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LUKASIEWICZ FUZZY IDEALS APPLIED TO BCK/BCI-ALGEBRAS

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Abstract In fuzzy logic, a fuzzy set $\mathfrak{FS}$ is defined as a set where each element has a degree of membership between 0 and 1. In Lukasiewicz fuzzy $\mathfrak{LF}$ logic, the membership function based on the Lukasiewicz t-norm and t-conorm operations. The possibility of $\mathfrak{LF}$ sets are applied to BCK/BCI-algebras. Additionally, the thought of $\mathfrak{LF}$ ideals is presented and its different properties are explored. Three sorts of subsets assumed $\in$-set, q-set and O-set are developed, and the circumstances under which they can be ideals are examined.

1 Introduction

$\mathfrak{FS}$s are a mathematical framework that extends the traditional notion of sets by allowing elements to have degrees of membership. Unlike classical sets where an element either belongs or does not belong to a set, $\mathfrak{FS}$s allow for partial membership based on a degree of resemblance. In classical set theory, an element can be represented as a crisp set, denoted as x, where x either belongs to the set or does not belong to the set. In contrast, a $\mathfrak{FS}$ allows for degrees of membership. The level of membership is a worth somewhere in the range of 0 and 1, where 0 addresses non-participation and 1 addresses full membership. For example, in a $\mathfrak{FS}$ representing the height of people, an individual's height could have a membership value of 0.8, indicating that they are 80 % tall.

$\mathfrak{FS}$s were introduced by Lotfi Zadeh [1] in 1965 as a way to model and reason with imprecise and uncertain information. They have found applications in various fields, including artificial intelligence, control systems, decision-making, pattern recognition, and data analysis. The degree of membership in a $\mathfrak{FS}$ is typically represented by a membership function, which assigns a membership value to each element in the set's universe of discourse. The shape of the membership function can vary depending on the application and the desired behavior. Commonly used membership functions include triangular, trapezoidal, Gaussian, and sigmoidal functions. Iséki and Tanaka first described *BCK*/it BCI algebras in cite(a2) to generalise the set difference in set theory. Fuzzy ideal($\mathfrak{FI}$)s and fuzzy subalgebras in *BCK* algebras were researched by Jun et al.[7] in 1999.

$\mathfrak{LF}$ sets, also known as $\mathfrak{LF}$ sets, are a type of $\mathfrak{FS}$ that uses the Lukasiewicz t-norm and t-conorm operations to compute the intersection and union of $\mathfrak{FS}$s, respectively. They are named after Jan Lukasiewicz, a Polish logician who introduced these operations. $\mathfrak{LF}$ sets have been widely used in fuzzy logic, $\mathfrak{FS}$ theory, and fuzzy control systems. They provide a fuzzy reasoning framework that is compatible with Lukasiewicz logic, a three-valued logic system that extends classical two-valued logic by introducing a third truth value, namely "unknown" or "indeterminate". This logic allows for reasoning with uncertain or incomplete information, making it suitable for various applications involving vagueness and imprecision. In *BCK/BCI*- algebras, Jun [10] investigated fuzzy subalgebras with thresholds. Jun et al. [8, 11, 12] introduced Lukasiewicz fuzzy subalgebras in BCI/BCK-algebras, $\mathfrak{FI}$s and fuzzy subalgebras of BCK-algebras, and crossing cubic ideals of BCK-algebras. Balamurugan et al. [13, 14, 15] contemplated different parts of BCK/BCI-algebras in light of ideal hypothesis.

Utilizing the Lukasiewicz t-norm idea, we construct the thought of $\mathfrak{LF}$ sets in light of a predetermined $\mathfrak{FS}$ and apply it to BCK/BCI-algebras in this work. We outline requirements that must be met for a $\mathfrak{LF}$ set to satisfy this criterion. We discuss descriptions of the $\mathfrak{LF}$ ideals. We build three different subsets, called in-set, q-set and O-set, and we determine under what circumstances they may be ideals.

