta_a(s),\theta_{b_1}(s)) - \nabla_2 f(\theta_a(s),\theta_{b_2}(s))|| \leq \epsilon(s)||\theta_{b_1}(s) - \theta_{b_2}(s)||, \tag{3.4}$$

$$\forall \theta_{b_1}(s), \theta_{b_2}(s) \in \mathcal{H};$$

now putting $\theta_{b_1}(s) = \theta_a(s)$ and $\theta_{b_2}(s) = \theta_b(s)$ and using Lemma (3.1) in above inequality we get,

$$||\nabla_2 f(\theta_a(s),\theta_b(s))|| \leq \epsilon(s)||\theta_a(s) - \theta_b(s)||.$$

Now again $\forall s \in \tau$, $\theta_a(s) \in \mathcal{H}$ and $\forall \theta_{b_1}(s), \theta_{b_2}(s) \in \mathcal{H}$, we have **(A3)** as,

$$f(\theta_a(s),\theta_{b_1}(s)) - f(\theta_a(s),\theta_{b_2}(s)) \geq \langle \nabla_2 f(\theta_a(s),\theta_{b_2}(s)), \eta(\theta_{b_1}(s),\theta_{b_2}(s))\rangle + \xi(s)||\theta_{b_1}(s) - \theta_{b_2}(s)||^2,$$

once again putting $\theta_{b_1}(s) = \theta_a(s)$ and $\theta_{b_2}(s) = \theta_b(s)$ in above inequality, we have,

$$f(\theta_a(s),\theta_a(s)) - f(\theta_a(s),\theta_b(s)) \geq \langle \nabla_2 f(\theta_a(s),\theta_b(s)), \eta(\theta_a(s),\theta_b(s))\rangle + \xi(s)||\theta_a(s) - \theta_b(s)||^2,$$

now by **(A4)** and skew property of $\eta(.,.)$ then above inequality transformed as

$$\begin{aligned} f(\theta_a(s),\theta_b(s)) &\leq \langle \nabla_2 f(\theta_a(s),\theta_b(s)), \eta(\theta_b(s),\theta_a(s))\rangle - \xi(s)||\theta_a(s) - \theta_b(s)||^2 \\ &\leq ||\nabla_2 f(\theta_a(s),\theta_b(s))||||\eta(\theta_b(s),\theta_a(s))|| - \xi(s)||\theta_a(s) - \theta_b(s)||^2, \end{aligned}$$

using Lipschitz continuity of $\eta(.,.)$ and (3.4) as

$$f(\theta_a(s),\theta_b(s)) \leq (\mu(s)\epsilon(s) - \xi(s))||\theta_a(s) - \theta_b(s)||^2.$$

□

Lemma 3.4. $\eta(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s))) = 0$ *if and only if $\theta_a(s)$ is a solution of RGVLIP(2.1) $\forall s \in \tau$, $\theta_a(s) \in \mathcal{H}$.*

Proof. Let us suppose,

$$\eta(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s))) = 0,$$

which is equivalent to

$$\phi(s,\theta_a(s)) = R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s)),$$

equivalently,

$$\phi(s,\theta_a(s)) = argmin_{\theta_b(s)}(\psi(\theta_b(s)) + \frac{1}{2\nu(s)}||\theta_b(s) - (\phi(s,\theta_a(s)) - \nu(s)\rho(s))||^2),$$

by optimal conditions, equivalently

$$0 \in \partial\psi(\phi(s,\theta_a(s)) + \frac{1}{\nu(s)}(\phi(s,\theta_a(s)) - (\phi(s,\theta_a(s)) - \nu(s)\rho(s)))$$

Which implies,

$$-\rho(s) \in \partial\psi(\phi(s,\theta_a(s))),$$

by defintion of sub-diffrential

$$\psi(\theta_b(s)) \geq \psi(\phi(s,\theta_a(s))) - \langle \rho(s), \eta(\theta_b(s), \phi(s,\theta_a(s))),$$

this shows $\theta_a(s)$ is a solution of RGVLIP(2.1). The converse is obvious. □

Theorem 3.5. *If $f(.,.)$ satisfies properties **(A1-A4)**; then we have*

$$G_{\nu(s)}(\theta_a(s)) \geq k_1(s)||\eta(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s)))||^2, \tag{3.5}$$

provided $k_1(s) > 0$ that is

$$\zeta(s) + \xi(s) > \epsilon(s)\mu(s) + \frac{\lambda(s)}{2}.$$

Futhermore $G_{\nu(s)}(\theta_a(s)) = 0$ if and only if $\theta_a(s)$ is the solution of RGVLIP(2.1).

