Palestine Journal of Mathematics

Vol 13(Special Issue II)(2024) , 38–44

On qclean and almost qclean Rings

R. G. Ghumde[1] and M. K. Patel[2]

Communicated by Mohammad Ashraf

Dedicated to Prof. B. M. Pandeya on his 78th birthday

MSC 2020 Classification: 16U99, 16E50, 16S34.

Keywords and phrases: Clean, almost clean, nil clean, qclean.

Acknowledgement: Authors would like to thank the referee for their valuable comments, which helps to improve the presentation of the article. M. K. Patel wishes to thank National Board for higher Mathematics, with file No.: 02211/3/2019 NBHM (R:P:) RDII/1439, for financial assistantship.

Abstract Nicholson introduced clean rings, where each element of ring is the sum of an idempotent and unit element. Motivated by this structure we introduce here two classes of rings, qclean and almost qclean ring. In this article we focus to study the fundamental properties of both qclean and almost qclean rings, also proved that if a ring R is (almost) qclean, then power series ring $R[[x]]$ is (almost) qclean. Apart from this we established for a commutative torsionless cancellative monoid T, a commutative graded ring $R = \bigoplus_{\beta \in T} T_\beta$ is almost qclean, if T_0 is almost qclean and each T_β is torsion-free T_0 module.

1 Introduction

An element q of ring is called qpotent if $q^4 = q$. From the definition of qpotent element it is clear that every idempotent element is qpotent but the converse need not be true. A non-zero element $a \in R$ is called regular if $ab = 0$ implies $b = 0$ and $ba = 0$ implies $b = 0$. These are some special elements of the rings, other than these special elements there are many special elements in the ring like nilpotent elements, idempotent elements, unit elements, periodic elements, Jacobson radicals etc. Many mathematicians used the properties of these special elements for studying the various types of rings such as Boolean rings, clean rings, almost clean rings, nil-clean rings, J-clean rings, r-clean rings etc. However, first such ring is studied by Nicholson [9] in his study of lifting idempotents and exchange rings, where he introduces new class of ring known as clean ring. A ring R is clean ring, if each element of ring can be expressed as sum of unit and idempotent element. Nicholson also investigated the relationship between Boolean and clean rings. Motivated by this work Ahn and Anderson [1] introduces two classes of rings known as weakly clean rings and almost clean rings. In weakly clean ring, each element of ring can be written as either sum or difference of a unit and an idempotent. While in almost clean ring, each element of ring can be expressed as sum of regular element and an idempotent. On similar line the notion of nil clean rings is introduced by A. J. Diesel [6], defined as if each element of ring can be written as sum of an idempotent and nilpotent element. Most of the authors classify the rings on the basis of properties of special elements of rings and studied their valuable properties. Many authors have discussed such type of classifications in [[2], [3], [4], [7], [10]] and discussed several important results of clean ring, r-clean ring, weakly clean ring, matrix ring over clean ring and rings in which every element expressed as a sum of tripotents respectively.

In this paper we continue the study of such types of rings in which each element of ring can be expressed by using qpotent elements. In this direction we study the qclean and almost qclean rings. In section 2, we start by proving various properties of qclean rings. The main focus of this section is to prove the class of qclean ring is closed under various fundamental structures of rings like direct product, polynomial, homomorphic images etc. We discuss the matrix ring is qclean whenever base ring is qclean and also discusses the results on ideal extensions in context of qclean rings. In section 3, some similar type of things discusses for almost qclean rings.

Throughout this paper, we consider R as an associative ring with unity. The set of regular elements, the set of qpotent elements, the set of unit elements, the set of zero divisors and Jacobson radical are denoted by $Reg\,(R)$, $Q\,(R)$, $U\,(R)$, $Z(R)$ and $J\,(R)$.

2 qclean Rings

We begin this section with the definition and example of qclean ring.

Definition 2.1. An element $r \in R$ is called qclean if r is a sum of unit element and qpotent element. A ring R is said to be qclean if every element of R is qclean i.e. every element of R is expressed as a sum of unit element and qpotent element of R.

From the above definition it is clear that every clean element is a qclean but the converse is not true. This is clear by the following example.