2 Preliminaries

A BCK/BCI-algebra is a significant class of legitimate algebras that was developed by Iséki (See [2]) and was thoroughly studied by a number of scholars. We are reminded of the concepts and fundamental findings that this work needs. For additional details on BCK/BCI-algebras, consult the books listed in [3, 9]. If the criteria are fulfilled and a fixed Q has a special component 0 and a binary " $*$ ":
(I_1) $(\forall q,\ v,\ e \in Q)$ $(((q*v)*(q*e))*(e*v)=0)$,
(I_2) $(\forall q,\ v \in Q)$ $((q*(q*v))*v=0)$,
(I_3) $(\forall q \in Q)$ $(q*q=0)$,
(I_4) $(\forall q,\ v \in Q)$ $(q*v=0,\ v*q=0 \Rightarrow q=v)$,
then we say that Q is a *BCI-algebra*. If a BCI-algebra Q satisfies:
(K) $(\forall q \in Q)$ $(0*q=0)$,
then Q is called a *BCK-algebra*.
The leq order relation within a BCK/BCI-algebra Q is defined as follows:

$$(\forall q,\ v \in Q, q \leq v \Leftrightarrow q*v=0). \tag{2.1}$$

The following criteria are met by each Q in the BCK/BCI algebra. (See [3, 9]):

$$(\forall q \in Q, q*0=q), \tag{2.2}$$

$$(\forall q,\ v,\ e \in Q, q \leq v \Rightarrow q*e \leq v*e,\ e*b \leq e*q), \tag{2.3}$$

$$(\forall q,\ v,\ e \in Q, (q*e)*(v*e) \leq q*v). \tag{2.4}$$

$$(\forall q,\ v,\ e \in Q, (q*v)*e=(q*e)*v)). \tag{2.5}$$

Every BCI-algebra Q satisfies (See [8]):

$$(\forall q,\ v \in Q, q*(q*(q*v))=q*v), \tag{2.6}$$

$$(\forall q,\ v \in Q, 0*(q*v)=(0*q)*(0*v)). \tag{2.7}$$

A I subset of a BCK/BCI-algebra Q is referred to as a *ideal* of Q (See [8, 11]) if it meets:

$$(\forall q,\ v \in I)\ (q*v \in I, v \in I \Rightarrow q \in I), \tag{2.8}$$

A $\mathfrak{FS}$ $\dot{m}$ in a set Q of the form

$$\dot{m}(v) := \begin{cases} s \in (0,1] & \text{if } v = q, \\ 0 & \text{if } v \neq q, \end{cases}$$

is said to be a *fuzzy point* with support q and a value of s and is indicated by $[q/s]$.
For a $\mathfrak{FS}$ $\dot{m}$ in a set Q, we say that a fuzzy point $[q/s]$ is
(i) *contained* in $\dot{m}$,shown by $[q/s] \in \dot{m}$([3]), if $\dot{m}(q) \geq s$,
(ii) *quasi-coincident* with $\dot{m}$, shown by $[q/s]q\dot{m}$, if $\dot{m}(q)+s>1$.
A $\mathfrak{FS}$ $\dot{m}$ within a BCK/BCI-algebra Q is called a $\mathfrak{FI}$ of Q, if it meets:

$$(\forall q,\ v \in Q, \dot{m}(q) \geq \min\{\dot{m}(q*v),\ \dot{m}(v)\}). \tag{2.9}$$

3 Lukasiewicz fuzzy ideals

Definition 3.1. Let $\dot{m}$ be a $\mathfrak{FS}$ in Q, and let $\delta \in [0,1]$. A function

$$Ł_{\dot{m}}^{\delta}\colon Q \to [0,\ 1], q \mapsto \max\{0, \dot{m}(q) + \delta - 1\}$$

is called an δ-$\mathfrak{LF}$ *set* of $\dot{m}$ in Q.