Proof. Since

$$\phi(s,\theta_a(s)) - \nu(s)\rho(s) \in (I + \nu(s)\partial\psi)(I + \nu(s)\partial\psi)^{-1}(\phi(s,\theta_a(s)) - \nu(s)\rho(s)),$$

which is equivalent to

$$-\rho(s)+\frac{1}{\nu(s)}(\phi(s,\theta_a(s))-R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s))-\nu(s)\rho(s))) \in \partial\psi(R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s))-\nu(s)\rho(s)),$$

by definition of sub-differential

$$\langle \rho(s) - \frac{1}{\nu(s)}(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s))), \eta((\theta_b(s),$$
$$R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s)))) + \psi(\theta_b(s)) - \psi(R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s))) \geq 0$$

now replacing $\theta_b(s) = \phi(s,\theta_a(s))$, we have

$$\langle \rho(s) - \frac{1}{\nu(s)}(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s))), \eta((\phi(s,\theta_a(s)),$$
$$R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s))) + \psi(\phi(s,\theta_a(s))) - \psi(R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s))) \geq 0,$$

we can write above as

$$G_{\nu(s)}(\theta_a(s)) \geq \frac{1}{\nu(s)}\langle(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s))), \eta((\phi(s,\theta_a(s)),$$
$$R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s))) - \frac{1}{\nu(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s))))-$$
$$\frac{\lambda(s)}{2\nu(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s)))||^2,$$

now by using the definition of η-strongly monotone and Lemma (3.3)

$$G_{\nu(s)}(\theta_a(s))$$
$$\geq \frac{1}{\nu(s)}(\zeta(s) - (\epsilon(s)\mu(s) - \xi(s)) - \frac{\lambda(s)}{2})||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s)))||^2,$$

which can be written by the definition of Lipschitz continuity of η as

$$G_{\nu(s)}(\theta_a(s))$$
$$\geq \frac{1}{\nu(s)\mu^2(s)}(\zeta(s)-(\epsilon(s)\mu(s)-\xi(s))-\frac{\lambda(s)}{2})||\eta(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s)))||^2,$$

Let $k_1(s) = \dfrac{1}{\nu(s)\mu^2(s)}(\zeta(s) - (\epsilon(s)\mu(s) - \xi(s)) - \dfrac{\lambda(s)}{2})$ and provided $k_1(s) > 0$ then the result follows as

$$G_{\nu(s)}(\theta_a(s)) \geq k_1(s)||\eta(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu(s)}(\phi(s,\theta_a(s)) - \nu(s)\rho(s)))||^2,$$

provided $\zeta(s) + \xi(s) > \epsilon(s)\mu(s) + \dfrac{\lambda(s)}{2}$ (or, $k_1(s) > 0$).
The second part of the theorem is obvious by Lemma (3.4). □

4 Reverse Schwarz Inequality

We know the basic elementry inequality for real numbers $a \geq 0$ and $b \geq 0$ such that for $\xi > 0$ is

$$\xi a^2 + \frac{b^2}{\xi} \geq 2ab \tag{4.1}$$

Motivating by the work of [23, 24, 25, 26] in the field of reverse Schwarz inequality, we generalized the following result in fuzzy settings:

Theorem 4.1. *Let $\mathcal{H}$ be a real Hilbert space and $m, M : \tau \longrightarrow (0,\infty)$ be measurable functions then for every measurable maps $\theta_a, \theta_b : \tau \longrightarrow \mathcal{H}$ such that*

$$\langle M(s)\theta_b(s) - \theta_a(s), m(s)\theta_a(s) - \theta_b(s)\rangle > 0, \tag{4.2}$$

Then

$$0 \leq ||\theta_a(s)||.||\theta_b(s)|| \leq \frac{1}{2}.\frac{M(s)+m(s)}{\sqrt{M(s)m(s)}}\langle\theta_a(s),\theta_b(s)\rangle. \tag{4.3}$$