Example 2.2. Let $F_4 = \{a + b\alpha | \alpha^2 = \alpha + 1, a, b \in \mathbb{Z}_2\}$. One can note that F_4 is a field of order 4. Now, consider $M_2(F_4) = (\begin{pmatrix} a & b \\ c & d \end{pmatrix} |a, b, c, d \in F_4)$. It is easy to see that $M_2(F_4)$ is a ring with usual matrix addition and multiplication. In this ring $\begin{pmatrix} \alpha & \alpha \\ \alpha & \alpha \end{pmatrix}$ is a qclean element as it can be expressed as $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$, where the first element is qpotent and the second element is invertible.

Now we discuss some elementary properties of qclean rings.

Proposition 2.3. *Let R be a qclean ring with characteristic two. Then $r \in R$ is qclean if and only if $1 - r$ is qclean.*

Proof. Let $r \in R$ be a qclean element with characteristic 2, so we can express $r = u + q$, where $u \in U\,(R)$ and $q \in Q(R)$. Then $1-r = -u+(1-q)$. But $(1 - q)^4 = 1-4q+\frac{3.4}{2}q^2+4q^3-q^4 = 1 - q^4 = 1 - q$. This imply $(1 - q) \in Q(R)$, so $1 - r$ is qclean. Conversely, if $1 - r$ is qclean, so $1 - r = u + q$, where $u \in U\,(R)$ and $q \in Q(R)$. Hence $r = -u + (1 - q)$, so $-u \in U\,(R)$ and $1 - q \in Q\,(R)$. □

Corollary 2.4. *If $r \in J\,(R)$ then r is qclean.*

Proof. As $r \in J(R)$ and hence $1 - r \in U(R)$. This implies $1 - r$ is qclean. Therefore, by Proposition 2.3, r is qclean. □

Proposition 2.5. *A ring R is qclean if and only if any element $r \in R$ can be expressed as $r = u - q$, where $u \in U(R)$ and $q \in Q(R)$.*

Proof. Proof is a routine exercise and can be verified easily. □

Theorem 2.6. *Every homomorphic image of qclean is qclean.*

Proof. Let R be qclean ring and $f : R \to R'$ be a homomorphism. Let $r' \in f(R)$ then for some $r \in R$, we have $r' = f(r)$. As R is a qclean, $r = u + q$, where $u \in U(R), q \in Q(R)$. Then $f\,(r) = f\,(u + q) = f\,(u) + f\,(q)$. Here $f\,(q) = f\,(q^4) = (f\,(q))^4$ and hence $f\,(q)$ is qpotent element. Now we prove $f(u)$ is invertible. For this consider v be the inverse of an element u in R, hence $f\,(u)\,f\,(v) = f\,(uv) = f\,(1) = 1$ and similarly $f\,(v)\,f\,(u) = 1$. This shows, $f(u)$ is an invertible element and hence $f(R)$ is qclean. □

Theorem 2.7. *Let $R_1, R_2, R_3...$ are rings and construct $R = \prod_{i\in I} R_i$. Then R is qclean if and only if each $R_{i's}$ are q clean.*

Proof. It is evident that $Q(R) = \prod_{i\in I} Q(R_i)$ and $U\,(R) = \prod_{i\in I} U(R_i)$. Hence the result follows. □

Theorem 2.8. *Let R be a ring, then the following statements are equivalent:*
(**i**) *R is qclean.*
(**ii**) *$R[[x]]$ is qclean.*

Proof. $(i) \Rightarrow (ii)$ Let $f = b_0 + b_1x + b_2x^2 + \ldots . \in R[[x]]$ where $b_i \in R$. As R is qclean, so $b_0 = u + q$ where $u \in U(R)$ and $q \in Q(R)$ then $f = q + u + b_1x + b_2x^2 + \ldots$ is qclean because $u + b_1x + b_2x^2 + \ldots \in U(R[[x]])$. Hence the result follows.
$(ii) \Rightarrow (i)$ As R is a homomorphic image of $R[[x]]$, hence by Theorem 2.6, the result follows. □

However, the similar type of result is not true for the polynomial ring $R[x]$. It is followed from the following proposition.

Proposition 2.9. *Let R be commutative ring then $R[x]$ is not qclean.*

Proof. Consider $x = u + q$, where $u \in U(R[x])$ and $q \in Q(R[x])$. Suppose $q = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and R is commutative with $q^4 = q$, hence $a_i = 0, \forall i \in \{1, 2, ..., n\}$. This implies $q \in R$. Thus $-q + x \in U(R[x])$. It is easy to see that this is impossible. □

It is always interesting to classify the matrix ring on the basis of base ring. Now we prove, whenever R is qclean then matrix ring $M_n(R)$ is also qclean.