Let $Ł_{\dot{m}}^{\delta}$ be an δ-$\mathfrak{LF}$ set of a $\mathfrak{FS}$ $\dot{m}$ in Q. If $\delta = 1$, then

$$Ł_{\dot{m}}^{\delta}(q) = \max\{0,\ \dot{m}(q) + 1 - 1\} = \max\{0,\ \dot{m}(q)\} = \dot{m}(q), \text{ for all } q \in Q.$$

If $\delta = 0$, then

$$Ł_{\dot{m}}^{\delta}(q) = \max\{0,\ \dot{m}(q) + 0 - 1\} = \max\{0,\ \dot{m}(q) - 1\} = 0, \text{ for all } q \in Q.$$

Consequently, in direction the δ-$\mathfrak{LF}$ set, the value of δ can ever be reasoned to be in $(0,1)$. Let $\dot{m}$ be a $\mathfrak{FS}$ in a set Q and $\delta \in (0,1)$. If $\dot{m}(q) + \delta \leq 1\ \forall\ q \in Q$, then the δ-$\mathfrak{LF}$ set $Ł_{\dot{m}}^{\delta}$ of $\dot{m}$ in Q is the 0-constant function, (i.e), $Ł_{\dot{m}}^{\delta}(q) = 0$ for all $q \in Q$. Subsequently, for the δ-$\mathfrak{LF}$ set to have a huge structure, a $\mathfrak{FS}$ $\dot{m}$ in Q and $\delta \in (0,1)$ should be set to delight the accompanying condition:

$$(\exists q \in Q)\ (\dot{m}(q) + \delta > 1). \tag{3.1}$$

Proposition 3.2. *If $\dot{m}$ is a $\mathfrak{FS}$ in a set Q and $\delta \in (0,1)$, then its δ-$\mathfrak{LF}$ set $Ł_{\dot{m}}^{\delta}$ satisfies:*

$$(\forall q,\ v \in Q)\ (\dot{m}(q) \geq \dot{m}(v) \Rightarrow Ł_{\dot{m}}^{\delta}(q) \geq Ł_{\dot{m}}^{\delta}(v)), \tag{3.2}$$

$$(\forall q \in Q)\ ([q/\delta]q\dot{m} \Rightarrow Ł_{\dot{m}}^{\delta}(q) = \dot{m}(q) + \delta - 1), \tag{3.3}$$

$$(\forall q \in Q)\ (\forall \delta \in (0,1))\ (\delta \geq \epsilon \Rightarrow Ł_{\dot{m}}^{\delta}(q) \geq Ł_{\dot{m}}^{\epsilon}(q)). \tag{3.4}$$

Proof. Straightforward. □

Proposition 3.3. *If $\dot{g}$ and $\dot{m}$ are $\mathfrak{FS}$s in a set Q, then*

$$(\forall \delta \in (0,1))\ (Ł_{\dot{g}\cup\dot{m}}^{\delta} = Ł_{\dot{g}}^{\delta} \cup Ł_{\dot{m}}^{\delta} \text{ and } Ł_{\dot{g}\cap\dot{m}}^{\delta} = Ł_{\dot{g}}^{\delta} \cap Ł_{\dot{m}}^{\delta}). \tag{3.5}$$

Proof. For every $q \in Q$, we have

$$\begin{aligned} Ł_{\dot{g}\cup\dot{m}}^{\delta}(q) &= \min\{0,\ (\dot{g} \cup \dot{m})(q) + \delta - 1\} \\ &= \min\{0,\ \max\{\dot{g}(q),\ \dot{m}(q)\} + \delta - 1\} \\ &= \min\{0, max\{\dot{g}(q) + \delta - 1, \dot{m}(q) + \delta - 1\}\} \\ &= \max\{\min\{0, \dot{g}(q) + \delta - 1\}, min\{0, \dot{m}(q) + \delta - 1\}\} \\ &= \max\{Ł_{\dot{g}}^{\delta}(q),\ Ł_{\dot{m}}^{\delta}(q)\} = (Ł_{\dot{g}}^{\delta} \cup Ł_{\dot{m}}^{\delta})(q) \end{aligned}$$

