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A note on Nil qclean rings

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Abstract A. J. Diesl introduced the notion of nil clean ring, where each element of ring is the sum of an idempotent and nilpotent element. Motivated by this structure, we introduce here the notion of nil qclean by replacing idempotent element by qpotent element in nil clean ring. In this article, we focus to study the fundamental properties of nil qclean rings and also proved that the ring of upper triangular matrices is nil qclean whenever base ring is nil qclean. Apart from this, we find some conditions on ring so that the ring of ideal extension is nil qclean.

1 Introduction

In last two to three decades many mathematician have considerable interest in classifications of those rings whose elements can be expressed as sum of some special elements of ring. Idempotent element, regular element (element has non zero divisor), element from Jacobson radical, nilpotent element, periodic element etc are the some element in the list of special elements of rings. Several researchers investigated variants of such type of rings in various articles [[1], [2], [4], [5], [10], [12]]. Few rings in these variants are clean and strongly clean ring, nil clean, J-clean, semi-potent ring, almost clean ring, precious ring etc. In the direction of defining the notion of rings whose each element is sum of special elements, the first such a ring is introduced by Nicholson. Nicholson [10] introduces new class of rings known as clean rings. In this ring, every element can be expressed as sum of idempotent and unit element. In the same article author investigated that this notion of clean ring enjoy the exchange property of rings. We get the further investigation on clean ring in [[1], [2], [9], [12]]. Connected to the notion of clean ring, Diesl [7] introduces the notion of nil clean rings. While defining the structure of nil clean ring Diesl replaced unit element by nilpotent element in the notion of clean rings. In the same article Diesl discusses some fundamental properties of nil qclean rings and also studied these fundamental properties for different ring extensions. Further many researchers investigated this notion of nil clean rings in [[5], [6], [11]].

In this paper we continue the study of ring whose elements can be expressed as sum of special elements, and we take qpotent element as a special element. An element q of ring R is known as qpotent, if $q^4 = q$. Clearly idempotent element is a qpotent but converse need not be true. For this consider the ring $\mathbb{Z}_7$, here the element 4 is qpotent but not an idempotent. By using qpotent element here we introduces the notion of nil qclean rings. For this we replace idempotent element by qpotent element in the notion of nil clean rings. One can note that the concept of nil qclean ring is generalization of notion of nil clean ring. The main objective of this paper is to prove the class of nil qclean ring is closed under various fundamental structures of rings like quotient structure, direct product, homomorphic images etc. In this paper we prove the ring of upper triangular matrices is nil qclean whenever base ring is nil qclean and also discusses some results on ideal extensions.

In this paper, $Q(R)$ represents the set of qpotent elements, $U(R)$ represents the set of units (infact, this set form a group under multiplication), $N(R)$ represents the set of nilpotent elements and $C(R)$ is the center of the ring R.

2 nil qclean rings

We start this section by giving the definitions and example of nil qclean ring and qclean ring.

Definition 2.1. An element $a \in R$ is known as nil qclean if it is a sum of a qpotent and nilpotent element. In a ring R, if each element is nil qclean, then ring is said to be nil qclean i.e. each element of R, expressed as a sum of nilpotent and qpotent element of R.

From the above definition it is easy to see that given any nil clean element is always nil qclean but the converse need not be true. This follows from the example mention below.

Example 2.2. Consider the field $F_4 = \{a + b\alpha | \alpha^2 = \alpha + 1, a, b \in \mathbb{Z}_2\}$ and $R = (\begin{pmatrix} c & d \\ 0 & c \end{pmatrix} | c, d \in F_4)$, It is easy to see that R is a ring with usual matrix addition and multiplication. In this ring all scalar matrices are qpotent, matrices with 0 on diagonal entries are nilpotent and idempotent elements of this ring are only identity and null matrix. Note that $\begin{pmatrix} \alpha & \alpha \\ 0 & \alpha \end{pmatrix}$ is nil qclean but not nil clean. In fact each element of this ring R is nil qclean and hence R is nil qclean but not nil clean.