Proposition 2.10. *If $q^4 = q \in R$ and q^3Rq^3 and $(1-q^3)R(1-q^3)$ are qclean rings, then R is qclean.*

Proof. As $q^4 = q$, so it is easy to see that q^3 is an idempotent element. Now, by Pierce decomposition of the ring, we have
$R = \begin{bmatrix} q^3Rq^3 & q^3R(1-q^3) \\ (1-q^3)Rq^3 & (1-q^3)R(1-q^3) \end{bmatrix}$. Let $P = \begin{bmatrix} l & x \\ y & m \end{bmatrix} \in R$, as $l \in q^3Rq^3$ implying $l = f + u$, where $u \in U(q^3Rq^3)$ and $f \in Q(q^3R\,q^3)$. Suppose u' is an inverse of u then $yu'x \in (1-q^3)Rq^3q^3Rq^3q^3R(1-q^3) \subseteq (1-q^3)R(1-q^3)$, and $m \in (1-q^3)R(1-q^3)$ gives
$m - yu'x \in (1-q^3)R(1-q^3)$, so we can write $m - yu'x = f' + v$, where $f' \in Q((1-q^3)R(1-q^3))$ and $v \in U((1-q^3)R(1-q^3))$ with inverse v'. This implies
$P = \begin{bmatrix} f+u & x \\ y & f' + v + yu'x \end{bmatrix} = \begin{bmatrix} f & 0 \\ 0 & f' \end{bmatrix} + \begin{bmatrix} u & x \\ y & v + yu'x \end{bmatrix}$. Clearly, $\begin{bmatrix} f & 0 \\ 0 & f' \end{bmatrix}^4 = \begin{bmatrix} f^4 & 0 \\ 0 & f'^4 \end{bmatrix} = \begin{bmatrix} f & 0 \\ 0 & f' \end{bmatrix}$ and hence $\begin{bmatrix} f & 0 \\ 0 & f' \end{bmatrix} \in Q(R)$. Now, we prove $\begin{bmatrix} u & x \\ y & v + yu'x \end{bmatrix}$ is a unit. For this consider
$\begin{bmatrix} q^3 & 0 \\ -yu' & 1-q^3 \end{bmatrix} \begin{bmatrix} u & x \\ y & v + yu'x \end{bmatrix} \begin{bmatrix} q^3 & -u'x \\ 0 & 1-q^3 \end{bmatrix} = \begin{bmatrix} q^3u & q^3x \\ -y + (1-q^3)y & -yu'x + (1-q^3)(v + yu'x) \end{bmatrix}$
$\begin{bmatrix} q^3 & -u'x \\ 0 & 1-q^3 \end{bmatrix}$. But $q^3u = u$ because $u \in q^3Uq^3$ and as $x \in q^3R(1-q^3)$ this implies $q^3x = x$.
Also, $-y + (1-q^3)y = -q^3y = 0$ because $y \in (1-q^3)Rq^3$. On similar line we can show that $-yu'x + (1-q^3)(v + yu'x) = v$. Hence,
$\begin{bmatrix} q^3 & 0 \\ -yu' & 1-q^3 \end{bmatrix} \begin{bmatrix} u & x \\ y & v + yu'x \end{bmatrix} \begin{bmatrix} q^3 & -u'x \\ 0 & 1-q^3 \end{bmatrix} = \begin{bmatrix} u & x \\ 0 & v \end{bmatrix} \begin{bmatrix} q^3 & -u'x \\ 0 & 1-q^3 \end{bmatrix} = \begin{bmatrix} uq^3 & -xq^3 \\ 0 & v(1-q^3) \end{bmatrix}$.
But, $uq^3 = u,\ -xq^3 = 0$ and $v(1-q^3) = v$. Hence,
$\begin{bmatrix} q^3 & 0 \\ -yu' & 1-q^3 \end{bmatrix} \begin{bmatrix} u & x \\ y & v + yu'x \end{bmatrix} \begin{bmatrix} q^3 & -u'x \\ 0 & 1-q^3 \end{bmatrix} = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}$. Since $\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}, \begin{bmatrix} q^3 & 0 \\ -yu' & 1-q^3 \end{bmatrix}$ and $\begin{bmatrix} q^3 & -u'x \\ 0 & 1-q^3 \end{bmatrix}$ all are in $U(R)$. Hence $\begin{bmatrix} u & x \\ y & v + yu'x \end{bmatrix}$ is unit and the result follows. □

Proposition 2.11. *Let $q_1, q_2, \ldots . q_n$ be qpotents with q_i^3 and q_j^3 are mutually orthogonal for $i \neq j$ and $q_1 + q_2 + \ldots + q_n = 1$ then for each i, if $q_i^3Rq_i^3$ is qclean then R is qclean.*

Proof. This follows from Proposition 2.10 . □

The following result is a direct consequence of above proposition.