and

$$\begin{aligned} Ł_{\dot{g}\cap\dot{m}}^{\delta}(q) &= \max\{0,\ (\dot{g} \cap \dot{m})(q) + \delta - 1\} \\ &= \max\{0,\ \min\{\dot{g}(q),\ \dot{m}(q)\} + \delta - 1\} \\ &= \max\{0, min\{\dot{g}(q) + \delta - 1, \dot{m}(q) + \delta - 1\}\} \\ &= \min\{\max\{0, \dot{g}(q) + \delta - 1\}, max\{0, \dot{m}(q) + \delta - 1\}\} \\ &= \min\{Ł_{\dot{g}}^{\delta}(q),\ Ł_{\dot{m}}^{\delta}(q)\} = (Ł_{\dot{g}}^{\delta} \cap Ł_{\dot{m}}^{\delta})(q) \end{aligned}$$

which proves (3.5). □

In what follows, let Q be a BCK/BCI-algebra, and δ is a component element of $(0,1)$ except if diversely determined.

Definition 3.4. Let $\dot{m}$ be a $\mathfrak{FS}$ in Q. Then its δ-$\mathfrak{LF}$ set $Ł_{\dot{m}}^{\delta}$ in Q is called a δ-$\mathfrak{LF}$ *ideal* of Q assuming it fulfills:

$$[q * v/s_{\dot{a}}] \in Ł_{\dot{m}}^{\delta},\ [v/s_{\dot{b}}] \in Ł_{\dot{m}}^{\delta} \Rightarrow [q/\min\{s_{\dot{a}}, s_{\dot{b}}\}] \in Ł_{\dot{m}}^{\delta} \tag{3.6}$$

for all $q, v \in Q$ and $s_{\dot{a}}, s_{\dot{b}} \in (0,1]$.

Example 3.5. Consider a BCK-algebra $Q = \{0,\ \dot{a}_1,\ \dot{a}_2,\ \dot{a}_3,\ \dot{a}_4\}$ with a binary operation "$*''$ stated by Table 1.

Table 1. Cayley table for the binary operation "$*''$

$*$	0	$\dot{a}_1$	$\dot{a}_2$	$\dot{a}_3$	$\dot{a}_4$
0	0	0	0	0	0
$\dot{a}_1$	$\dot{a}_1$	0	$\dot{a}_1$	0	0
$\dot{a}_2$	$\dot{a}_2$	$\dot{a}_2$	0	0	0
$\dot{a}_3$	$\dot{a}_3$	$\dot{a}_3$	$\dot{a}_3$	0	0
$\dot{a}_4$	$\dot{a}_4$	$\dot{a}_3$	$\dot{a}_4$	$\dot{a}_1$	0

Define a $\mathfrak{F}\mathfrak{S}$ $\dot{m}$ in Q as follows:

$$\dot{m} : Q \to [0,1], q \mapsto \begin{cases} 0.74 \text{ if } q = 0, \\ 0.67 \text{ if } q = \dot{a}_1, \\ 0.61 \text{ if } q = \dot{a}_2, \\ 0.55 \text{ if } q = \dot{a}_3, \\ 0.40 \text{ if } q = \dot{a}_4. \end{cases}$$

Given $\delta := 0.55$, the δ-$\mathfrak{L}\mathfrak{F}$ set $Ł^{\delta}_{\dot{m}}$ of $\dot{m}$ in Q is given as follows:

$$Ł^{\delta}_{\dot{m}} : Q \to [0,1], q \mapsto \begin{cases} 0.29 & \text{if } q = 0, \\ 0.22 & \text{if } q = \dot{a}_1, \\ 0.16 & \text{if } q = \dot{a}_2, \\ 0.10 & \text{if } q = \dot{a}_3, \\ 0 & \text{if } q = \dot{a}_4. \end{cases}$$

It is routine to verify that $Ł^{\delta}_{\dot{m}}$ is an δ-$\mathfrak{L}\mathfrak{F}$ ideal of Q.