Definition 2.3. An element $a \in R$ is called qclean if a is a sum of unit and qpotent element. If all the elements in ring R are qclean then ring R is said to be qclean i.e. given any element of R is expressed as a sum of unit and qpotent element of R.

The following proposition discusses the relationship between qclean and nil qclean rings.

Proposition 2.4. *Every nil qclean ring is qclean.*

Proof. Suppose R is a nil qclean ring and let $a \in R$ then $a - 1 = q + \eta$ where $q \in Q(R)$ and $\eta \in N(R)$. Therefore, $a = q + (1 + \eta)$. But $(1 + \eta)$ is unit, hence a is qclean. □

In next two propositions, we prove some elementary results on nil qclean rings.

Proposition 2.5. *Let R be a ring with characteristic 2, then $a \in R$ is nil qclean if and only if $(1 - a)$ is nil qclean.*

Proof. Since R is nil qclean, for $a \in R$ we have $a = q + \eta$, where $q \in Q(R)$ and $\eta \in N(R)$. Clearly $(1 - a) = (1 - q) + (-\eta)$. Here $\eta \in N(R)$ and the characteristic of ring is 2, so $(1 - q) \in Q(R)$. Similarly we can prove the converse and hence the result follows. □

Proposition 2.6. *Let R be a ring. Then R is nil qclean if and only if any element $a \in R$ can be expressed as $a = -q + \eta$.*

Proof. $(\Rightarrow)$ Suppose R is a nil qclean and hence $-a \in R$ can be expressed as $-a = q + \eta$ where $q \in Q(R)$ and $\eta \in N(R)$. Clearly $a = -q + (-\eta)$ where $(-\eta) \in N(R)$ and $q \in Q(R)$.
$(\Leftarrow)$ We prove this part by using similar arguments. □

Next, we prove some results on various ring constructions concerning nil qclean property.

Proposition 2.7. *An homomorphic image of nil qclean ring is nil qclean.*

Proof. Let R' be an homomorphic image of nil qclean ring R. Therefore, for any $a' \in R'$ there exists $a \in R$ such that $f(a) = a'$. Since R is nil qclean, $a = q + \eta$ where $q \in Q(R)$ and $\eta \in N(R)$. Hence, $a' = f(a) = f(q + \eta) = f(q) + f(\eta)$. Clearly, as f is a ring homomorphism so $(f(q))^4 = f(q^4) = f(q)$ and $f(q) \in Q(R')$ also it is easily seen that $f(\eta) \in N(R')$. Hence R' is nil qclean. □

But the converse of Proposition 2.7 need not be true. For this we consider a ring of integers $\mathbb{Z}$, the non trivial homomorphic image of $\mathbb{Z}$ is $\mathbb{Z}_m$, which is a nil qclean for some m but in $\mathbb{Z}$ as 0 is the only nilpotent element. So $\mathbb{Z}$ is not nil qclean.

The another important structure of ring is quotient structure. Now we investigate some conditions over ideals of ring which imply that, whenever given ring is nil qclean then the quotient ring is also nil qclean and vice versa.

Proposition 2.8. *Let I be a nil ideal of ring R with qpotent element of R/I lifted by modulo I, then the following statements are equivalent:*
(**i**) *R is a nil qclean;*
(**ii**) *R/I is nil qclean ring.*

Proof. $(i) \Rightarrow (ii)$ From Proposition 2.7 this part is trivial.
$(ii) \Rightarrow (i)$ Suppose $\bar{R} = R/I$ be a nil qclean ring, hence for $a \in R$, we have $\bar{a} = \bar{\eta} + \bar{q}$ where $\bar{\eta} \in N(\bar{R})$ and $\bar{q} \in Q(\bar{R})$. But qpotent elements lifted by modulo I and therefore $a = q + \eta + y$ where $y \in I$. Since $\eta \in N(R)$, so for some integer n, $\eta^n \in I$ and this imply $\eta + y \in N(R)$. So a is nil qclean and hence the result follows. □

For a ring R, we know that prime radical $P(R)$ is nil ideal. Then from Proposition 2.8, we can state, if a ring R is nil qclean, then $R/P(R)$ is also nil qclean.