Corollary 2.12. *Let R be a qclean ring, then matrix ring $M_n(R)$ is qclean.*

Proposition 2.13. *Let X and Y are rings and M be $Y-X$ module with formal triangular ring $\begin{bmatrix} X & 0 \\ M & Y \end{bmatrix}$ is qclean, then X and Y are qclean.*

Proof. Let $S = \begin{bmatrix} X & 0 \\ M & Y \end{bmatrix}$ be a qclean ring, consider an element $s \in S$ such that $s = \begin{bmatrix} x & 0 \\ m & y \end{bmatrix} = \begin{bmatrix} f_1 & 0 \\ f_2 & f_3 \end{bmatrix} + \begin{bmatrix} u_1 & 0 \\ u_2 & u_3 \end{bmatrix}$, where $\begin{bmatrix} f_1 & 0 \\ f_2 & f_3 \end{bmatrix} \in Q(S)$ and $\begin{bmatrix} u_1 & 0 \\ u_2 & u_3 \end{bmatrix} \in U(S)$. Thus, $x = f_1 + u_1$ and $y = f_3 + u_3$. But $\begin{bmatrix} f_1 & 0 \\ f_2 & f_3 \end{bmatrix}^4 = \begin{bmatrix} f_1 & 0 \\ f_2 & f_3 \end{bmatrix}$ this implies $f_1^4 = f_1$ and $f_2^4 = f_2$ and hence $f_1 \in Q(X)$ and $f_2 \in Q(Y)$. Also there exist $\begin{bmatrix} u_1' & 0 \\ u_2' & u_3' \end{bmatrix}$ such that $\begin{bmatrix} u_1 & 0 \\ u_2 & u_3 \end{bmatrix} \begin{bmatrix} u_1' & 0 \\ u_2' & u_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Hence $u_1 \in X$ and $u_3 \in Y$. Therefore, x and y are qclean elements. Hence X and Y are qclean. □

Let V be an $R-R$ bimodule over a ring R. This bimodule is also a ring with $(v_1 v_2) r = v_1 (v_2 r)$, $(v_1 r) v_2 = v_1 (r v_2)$ and $(r v_1) w = r (v_1 w)$, where $r \in R$ and $v_1, v_2 \in V$. Then the ideal extension $I(R;V) = \{(x,v) : x \in R, v \in V\}$ is a ring with multiplication $(x, v_1)(y, v_2) = (xy, xv_2 + v_1 y + v_1 v_2)$ and usual addition.

Now, we discuss some results on ideal extension.

Proposition 2.14. *Let I(R;V) be an ideal extension as defined above. Then the following statements hold:*
(**i**) *If $I(R;V)$ is qclean, then R is qclean.*
(**ii**) *If R is qclean and for any $v_1 \in V$, if there exist $v_2 \in V$ such that $v_1 + v_2 + v_1 v_2 = 0$, then $I(R;V)$ is qclean.*

Proof. (**i**) Let $(r, 0) \in I(R;V)$ where $r \in R$. Since $I(R;V)$ is qclean, $(r,0) = (u_1, u_2) + (q_1, q_2)$, where $(q_1, q_2) \in Q(I(R;V))$ and $(u_1, u_2) \in U(I(R;V))$. But $(q_1, q_2)^4 = (q_1, q_2)$, this gives $\left(q_1^4, q_1^3 q_2 + q_1^2 q_2 q_1 + q_1^2 q_2^2 + q_1 q_2 q_1^2 + q_2 q_1^3 + q_2^2 q_1^2 + \left(q_1 q_2 + q_2 q_1 + q_2^2\right)^2\right) = (q_1, q_2)$. Therefore, $q_1^4 = q_1$ and hence $q_1 \in Q(R)$. Similarly we can prove that $u_1 \in U(R)$. Hence, $r = u_1 + q_1$ is a qclean.
(**ii**) Since R is qclean, for $r \in R$ we have $r = u + q$, where $q \in Q(R)$ and $u \in U(R)$. Therefore for $(r, v) \in I(R;V)$ can be written as $(r,v) = (q+u, v) = (u, v) + (q, 0)$. Clearly, $(q,0)^4 = (q^4, 0) = (q, 0)$ and hence $(q, 0) \in Q(I(R;V))$. Now to prove (u, v) is invertible in $I(R;V)$, consider $(u,v) = (u,0)(1,v')$, where $v' = u^{-1} v$. Clearly, $(u, 0)$ is invertible in $I(R;V)$. Since for $v' \in V$ there exist $v_2' \in V$ such that $v' + v_2' + v' v_2' = 0$, this implies $(1, v')(1, v_2') = 1$. Therefore $(1, v')$ is invertible in $I(R;V)$ and hence (u, v) is invertible in $I(R;V)$. This completes the proof. □