Proof. Consider

$$Z(s) = \langle M(s)\theta_b(s) - \theta_a(s), m(s)\theta_a(s) - \theta_b(s)\rangle,$$

or

$$Z(s) = M(s)\langle\theta_a(s),\theta_b(s)\rangle + m(s)\langle\theta_a(s),\theta_b(s)\rangle - ||\theta_a(s)||^2 - M(s)m(s)||y||^2.$$

Also as assumption (4.2), $Z(s) > 0$ therefore

$$M(s)\langle\theta_a(s),\theta_b(s)\rangle + m(s)\langle\theta_a(s),\theta_b(s)\rangle \geq ||\theta_a(s)||^2 + M(s)m(s)||y||^2.$$

equivalently

$$\frac{M(s)+m(s)}{\sqrt{M(s)m(s)}}\langle\theta_a(s),\theta_b(s)\rangle \geq \sqrt{M(s)m(s)}||\theta_a(s)||^2 + \frac{1}{\sqrt{M(s)m(s)}}||\theta_b(s)||^2,$$

now using inequality (4.1) we have

$$\frac{M(s)+m(s)}{\sqrt{M(s)m(s)}}\langle\theta_a(s),\theta_b(s)\rangle \geq 2||\theta_a(s)||.||\theta_b(s)|| \geq 0.$$

This proves the result (4.3). □

The above inequality can also be written as

$$\langle\theta_a(s),\theta_b(s)\rangle \geq \mathcal{K}(s)||\theta_a(s)||.||\theta_b(s)|| \geq 0, \tag{4.4}$$

where $\mathcal{K}(s) = \dfrac{2\sqrt{M(s)m(s)}}{M(s)+m(s)} > 0.$

5 D-Gap Function

In this section, we constructed a D-gap function by taking difference of two GRGF (3.1) with parameters $\nu_1, \nu_2 : \tau \longrightarrow (0, \infty)$ such that $\forall s \in \tau$, $\nu_1(s) > \nu_2(s)$ that is $\forall\, \theta_a(s) \in \mathcal{H}$

$$DG_{\nu_1(s),\nu_2(s)}(\theta_a(s)) = G_{\nu_1(s)}(\theta_a(s)) - G_{\nu_2(s)}(\theta_a(s)),$$

or,

$$\begin{aligned} DG_{\nu_1(s),\nu_2(s)}(\theta_a(s)) = \langle \rho(s), \eta(R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))\rangle \\ + \psi(R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)) - \psi(R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)) \\ - \frac{1}{\nu_1(s)} f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s))) \\ + \frac{1}{\nu_2(s)} f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))) \\ - \frac{\lambda(s)}{2\nu_1(s)} ||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))||^2 \\ + \frac{\lambda(s)}{2\nu_2(s)} ||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2. \end{aligned} \quad (5.1)$$

For measurable functions, $\nu_1, \nu_2 : \tau \longrightarrow (0, \infty)$, consider the proximal map $R^{\theta_a(s)}$ satisfied the following property:
(P1): Consider measurable functions $n, N : \tau \longrightarrow (0, \infty)$ such that $\forall s \in \tau$, $\theta_a(s), \theta_b(s), \rho(s) \in \mathcal{H}$;

$$\langle N(s)\mathcal{X}(s) - \mathcal{Y}(s), n(s)\mathcal{Y}(s) - \mathcal{X}(s)\rangle > 0,$$

where
$\mathcal{X}(s) = R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)) + R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)) - 2\theta_b(s)$ and $\mathcal{Y}(s) = R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s))$.

Theorem 5.1. *Consider f follows the properties **(A1-A4,A6)**; then*

$$\begin{aligned} DG_{\nu_1(s),\nu_2(s)}(\theta_a(s)) \geq k_2||R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))||^2 \\ + k_3||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2, \end{aligned}$$

and

$$\begin{aligned} DG_{\nu_1(s),\nu_2(s)}(\theta_a(s)) \leq k_4||\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))||^2 \\ + k_5||\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))|| \end{aligned}$$

$$\begin{aligned} \times ||R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s))|| \\ + k_6||R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s))||^2, \end{aligned}$$