Next we prove the property of being nil qclean is closed under the direct product.

Proposition 2.9. *If $R = \prod_{i\in I} R_i$ where $(R_i)_{i\in I}$ be a family of rings, then we have following assertions:*
(**i**) *If R is nil qclean, then each R_i is nil qclean.*
(**ii**) *If I is finite then, R is nil qclean if and only if each R_i is nil qclean.*

Proof. (**i**) It follows directly from Proposition 2.7
(**ii**) It is easy to see that finiteness of I implies $\prod_{i\in I} N(R_i) = N(R)$ and hence $\prod_{i\in I} N(R_i) \subseteq N(R)$. Now assume each R_i is nil qclean and let $(x_i) \in R$ where $x_i \in R_i$. Therefore, $x_i = \eta_i + q_i$, where $\eta_i \in N(R_i)$ and $q_i \in Q(R_i)$. Clearly, $(\eta_i) \in \prod_{i\in I} N(R_i) \subseteq N(R)$, this gives $(\eta_i) \in N(R)$ and $(q_i) \in Q(R)$. Hence R is nil qclean. The converse part follows from (i). □

Next we study the matrix rings in context of nil qclean rings. For this we consider rings X and Y with M be an $Y - X$ bi-module and find some conditions on rings X and Y so that the upper triangular matrix ring $\begin{pmatrix} X & M \\ 0 & Y \end{pmatrix}$ is nil qclean.

Proposition 2.10. *In a ring R, if q is a central qpotent element of R and q^3Rq^3 and $(1-q^3)R(1-q^3)$are both nil qclean, then so is R.*

Proof. Since q is a qpotent element, so q^3 is an idempotent element. Therefore by Pierce decomposition we can write, $R = q^3Rq^3 \bigoplus q^3R(1-q^3) \bigoplus (1-q^3)Rq^3 \bigoplus (1-q^3)R(1-q^3)$. But qpotent elements are central and hence we have $R = q^3Rq^3 \bigoplus (1-q^3)R(1-q^3) \cong \begin{pmatrix} q^3Rq^3 & 0 \\ 0 & (1-q^3)R(1-q^3) \end{pmatrix}$. So for any $a \in R$, we can write $a = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ where $x \in q^3Rq^3$ and $y \in (1-q^3)R(1-q^3)$. But by given hypothesis x and y are nil qclean. Thus, $x = q_1 + \eta_1$ and $y = q_2 + \eta_2$ where $q_1, q_2 \in Q(R)$ and $\eta_1, \eta_2 \in N(R)$. So, $a = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} q_1 + \eta_1 & 0 \\ 0 & q_2 + \eta_2 \end{pmatrix} = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} + \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}$. Clearly $\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \in Q(R)$ and $\begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix} \in N(R)$, it follows that R is nil qclean. □

Proposition 2.11. *Let $q_1, q_2, ...q_n$ be an orthogonal central qpotents with $q_1 + q_2 + ... + q_n = 1$, and if for each i, $q_i^3Rq_i^3$ and $(1-q_i^3)R(1-q_i^3)$ are nil qclean, then R is nil qclean.*

Proof. This follows from Proposition 2.10 by induction. □

Proposition 2.12. *Let* $M = \begin{pmatrix} X & S \\ T & Y \end{pmatrix}$ *be the ring of Morita context, where* T *is* $X - Y$ *bi-module and* S *is a* $Y - X$ *bi-module of rings* X *and* Y*. Then* M *is nil qclean if and only if* X *and* Y *are nil qclean.*