3 Almost qclean rings

We start this section by giving the definition and example of an almost qclean rings.

Definition 3.1. An element $r \in R$ is called almost qclean ring if r is a sum of regular element and qpotent element. If every element of a ring R is almost qclean, then ring R is called as almost qclean ring. Clearly, qclean ring is almost qclean but the converse need not be true.

Now, we discuss some properties of almost qclean rings.
In previous section we proved that If R is qclean then each of its homomorpic image is also qclean. But similar type of result need not be true for almost qclean rings. An example in this direction is given below:

Example 3.2. Let $R = \mathbb{R}[a, b]$ and $R' = R/(a-1)\cap(a)\cap(a+1)\cap(b)$. Clearly $Q\,(R') = \{0', 1'\}$. Here $a', (a-1)' and\,(a+1)'$ are zero divisor elements of R' and which are not almost qclean, hence R' is not almost qclean.

Proposition 3.3. *Let R be a ring with characteristic 2, then $r \in R$ is almost qclean if and only if $1 - r$ is almost qclean.*

Proof. Proof is similar to the proof of Proposition 2.3. □

Now, we prove the direct product preserves the property of an almost qclean.

Proposition 3.4. *Let $R_1, R_2, R_3...$ are rings and construct $R = \prod_{i\in I} R_i$. Then R is almost qclean if and only if each R_i is an almost qclean.*

Proof. Let $r = (r_1, r_2, \ldots .r_i, \ldots)$ where $r_i \in R_i$ for each $i \in I$, can be expressed as $r = x + q$ where $x \in Reg\,(R)$ and $q \in Q\,(R)$. But $\prod_{i\in I} Reg(R_i) = Reg\,(R)$ this implies $x_i \in Reg(R_i)$ for each $i \in I$. Similarly if $q = (q_1, q_2, \ldots .., q_i) \in Q\,(R)$ then $q_i \in Q\,(R_i)$ for each $i \in I$. Hence, we get $r_i = x_i + q_i$, where $x_i \in Reg(R_i)$ and $q_i \in Q\,(R_i)$ for each $i \in I$. So R_i is almost qclean. Conversely, let $r = (r_i) \in R = \prod_{i\in I} R_i$. For each $i \in I$, we can write $a_i = r_i + q_i$ where $r_i \in Reg(R_i)$ and $q_i \in Q(R_i)$. Since $r_i \in Reg(R_i)$, so there exist $s_i \neq 0 \in R_i$ such that $s_i r_i = 0$. Hence $r = (r_i) \in Reg\,(R)$. Clearly $q = (q_i) \in Q\,(R)$. Hence r is an almost qclean. □

Proposition 3.5. *If ring R is almost qclean, then $R[[x]]$ is almost qclean.*

Proof. Let $f = a_0 + a_1x + \ldots \in R\,[[X]]$. Since R is an almost qclean, we can write $a_0 = r_0 + q_0$ where $r_0 \in Reg\,(R)$ and $q_0 \in Q\,(R)$. Therefore, $f = r_0 + q_0 + a_1x + a_2x^2 + \ldots = q_0 + r_0 + a_1x + a_2x^2 + \ldots = q_0 + h(x)$, where $h\,(x) = r_0 + a_1x + a_2x^2 + \ldots$. If $h(x)$ is not regular then there exist $g\,(x) = b_0 + b_1x + b_2x^2 + \ldots$ such that $h\,(x)\,g\,(x) = 0$. Therefore $r_0 g\,(x) = 0$ and hence $r_0 g_i = 0 \forall i$, and this gives $g_i = 0$ for all i. This is not possible. So $h\,(x) \in reg(R\,[[x]])$. Also, $q \in Q\,(R) \subseteq Q(R\,[[x]])$. Hence $R\,[[x]]$ is an almost qclean. □