where $k_2(s) = \dfrac{\xi(s)}{\nu_1(s)}$, $k_3(s) = \dfrac{\lambda(s)}{2}(\dfrac{1}{\nu_2(s)} - \dfrac{1}{\nu_1(s)})$, $k_4(s) = \dfrac{\lambda(s)}{2\nu_2(s)} + (\dfrac{1}{\nu_2(s)} - \dfrac{1}{\nu_1(s)})(\mu(s)\epsilon(s) - \xi(s))$, $k_5 = \dfrac{\mu(s)}{2\nu_2(s)}$ *and* $k_6(s) = \dfrac{\lambda(s)}{2\nu_2(s)}$ *i.e.* $k_i(s) > 0$, $\forall i = 2, 3, 4, 5$.

Proof. Since

$$\phi(s,\theta_a(s))-\nu_1(s)\rho(s)\in(I+\nu_1(s)\partial\psi)(I+\nu_1(s)\partial\psi)^{-1}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s)),$$

equivalently write

$$-\rho(s)+\frac{1}{\nu_1(s)}(\phi(s,\theta_a(s))-R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s)))\in\partial\psi(R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s)),$$

by definition of sub-differential, we have

$$\langle\rho(s)-\frac{1}{\nu_1(s)}(\phi(s,\theta_a(s))-R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s))),$$
$$\eta((\theta_b(s),R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s))))\rangle+\psi(\theta_b(s))-\psi(R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s)))\geq 0,$$

or

$$\langle\rho(s),\eta((\theta_b(s),R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s))))\rangle$$
$$+\psi(\theta_b(s))-\psi(R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s)))$$

$$\geq\langle\frac{1}{\nu_1(s)}(\phi(s,\theta_a(s))-R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s))),$$
$$\eta((\theta_b(s),R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s))))\rangle,$$

let $X(s)=\phi(s,\theta_a(s))-R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s))$ and $Y(s)=\eta((\theta_b(s),R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s))))$ then for measurable functions $m,M:\tau\longrightarrow(0,\infty)$ such that by theorem (4.1) (reverse Schwarz inequality (4.4)) the RHS of the above inequality is non-negative therefore the above inequality is converted as

$$\langle\rho(s),\eta((\theta_b(s),R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s))))\rangle+\psi(\theta_b(s))$$
$$-\psi(R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s)))\geq 0,$$

put $\theta_b(s)=R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s))-\nu_2(s)\rho(s)))$ in above inequality, we get

$$\langle\rho(s),\eta((R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s))-\nu_2(s)\rho(s))),R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s)))\rangle$$
$$+\psi(R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s))-\nu_2(s)\rho(s)))-\psi(R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s)))\geq 0,$$

from (5.1) the above inequality is written in terms of D-gap function as

$$\begin{aligned}DG_{\nu_1(s),\nu_2(s)}(\theta_a(s))\geq&-\frac{1}{\nu_1(s)}f(\phi(s,\theta_a(s)),R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s)))\\&+\frac{1}{\nu_2(s)}f(\phi(s,\theta_a(s)),R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s))-\nu_2(s)\rho(s)))\\&-\frac{\lambda(s)}{2\nu_1(s)}||(\phi(s,\theta_a(s))-R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s))-\nu_1(s)\rho(s)))||^2\\&+\frac{\lambda(s)}{2\nu_2(s)}||(\phi(s,\theta_a(s))-R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s))-\nu_2(s)\rho(s)))||^2.\end{aligned}\tag{5.2}$$

From the above inequality (5.2), we take last two terms as

$$\frac{1}{\nu_2(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2 - \frac{1}{\nu_1(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))||^2$$

$$= \frac{1}{\nu_2(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2 - \frac{1}{\nu_2(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))||^2$$

$$+ \frac{1}{\nu_2(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))||^2 - \frac{1}{\nu_1(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))||^2$$

$$= (\frac{1}{\nu_2(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2 - ||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))||^2) + (\frac{1}{\nu_2(s)} - \frac{1}{\nu_1(s)})||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2. \quad (5.3)$$