Proof. Let $A = \begin{pmatrix} x & S \\ T & y \end{pmatrix} \in M$. Since $x \in X$ and $y \in Y$are nil qclean, $x = q_1 + \eta_1, y = q_2 + \eta_2$. Then $A = Q_1 + N_1$ where $Q_1 = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}$ and $N_1 = \begin{pmatrix} \eta_1 & S \\ T & \eta_2 \end{pmatrix}$. Clearly, $Q_1 \in Q(M)$ and $N_1 \in N(M)$. Hence A is nil qclean.
To prove converse, consider $I = \left\{ \begin{pmatrix} 0 & s \\ t & y \end{pmatrix} : y \in Y, s \in S \text{ and } t \in T \right\}$. Clearly, $X \cong M/I$. Hence by Proposition 2.7, X is nil qclean. Similarly we can show Y is nil qclean. □

Proposition 2.13. *Let* X *and* Y *are rings with* S *be a* $Y - X$ *bi-module. Then the triangular matrix ring* $\begin{pmatrix} X & S \\ 0 & Y \end{pmatrix}$ *is nil qclean if and only if* X *and* Y *are nil qclean.*

Proof. The proof follows from Proposition 2.12 by putting $T = 0$. □

Next, we use Proposition 2.12 and 2.13 to prove some results on the ring of upper triangular matrices.

Proposition 2.14. *Let* R *be a ring. Then the ring of upper triangular matrices* $UM_n(R)$ *is nil qclean if and only if R is nil qclean.*

Proof. We use induction on $'n'$ to prove this result. Assume that the result is true for $n-1$. So, $UM_n(R) = \begin{pmatrix} R & S \\ 0 & UM_{n-1} \end{pmatrix}$, where $S = \{(s_1, s_2, ..., s_{n-1}) : s_i \in R\}$. Clearly S is an $UM_{n-1}(R) - R$ bi-module. Hence by Proposition 2.13, the result follows. □

Note that the similar result is also true for the ring of Lower triangular matrices.

In next proposition we impose some conditions on the ring, so that the elements of ring will be nil qclean.

Proposition 2.15. *Let* R *be a ring with characteristic* 2 *and qpotent elements are commute with nilpotent elements. Then an element* $b \in R$ *is nil qclean if and only if* $b^4 - b \in N(R)$*.*

Proof. Suppose R is nil qclean, so from given hypothesis for $b \in R$, $b = q + \eta$ where $q \in Q(R), \eta \in N(R)$ and $q\eta = \eta q$. Therefore, $b^4 - b = (q+\eta)^4 - (q+\eta) = q^4 + \eta^4 - q - \eta = \eta^4 - \eta \in N(R)$.
To prove converse, first we consider $\eta = 1 - b^3$ then $b\eta = \eta b = b - b^4 \in N(R)$. So there exist $n \in \mathbb{N}$ such that $b^n\eta^n = \eta^n b^n = (b\eta)^n = 0$. Now we select $k \in \mathbb{N}$ such that $4k \geq n$. But the characteristic of ring R is 2, so we have $1 = (\eta + b^3)^{4k} = (\eta)^{4k} + (b^3)^{4k} = \eta^{4k} + (b^{4k})^3$. Put $q = b^{4k}$ and $f = \eta^{4k}$, this gives $q^3 + f = 1$ and $qf = (b\eta)^{4k} = 0$. But $q = q(q^3 + f) = q^4$. Thus, $q = b^{4k}$ is qpotent and $(b-q)^{4k} = b^{4k} - q^{4k} = q - q^{4k} = 0$, this gives $b - q \in N(R)$. Hence we can write $b = q + (b - q)$. So b is nil qclean. □

Let R be a ring with M be an $R - R$ bi-module, then the ideal extension of R is denoted by $I(R; M)$ which is also form a ring with usual addition and multiplication defined as $(r, m)(s, m') = (rs, rm' + ms + mm')$ where $r, s \in R$ and $m, m' \in M$.

In the next few propositions we discuss the results on ideal extentions.

Proposition 2.16. *Let* R *be a ring and* $I(R; M)$ *be nil qclean ring, then* R *is nil qclean.*

Proof. Clearly, $R \cong I(R;M)/(0 \bigoplus M)$ then by Proposition 2.7 the result follows. □

One can naturally ask whether the converse of above proposition is true or not. In the following proposition some conditions are given for existence of the converse.