Proposition 3.6. *Let R be a ring which embedded in a ring P, also R and P has the same qpotent, then R is almost qclean.*

Proof. Let R be a ring which embeds in a ring P and has the same qpotent as R. Therefore for $x \in R$, we have $x \in P$ and hence $x = u + q$ where $q \in Q(P)$ and $u \in U\,(P)$. By given condition $q \in Q(R)$ and this implies $u = x - q \in R$. Now to prove $u \in Reg(R)$, we start with the assumption that u is not regular and therefore there exist $y \in R$ such that $yu = 0,\ y \neq 0$. Since y and $u \in R$, so $y, u \in P$, and hence $yu \in P$. Also as $u \in U\,(P)$, this implies there exist $u' \in P$ such that $uu' = 1$. Therefore, $yu = 0$ gives $(yu)\,u' = 0, y = 0$ is a contradiction. Hence R is almost qclean. □

Proposition 3.7. *Let R be a ring. Then the following statements are equivalent:*
(**i**) *R is almost qclean.*
(**ii**) *$R[[x]]/((x^n))$ is an almost qclean.*

Proof. (**i**) $\Longrightarrow$ (**ii**) For a ring R, consider $S = R[[x]]/((x^n))$. First we start by assuming R is almost qclean and will prove S is an almost qclean . Let $f = f_0 + f_1x + \ldots + f_{n-1}x^{n-1} + (x^n)$ be an element of S. Since R is qclean, $f = (r_0 + f_1x + \ldots + f_{n-1}x^{n-1} + (x^n)) + (q_0 + (x^n)$, where $r_0 \in Reg(R)$, $q_0 \in Q(R)$. Suppose that $(r_0 + f_1x + f_2x^2 + \ldots + f_{n-1}x^{n-1} + (x^n))(g_0 + g_1x + g_2x^2 + \ldots + g_{n-1}x^{n-1} + (x^n)) = 0 + (x^n)$. This gives $r_0g_0 = 0, r_0g_1 + f_1g_0 = 0, \ldots, g_{n-1}r_0 + g_{n-2}f_1 + \ldots + g_0f_{n-1} = 0$. This gives $g_0 = g_1 = \ldots = g_{n-1} = 0$. Therefore $(r_0 + f_1x + f_2x^2 + \ldots + f_{n-1}x^{n-1}) + (x^n) \in Reg(S)$. Clearly, $q_0 + (x^n) \in Q(S)$. Hence S is a qclean.
(**ii**) $\Longrightarrow$ (**i**) Let S be an almost qclean and hence for $r \in R$, we have $r + (x^n) = (f + (x^n)) + (q + (x^n))$, clearly $r = f_0 + q_0$ and it is easy to see that $q_0 \in Q(R)$. Now we prove f_0 is an

regular element of R. Assume f_0 is not regular element of R then there exist $g_0 \neq 0 \in R$ such that $f_0g_0 = 0$. Let $g = g_0x^{n-1} \in R[[x]]$, then $(f + (x^n))(g + (x^n)) = 0 + (x^n)$, which is a contradiction because $f + (x^n) \in Reg(S)$. Hence R is an almost qclean. □

It is well known that if ring R is commutative clean ring, then $R/nil(R)$ is also clean. However, we can prove that if R is almost qclean, then $R/nil(R)$ is also qclean.

Proposition 3.8. *If R is a commutative almost qclean, then $R/nil(R)$ is almost qclean.*

Proof. Since R is almost qclean, then each element x in R can be expressed as $x = r + q$ where $r \in Reg(R)$ and $q \in Q(R)$. Hence $x' \in R'$ where $R' = R/nil\,(R)$ can be expressed as $r' + q'$. It is easy to see that $q' \in Q\,(R')$. So, now we prove $r' \in Reg\,(R')$. Consider $r'a' = 0'$, this implies $ra \in nil(R)$. Therefore there exists a natural number n such that $(ra)^n = 0$. But $r \in Reg(R)$ so $a^n = 0$, this implies $a \in nil(R)$. Hence $a' = 0'$ in R and therefore $r' \in Reg\,(R')$. □

Let $(T, +)$ be a torsion-less commutative cancellative monoid (torsion-less mean if $nt_1 = nt_2 \to t_1 = t_2$ where $t_1, t_2 \in T$, $n \in \mathbb{N}$). It is easy to see that $R[x;T] = \left\{\sum_{i=1}^{n} a_i x_i^{t_i} \,|\, t_i \in T, a_i \in R\right\}$ is the monoid ring of R over T. Note that if T is a torsion-less cancellative commutative monoid and R is an integral domain then $R[x;T]$ is also an integral domain. Also the map $R[x;T] \to R$ defined by $a_0 + \sum_{t_i \neq 0} a_i x^{t_i} \to a_0$ is a ring homomorphism whenever T is a cancellative commutative monoid with $T \cap (-T) = 0$.