We take the first term from the above inequality and will show that this is non-negative hence taking first term from (5.3) as

$$||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2 - ||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))||^2$$

$$= \langle \phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)), \phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))\rangle - \langle \phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)), \phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s))\rangle$$

$$= \langle \phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)), \phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))\rangle - \langle \phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)), \phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))\rangle + \langle \phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)), \phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))\rangle - \langle \phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)), \phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s))\rangle$$

$$= \langle R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)), \phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))\rangle$$

$$+ \langle \phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))\rangle$$

$$= \langle R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)),$$
$$2\phi(s,\theta_a(s)) - (R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))\rangle,$$

using the property **(P1)** of the proximal map $R^{\theta_a(s)}$ and theorem (4.1) (reverse Schwarz inequality (4.4)) then RHS of the above equation is non-negative hence

$$||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2$$
$$- ||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))||^2 \geq 0.$$

Therefore (5.3) converted as

$$\frac{1}{\nu_2(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2$$
$$- \frac{1}{\nu_1(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))||^2$$
$$\geq (\frac{1}{\nu_2(s)} - \frac{1}{\nu_1(s)})||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2. \tag{5.4}$$

Now take the first two terms from (5.2)

$$\frac{1}{\nu_2(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))$$
$$- \frac{1}{\nu_1(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))$$

$$= \frac{1}{\nu_2(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))$$
$$- \frac{1}{\nu_1(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))$$

$$+ \frac{1}{\nu_1(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))$$
$$- \frac{1}{\nu_1(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))$$

$$= (\frac{1}{\nu_2(s)} - \frac{1}{\nu_1(s)})f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_(s)\rho(s)))$$
$$+ \frac{1}{\nu_1(s)}(f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))))$$
$$- f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))),$$

$$\geq \frac{1}{\nu_1(s)}(f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))))$$
$$- f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))),$$

by**(A3)**

$$\frac{1}{\nu_2(s)} f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))$$
$$- \frac{1}{\nu_1(s)} f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))$$

$$\geq \frac{1}{\nu_1(s)} \langle \nabla_2 (f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))), \eta(R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)),$$
$$R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))\rangle + \frac{\xi(s)}{\nu_1(s)} ||R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s))$$
$$- R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))||^2,$$

by **(A6)** and theorem 4.1 (reverse Schwaz inequality (4.4)) then the first term of the above inequality is non-negative therefore

$$\frac{1}{\nu_2(s)} f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))$$
$$- \frac{1}{\nu_1(s)} f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))$$
$$\geq \frac{\xi(s)}{\nu_1(s)} ||R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))||^2, \quad (5.5)$$

using (5.4) and (5.5) then (5.2) can be written as

$$DG_{\nu_1(s),\nu_2(s)}(\theta_a(s))$$
$$\geq \frac{\xi(s)}{\nu(s)} ||R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))||^2$$
$$+ (\frac{1}{\nu_2(s)} - \frac{1}{\nu_1(s)}) ||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2,$$

or

$$DG_{\nu_1(s),\nu_2(s)}(\theta_a(s))$$
$$\geq k_2 ||R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))||^2$$
$$+ k_3 ||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2.$$

Now similarly for $\nu_2(s)$

$$-\rho(s) + \frac{1}{\nu_2(s)} (\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))) \in \partial\psi(R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)),$$

again by definition of sub-differential

$$\langle \rho(s) - \frac{1}{\nu_2(s)} (\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))),$$
$$\eta((\theta_b(s), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))))\rangle + \psi(\theta_b(s)) - \psi(R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))) \geq 0,$$

or

$$\langle \rho(s), \eta(R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))), \theta_b(s))\rangle - \psi(\theta_b(s)) + \psi(R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))$$
$$\leq \frac{1}{\nu_2(s)} \langle (\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))), \eta(R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)), \theta_b(s))\rangle,$$

now putting $\theta_b(s) = R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s))$ and using Cauchy-Schwarz inequality on RHS, we have,

$$\langle \rho(s), \eta(R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s))\rangle - \psi(R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s))) + \psi(R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))$$

$$\leq \frac{1}{\nu_2(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))|| \times ||\eta(R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))||,$$

now above inequality can be written in the form of a D-gap function (5.1) and use Lipschitz continuity of η on RHS, we have

$$DG_{\nu_1(s),\nu_2(s)}(\theta_a(s)) \leq \frac{\mu(s)}{\nu_2(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))|| \times ||R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s))||$$

$$+ \frac{1}{\nu_2(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))) - \frac{1}{\nu_1(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))$$

$$+ \frac{\lambda(s)}{2\nu_2(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2 - \frac{\lambda(s)}{2\nu_1(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))||^2. \quad (5.6)$$