Proposition 2.17. *Let R be a ring with $N(R) \subseteq C(R)$ and for any $m \in M$, for some $n \in \mathbb{N}$ we have $\binom{n}{n-1}\eta^{n-1}m + \binom{n}{n-2}\eta^{n-2}m^2 + ... \binom{n}{1}\eta m^{n-1} + m^n = 0$, then $I(R;M)$ is nil qclean.*

Proof. Let $(r,m) \in I(R;M)$. Since R is nil qclean, so $(r,m) = (q,0) + (\eta,m)$ where $q \in Q(R)$ and $\eta \in N(R)$. By given condition, for $n \in \mathbb{N}$ we have $\binom{n}{n-1}\eta^{n-1}m + \binom{n}{n-2}\eta^{n-2}m^2 + ... \binom{n}{1}\eta m^{n-1} + m^n = 0$ and since η is nilpotent, so for some $k \in \mathbb{N}$ we have $\eta^k = 0$. If we take $l = max(n,k)$ then $(\eta,m)^l = (\eta^l,0) = (0,0)$. So $(\eta,m) \in I(R;M)$ and clearly, $(q,0) \in Q(I(R;M))$. Hence the result follows. □

From the notation of nil qclean ring one can observe that whenever ring R is nil qclean, then polynomial ring $R[x]$ need not be nil qclean ring. But following result holds.

Proposition 2.18. *If R is a commutative ring then, R is nil qclean if and only if $R[x]/(x^n)$ is nil qclean.*

Proof. ($\Rightarrow$) Suppose R is nil qclean and let $f \in R[x]/(x^n)$ where $f = \sum_{i=0}^{n-1} r_i x^i + (x^n)$. But R is nil qclean, so $r_0 = q_0 + \eta_0$ where $q_0 \in Q(R)$ and $\eta_0 \in N(R)$. Clearly $q_0 + (x^n) \in Q(R[x]/(x^n))$ and $\eta_0 + \sum_{i=1}^{n-1} r_i x^i + (x^n) \in N(R[x]/(x^n))$. This implies f is nil qclean.
($\Leftarrow$) Note that $(R[x]/(x^n))/(x + (x^n)) \cong R$. Hence by Proposition 2.7 the result follows. □

References

[1] M. S. Ahn, D. D. Anderson, Weakly clean rings and almost clean rings, Rocky Mountain J. Math., **36**; 783–798, 2006.

[2] D. D. Anderson, V. P. Camillo, Commutative rings whose elements are a sum of a unit and idempotent, Comm. Algebra, **30**, 3327–3336, 2002.

[3] D. Andrica, G. Calugareanu, A nil-clean 2×2 matrix over the integers which is not clean, J. Algebra Appl, **13**(6); 1450009 , 2014, doi: 10.1142/S0219498814500091.

[4] N. Ashrafi, E. Nasibi, r-clean rings, Math. Rep. (Bucur.), **15**, 125–132, 2013.

[5] H. Chen, M. Sheibani, On strongly nil clean rings, Comm. Algebra, **45(4)**,1719–1726, 2017.

[6] H. Chen, M. Sheibani, Strongly 2-nil clean rings, J. Algebra Appl, **16(09)**, 1750178, 2017, doi: 10.1142/S021949881750178X.

[7] A. J. Diesl, Nil clean rings, J. Algebra, **383** , 197–211, 2013.

[8] J. Han, W. K. Nicholson, Extensions of clean rings, Comm. Algebra, **29**, 2589–2595, 2001.

[9] W. K. Nicholson, Strongly clean rings and fitting lemma, Comm. Algebra, **27(8)**, 3583-3592, 1999.

[10] W. K. Nicholson, Lifting idempotents and exchange rings, Trans. Amer. Math. Soc., **229**, 269–278, 1977.

[11] A Stancu, A note on commutative weakly nil clean rings, J. Algebra Appl., **15(10)** 1620001, 2016, doi: 10.1142/S0219498816200012.

[12] Y. Ye , Semiclean rings, Comm. Algebra, **31**, 5609–5625, 2003.

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