Proposition 3.9. *If $q_0 + \sum_{t_i \neq 0} q_i x^{t_i} \in Q(R[x;T])$, where T is commutative torsion-less cancellative monoid with $T \cap (-T) = 0$, then $q_0 \in Q(R)$ and $q_i \in nil(R)$, for $i \neq 0$.*

Proof. Consider a mapping $f : R[x;T] \to R$ such that $f(a_0 + \sum_{t_i \neq 0} a_i x_i^{t_i}) \to a_0$. But as $T \cap (-T) = 0$, this implies f is a ring homomorphism. Thus, if $q_0 + \sum_{t_i \neq 0} q_i x^{t_i} \in Q(R[x;T])$, then $q_0 \in Q(R)$. Now, consider a map from $R[x;T] \to R[x;T]$ such that $a_0 + \sum_{t_i \neq 0} a_i x_i^{t_i} \to \bar{a_0} + \sum_{t_i \neq 0} \bar{a}_i x^{t_i}$, where $\bar{R} = R/P$ and P is a prime ideal. Clearly, the kernel of this map is $a_0 + \sum_{t_i \neq 0} a_i x^{t_i}\; a_0, a_i \in P$. As $\bar{R}$ is an integral domain and T is torsion-less cancellative monoid, this gives $Q(\bar{R}[x;T]) = Q(R)$. This implies $\bar{q_i} = \bar{0}$ for $i \neq 0, q_i \in P$. Hence $q_i \in \bigcap P = nil(R)$. □

Theorem 3.10. *Let T be commutative torsion-less cancellative monoid with $R = \bigoplus_{\beta \in T} T_\beta$ be a commutative graded ring. If T_0 is almost qclean and each T_β is torsion-free T_0 module, then R is almost qclean.*

Proof. Let $x \in R,\ x = \sum x_\beta$, where $x_\beta \in T_\beta$. Since T_0 is almost qclean, $x_0 = a + q$, where $a \in Reg(T_0)$ and $q \in Q(T_0)$. Therefore, $x = a + \sum_{\beta \neq 0} x_\beta + q$. Clearly $q \in Q\,(R)$. Now it suffices to prove that $a + \sum_{\beta \neq 0} x_\beta \in Reg\,(R)$. Let $y = a + \sum_{\beta \neq 0} x_\beta$ and assume $y \in Z\,(R)$, so there exists non zero element $\beta \in T_\beta$ with $y\beta = 0$ but then $a\beta = 0$ [7, Theorem 8.4], this contradict to T_β is a torsion free T_0-module. □

Theorem 3.11. *Let $R\,[x : T]$ be an almost qclean, where R is a commutative ring and T be a commutative torsion-less cancellative monoid with $T \cap (-T) \neq 0$, then R is an almost qclean.*

Proof. Since R is an almost qclean, so for $y \in R, y = a+q$, where $a = a_0+\sum_{t_i \neq 0} a_i x^{t_i} \in Reg\,(R\,[x;T])$ and $q = q_0 + \sum_{t_i \neq 0} q_i x^{t_i} \in Q\,(R\,[x;T])$. By Proposition 3.9, $q_0 \in Q\,(R)$ and $q_i \in nil\,(R)$, for $i \neq 0$. Hence $a_i = -q_i \in nil\,(R)$ for $i \neq 0$. Now to prove $a_0 \in Reg\,(R)$, assume that $a_0 \notin Reg\,(R)$ i.e. $a_0 \in Z\,(R\,[x;T])$. This says there exists a prime ideal P of $R([x;T])$ such that $a_0 \in P \subseteq Z\,(R\,[x;T])$ and $a_i \in nil\,(R)\,,\ i \neq 0$. This implies $a_i x^{t_i} \in nil\,(R\,[x;T]) \subseteq P$. Hence $a \in P$. But $P \subseteq Z\,(R\,[x : T])$, this gives $a \in Z\,(R\,[x;T])$ but this is not possible as $a \in Reg\,(R\,[x;T])$. Thus, $a_0 \in Reg\,(R)$ and hence R is almost qclean. □