As we proceed above, similarly we will solve the above inequality. So taking 2^{nd} and 3^{rd} terms first then

$$\frac{1}{\nu_2(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))) - \frac{1}{\nu_1(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))$$

$$= \frac{1}{\nu_2(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))) - \frac{1}{\nu_1(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))$$

$$+ \frac{1}{\nu_1(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))) - \frac{1}{\nu_1(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))$$

$$= (\frac{1}{\nu_2(s)} - \frac{1}{\nu_1(s)})f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_(s)\rho(s)))$$
$$- \frac{1}{\nu_1(s)}(f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))$$
$$- f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))),$$

by **(A3)** and Lemma (3.3)

$$\frac{1}{\nu_2(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))$$
$$- \frac{1}{\nu_1(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))$$

$$= (\frac{1}{\nu_2(s)} - \frac{1}{\nu_1(s)})(\mu(s)\epsilon(s) - \xi(s))||\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_(s)\rho(s))||^2$$
$$-\frac{1}{\nu_1(s)}\langle\nabla_2(f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))), \eta(R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)),$$

now using **(A6)** along with theorem (4.1) (reverse Schwarz inequality (4.4)), we have

$$\frac{1}{\nu_2(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))$$
$$- \frac{1}{\nu_1(s)}f(\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))$$
$$\leq (\frac{1}{\nu_2(s)} - \frac{1}{\nu_1(s)})(\mu(s)\epsilon(s) - \xi(s))||\phi(s,\theta_a(s)), R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_(s)\rho(s))||^2. \quad (5.7)$$

Again taking the last two terms of (5.6)

$$\frac{1}{\nu_2(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2$$
$$- \frac{1}{\nu_1(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s)))||^2$$

$$\leq \frac{1}{\nu_2(s)}||(\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)))||^2$$
$$+ \frac{1}{\nu_2(s)}||R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s))||^2, \quad (5.8)$$

now using (5.7) and (5.8) in inequality (5.6), we have

$$DG_{\nu_1(s),\nu_2(s)}(\theta_a(s)) \leq k_4||\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))||^2$$
$$+ k_5||\phi(s,\theta_a(s)) - R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s))||$$

$$\times ||R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s))||$$
$$+ k_6||R^{\theta_a(s)}_{\nu_2(s)}(\phi(s,\theta_a(s)) - \nu_2(s)\rho(s)) - R^{\theta_a(s)}_{\nu_1(s)}(\phi(s,\theta_a(s)) - \nu_1(s)\rho(s))||^2,$$

where $k_2(s) = \frac{\xi(s)}{\nu_1(s)}$, $k_3(s) = \frac{\lambda(s)}{2}(\frac{1}{\nu_2(s)} - \frac{1}{\nu_1(s)})$, $k_4(s) = \frac{\lambda(s)}{2\nu_2(s)} + (\frac{1}{\nu_2(s)} - \frac{1}{\nu_1(s)})(\mu(s)\epsilon(s) - \xi(s))$, $k_5 = \frac{\mu(s)}{2\nu_2(s)}$ and $k_6(s) = \frac{\lambda(s)}{2\nu_2(s)}$ i.e. $k_i(s) > 0$ $\forall i = 2, 3, 4, 5$. □

6 Application: Error Bounds

In this section, error bounds for the problem RGVLIP (2.1) have been evaluated.
In the following theorem, the error bound for RGVLIP (2.1) in the term of GRGF (3.1) is established.