Lemma 3.6. *Every δ-$\mathfrak{L}\mathfrak{F}$ ideal of Q fulfills the accompanying implication.*

$$(\forall q, v \in Q)\ (q \leq v \Rightarrow Ł^{\delta}_{\dot{m}}(q) \geq Ł^{\delta}_{\dot{m}}(v)).$$

Proof. Straightforward. □

Proposition 3.7. *Let $\dot{m}$ be a $\mathfrak{F}\mathfrak{I}$ of Q, and $Ł^{\delta}_{\dot{m}}$ be an δ-$\mathfrak{L}\mathfrak{F}$ ideal of Q. If the inequality holds $q * v \leq e$ in Q, then $Ł^{\delta}_{\dot{m}}$ satisfies:*

$$(\forall q \in Q)\ (Ł^{\delta}_{\dot{m}}(q) \geq min\{Ł^{\delta}_{\dot{m}}(v), Ł^{\delta}_{\dot{m}}(e)). \tag{3.7}$$

Proof. Let $Ł^{\delta}_{\dot{m}}$ be an δ-$\mathfrak{L}\mathfrak{F}$ ideal of Q, and let $q, v, e \in Q$ be such that $q*v \leq e$. Then $(q*v)*e = 0$, we have

$$\begin{aligned} Ł^{\delta}_{\dot{m}}(q) &\geq min\{Ł^{\delta}_{\dot{m}}(q * v), Ł^{\delta}_{\dot{m}}(v)\} \\ &= min\{min\{Ł^{\delta}_{\dot{m}}((q * v) * e), Ł^{\delta}_{\dot{m}}(e)\}, Ł^{\delta}_{\dot{m}}(v)\} \\ &= min\{min\{Ł^{\delta}_{\dot{m}}(0), Ł^{\delta}_{\dot{m}}(e)\}, Ł^{\delta}_{\dot{m}}(v)\} \\ &= min\{Ł^{\delta}_{\dot{m}}(v), Ł^{\delta}_{\dot{m}}(e)\}. \end{aligned}$$

□

Theorem 3.8. *Let $Ł^{\delta}_{\dot{m}}$ be an δ-$\mathfrak{L}\mathfrak{F}$ ideal of Q. Then*

$$Ł^{\delta}_{\dot{m}}(q * v) \geq min\{Ł^{\delta}_{\dot{m}}(q * e), Ł^{\delta}_{\dot{m}}(e * v)\},$$

for all $q, v, e \in Q$.

Proof. Note that $((q * v) * (q * e)) \leq (e * v)$. It follows for Lemma 3.6, that
$Ł^{\delta}_{\dot{m}}((q * v) * (q * e)) \geq Ł^{\delta}_{\dot{m}}(e * v)$.
Now, by Definition 3.4, we have
$Ł^{\delta}_{\dot{m}}(q * v) \geq min\{Ł^{\delta}_{\dot{m}}((q * v) * (q * e)), Ł^{\delta}_{\dot{m}}(q * e)\}$
$Ł^{\delta}_{\dot{m}}(q * v) \geq min\{Ł^{\delta}_{\dot{m}}(q * e), Ł^{\delta}_{\dot{m}}(e * v)\}$,
for all $q, v, e \in Q$. □

Theorem 3.9. *Let $Ł_m^\delta$ be an δ-$\mathfrak{LF}$ ideal of Q. Then*

$$Ł_m^\delta(q*(q*v)) \geq Ł_m^\delta(v),$$

for all $q, v \in Q$.