Theorem 3.12. *Let $R = S_0 \bigoplus S_1 \bigoplus S_2 \bigoplus \ldots\ldots.$ be a commutative graded ring and if for $1 \leq i \leq n-1, S_i$ is torsion free S_0 module with S_0 is almost qclean, then $(R/S_n \bigoplus S_{n+1} \bigoplus \ldots)$ is qclean.*

Proof. Let $a \in R,\ a = a_0 + a_1 + \ldots + a_{n-1} + S_n \bigoplus S_{n+1} \bigoplus \ldots..$, where $a_i \in S_i$. As S_0 is almost qclean, so $a_0 = r + q$, where $r \in Reg\,(S_0)$ and $q \in Q\,(S_0)$. It is easy to see that

$q + S_n \bigoplus S_{n+1} \bigoplus \ldots \in Q\left((R/S_n \bigoplus S_{n+1} \bigoplus \ldots)\right)$. Now we prove $y = r + a_1 + \ldots. + a_{n-1} + S_n \bigoplus S_{n+1} \bigoplus .... \in Reg\left((R/S_n \bigoplus S_{n+1} \bigoplus \ldots)\right)$. Suppose that $y \notin Reg((R/S_n \bigoplus S_{n+1} \bigoplus \ldots))$ i.e $y \in Z\left(R/S_n \bigoplus S_{n+1} \bigoplus \ldots\right)$.
But $(R/S_n \bigoplus S_{n+1} \bigoplus \ldots)$ is a graded ring, by [7, Theorem 8.4], there is a non-zero element $x_i \bigoplus S_n \bigoplus S_{n+1} \bigoplus ....$, $x_i \neq 0 \in S_i$ for some $0 \leq i \leq n-1$ such that $(y+S_n \bigoplus S_{n+1} \bigoplus ....)(x_i + S_n \bigoplus S_{n+1} \bigoplus ....) = 0+S_n \bigoplus S_{n+1} \bigoplus ......$ This gives $yx_i + S_n \bigoplus S_{n+1} \bigoplus ... = 0+S_n \bigoplus S_{n+1} \bigoplus ...$, hence $yx_i = 0$ which is not possible as S_i is torsion-free S_0 module. Hence the result follows. □

References

[1] M. S. Ahn, D. D. Anderson, Weakly clean rings and almost clean rings, Rocky Mountain J. Math., **36**; 783–798, 2006.

[2] D. D. Anderson, V. P. Camillo, Commutative rings whose elements are a sum of a unit and idempotent, Comm. Algebra **30**, 3327–3336, 2002.

[3] D. Andrica, G. Calugareanu, A nil-clean 2×2 matrix over the integers which is not clean, J. Algebra Appl, **13**(6); 1450009 , 2014.

[4] N. Ashrafi, E. Nasibi, r-clean rings, Math. Rep. (Bucur.), **15**, 125–132, 2013.

[5] S. Breaz, P. Danchev and Y. Zhou, Rings in which every element is either a sum or a difference of a nilpotent and an idempotent, J. Algebra Appl, **15**, 1650148, 2016.

[6] A. J. Diesl, Nil clean rings, J. Algebra, **383** , 197–211, 2013.

[7] J. Han, W. K. Nicholson, Extensions of clean rings, Comm. Algebra, **29**, 2589–2595, 2001.

[8] R. Gilmer, Commutative Semigroup Rings: Chicago Lectures in Mathematics. Chicago and London: University of Chicago press, 1984.

[9] W. K. Nicholson, Lifting idempotents and exchange rings, Trans. Amer. Math. Soc., **229**, 269–278, 1977.

[10] Z. Ying, T. Kosan and Y. Zhou, Rings in which every element is a sum of two tripotents, Canad. Math. Bull., **59(3)**, 661-672, 2016.

Author information

R. G. Ghumde[1] and M. K. Patel[2], [1]Department of Mathematics, Shri Ramdeobaba College of Engineering and Management, Nagpur, Maharashtra -440024, India;
[2] Department of Mathematics, National Institute of Technology Nagaland, Chumukedima-797103, India.
E-mail: `ghumderg@rknec.edu, mkpitb@gmail.com`