Theorem 6.1. *Let $\eta(.,.)$ be Lipschitz continuous and $\theta_{a_0}(s) \in \mathcal{H}$ be solution of the RGVLIP (2.1) $\forall s \in \tau$. Also consider f satisfies **(A1-A5)**. $\forall s \in \tau$ consider Γ be strongly monotonic random set-valued map with $\alpha : \tau \longrightarrow (0, \infty)$ measurable function. Then $\forall \theta_a(s) \in \mathcal{H}$,*

$$||\theta_a(s) - \theta_a(s)|| \leq \sqrt{\frac{1}{\alpha(s) + \frac{1}{\nu(s)}(\epsilon(s)\mu(s) - \xi(s)\frac{\lambda(s)}{2})} G_{\nu(s)}(\theta_a(s))}. \tag{6.1}$$

Proof. Since $\theta_{a_0}(s)$ solves RGVLIP(2.1) then,

$$\langle \rho_0(s), \eta(\theta_a(s), \theta_{a_0}(s)) \rangle + \psi(\theta_a(s)) - \psi(\theta_{a_0}(s)) \geq 0. \tag{6.2}$$

Now above inequality can be written in the form of GRGF $G_{\nu(s)}(\theta_a(s))$ as

$$G_{\nu(s)}(\theta_a(s)) \geq \langle \rho(s), \eta(\theta_a(s), \theta_{a_0}(s)) \rangle + \psi(\theta_a(s)) - \psi(\theta_{a_0}(s)) \\ - \frac{1}{\nu(s)} f(\theta_a(s), \theta_{a_0}(s)) - \frac{\lambda(s)}{2\nu(s)} ||\theta_a(s) - \theta_{a_0}(s)||^2,$$

now using (4.7), the above can be written as

$$G_{\nu(s)}(\theta_a(s)) \geq \langle \rho(s) - \rho_0(s), \eta(\theta_a(s), \theta_{a_0}(s)) \rangle - \frac{1}{\nu(s)} f(\theta_a(s), \theta_{a_0}(s)) - \frac{\lambda(s)}{2\nu(s)} ||\theta_a(s) - \theta_{a_0}(s)||^2,$$

now by definition of strongly monotonicity of Γ and using Lemma (3.3), we have

$$G_{\nu(s)}(\theta_a(s)) \geq (\alpha(s) + \frac{1}{\nu(s)}(\epsilon(s)\mu(s) - \xi(s)\frac{\lambda(s)}{2}))||\theta_a(s) - \theta_{a_0}(s)||^2,$$

this can be written as

$$||\theta_a(s) - \theta_{a_0}(s)|| \leq \sqrt{\frac{1}{\alpha(s) + \frac{1}{\nu(s)}(\epsilon(s)\mu(s) - \xi(s)\frac{\lambda(s)}{2})} G_{\nu(s)}(\theta_a(s))}.$$

□

In the following theorem, the error bound for RGVLIP (2.1) in the term of D-gap function (5.1) is established.

Theorem 6.2. *Let $\theta_{a_0}(s) \in \mathcal{H}$ be solution of the RGVLIP(2.1) $\forall s \in \tau$. Also consider f satisfies **(A1-A6)**. $\forall s \in \tau$, $\phi : \tau \times \mathcal{H} \longrightarrow \mathcal{H}$ be Lipschitz continuous and Γ be strongly ϕ-monotonic random set-valued map with $L, \lambda^* : \tau \longrightarrow (0, \infty)$ measurable functions respectively. Then $\forall \theta_a(s) \in \mathcal{H}$*

$$||\theta_a(s) - \theta_{a_0}(s)|| \leq \sqrt{\frac{1}{\lambda^*(s) + L^2(s)(\frac{1}{\nu_2(s)} - \frac{1}{\nu_1(s)})(\frac{\lambda(s)}{2} + \xi(s))} DG_{\nu_1(s),\nu_2(s)}(\theta_a(s))}. \tag{6.3}$$

Proof. Since $\theta_{a_0}(s)$ solves RGVLIP(2.1) then,

$$\langle \rho_0(s), \eta(\phi(s, \theta_a(s)), \phi(s, \theta_{a_0}(s))) \rangle + \psi(\phi(s, \theta_a(s))) - \psi(\phi(s, \theta_{a_0}(s))) \geq 0. \tag{6.4}$$

Also using property **(A4)**,

$$f(\phi(s, \theta_a(s)), \phi(s, \theta_{a_0}(s))) = f(\phi(s, \theta_a(s)), \phi(s, \theta_{a_0}(s))) - f(\phi(s, \theta_a(s)), \phi(s, \theta_a(s))),$$

now using property **(A3)** and Lemma (3.1),

$$f(\phi(s,\theta_a(s)),p_0(s,\theta_a(s)))=\xi(s)||(\phi(s,\theta_a(s))-\phi(s,\theta_{a_0}(s)))||^2. \tag{6.5}$$

Now D-gap function (5.1) can be written as,

$$DG_{\nu_1(s),\nu_2(s)}(\theta_a(s))\geq\langle\rho(s),\eta(\phi(s,\theta_a(s)),\phi(s,\theta_{a_0}(s)))\rangle+\psi(\phi(s,\theta_a(s)))-\psi(\phi(s,\theta_{a_0}(s)))+ (\frac{1}{\nu_2(s)}-\frac{1}{\nu_1(s)})f(\phi(s,\theta_a(s)),\phi(s,\theta_{a_0}(s)))+\frac{\lambda(s)}{2}(\frac{1}{\nu_2(s)}-\frac{1}{\nu_1(s)})||(\phi(s,\theta_a(s))-\phi(s,\theta_{a_0}(s)))||^2,$$

equivalently

$$DG_{\nu_1(s),\nu_2(s)}(\theta_a(s))\geq\langle\rho(s)-\rho_0(s),\eta(\phi(s,\theta_a(s)),\phi(s,\theta_{a_0}(s)))\rangle+\langle\rho_0(s),\eta(\phi(s,\theta_a(s)),\phi(s,\theta_{a_0}(s)))\rangle +\psi(\phi(s,\theta_a(s)))-\psi(\phi(s,\theta_{a_0}(s)))+(\frac{1}{\nu_2(s)}-\frac{1}{\nu_1(s)})f(\phi(s,\theta_a(s)),\phi(s,\theta_{a_0}(s)))+\frac{\lambda(s)}{2}(\frac{1}{\nu_2(s)}-\frac{1}{\nu_1(s)}) \times||(\phi(s,\theta_a(s))-\phi(s,\theta_{a_0}(s)))||^2,$$

by (6.4) and (6.5) we have,

$$DG_{\nu_1(s),\nu_2(s)}(\theta_a(s)) \geq\langle\rho(s)-\rho_0(s),\eta(\phi(s,\theta_a(s)),\phi(s,\theta_{a_0}(s)))\rangle+\xi(s)(\frac{1}{\nu_2(s)}-\frac{1}{\nu_1(s)})||\phi(s,\theta_a(s))-\phi(s,\theta_{a_0}(s))||^2 +\frac{\lambda(s)}{2}(\frac{1}{\nu_2(s)}-\frac{1}{\nu_1(s)})||(\phi(s,\theta_a(s))-\phi(s,\theta_{a_0}(s)))||^2,$$

or

$$DG_{\nu_1(s),\nu_2(s)}(\theta_a(s))\geq\langle\rho(s)-\rho_0(s),\eta(\phi(s,\theta_a(s)),\phi(s,\theta_{a_0}(s)))\rangle +(\xi(s)+\frac{\lambda(s)}{2})(\frac{1}{\nu_2(s)}-\frac{1}{\nu_1(s)})||(\phi(s,\theta_a(s))-\phi(s,\theta_{a_0}(s)))||^2,$$

now by definitions of Lipschitz continuity of ϕ and ϕ-strongly monotonicity, we have

$$DG_{\nu_1(s),\nu_2(s)}(\theta_a(s))\geq(\lambda^*(s)+(\xi(s)+\frac{\lambda(s)}{2})(\frac{1}{\nu_2(s)}-\frac{1}{\nu_1(s)})L^2(s))||\theta_a(s)-\theta_{a_0}(s)||^2,$$

therefore we have the required result. □

7 Conclusion and Remarks

This paper has dedicated our efforts to studying the theory of gap functions and error bounds within a fuzzy environment. Inspired by the work of [23, 24, 25, 26]; we establish a reverse Schwarz inequality for real Hilbert spaces and apply it within a fuzzy context. Furthermore, we introduce a novel version of the generalized regularized gap function for RGVLIP (2.1), drawing motivation from the contributions of Fukushima [20] and Wu et al. [21]. Subsequently, we construct the D-gap function for RGVLIP (2.1) utilizing the GRGF (3.1). Additionally, we derive error bounds with the aid of the constructed gap functions (3.1) and (5.1). Our results are both innovative and encompass many of the previously known outcomes.

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