Proof. Let $Ł_m^\delta$ *be an δ-$\mathfrak{LF}$ ideal of Q*. Then

$$\begin{aligned} Ł_m^\delta(q*(q*v)) &\geq min\{Ł_m^\delta((q*(q*v))*v), Ł_m^\delta(v)\} \\ &= min\{Ł_m^\delta((q*v)*(q*v)), Ł_m^\delta(v)\} \\ &= \min\{Ł_m^\delta(0), Ł_m^\delta(v)\} \\ &= Ł_m^\delta(v), \end{aligned}$$

for all $q, v \in Q$. □

Proposition 3.10. *Let $Ł_m^\delta$ be an δ-$\mathfrak{LF}$ ideal of Q. Then the following assertions are equivalent.*
(1) $(\forall q, v \in Q)\ Ł_m^\delta(q*v) \geq Ł_m^\delta((q*v)*v))$.
(2) $(\forall q, v, e \in Q)\ Ł_m^\delta((q*e)*(v*e)) \geq Ł_m^\delta((q*v)*e)))$.

Proof. Assume that the condition (1) is valid. Note that
$((q*(v*e))*e)*e = ((q*e)*(v*e))*e \leq\ (q*v)*e$
For all $q, v, e \in Q$ by using Definition 3.4. As a result of Lemma 3.6,
$Ł_m^\delta((q*v)*e)) \leq Ł_m^\delta(((q*(v*e)*e)*e)$
so from Definition 3.4. and (1) that

$$\begin{aligned} Ł_m^\delta((q*e)*(v*e)) &= Ł_m^\delta((q*(v*e))*e) \\ &\geq Ł_m^\delta(((q*(v*e))*e)*e) \\ &\geq Ł_m^\delta((q*v)*e). \end{aligned}$$

Thus (2) holds. Now suppose that (2) is valid. If we replace e by v in (2), then

$$Ł_m^\delta(q*v) = Ł_m^\delta((q*v)*0) = Ł_m^\delta((q*v)*(v*v)) \geq Ł_m^\delta((q*v)*v),$$

which proves (1). □

Theorem 3.11. *If $Ł_m^\delta$ be an δ-$\mathfrak{LF}$ ideal of Q, then for all $q, v_1, v_2, ..., v_n \in Q$,*

$$\prod_{i=1}^{n} q*v_i = 0 \Rightarrow Ł_m^\delta(q) \geq min\{Ł_m^\delta(v_1), Ł_m^\delta(v_2)...., Ł_m^\delta(v_n)\}, \tag{3.8}$$

where $\prod_{i=1}^{n} q*v_i = (...((q*v_1)*v_2)*...)*v_n$.

Proof. The proof is by induction on n. Let $Ł_m^\delta$ be an δ-$\mathfrak{LF}$ ideal of Q. Lemma 3.6 and Proposition 3.10 indicate that the requirement (3.8) is valid for n = 1, 2. Suppose $Ł_m^\delta$ meets condition (3.8) is valid for n = k, that is, for all $q, v_1, v_2, ..., v_k \in Q$, $\prod_{i=1}^{n} q*v_i = 0$ implies $Ł_m^\delta(q) \geq \min\{Ł_m^\delta(v_1), Ł_m^\delta(v_2), ..., Ł_m^\delta(v_k)\}$.
Let $q, v_1, v_2, ..., v_k, v_{k+1} \in Q$ be such that $\prod_{i=1}^{k+1} q*v_i = 0$. Then

$$Ł_m^\delta(q*v_1) \geq mi\{Ł_m^\delta(v_2), Ł_m^\delta(v_3), ..., Ł_m^\delta(v_{k+1})\}.$$

Since $Ł_m^\delta$ is a $\mathfrak{LF}$ ideal of Q, it derives from Definition 3.4,

$$Ł_m^\delta(q) \geq min\{Ł_m^\delta(q*v_1), Ł_m^\delta(v_1)\} \geq min\{Ł_m^\delta(v_1), Ł_m^\delta(v_2), Ł_m^\delta(v_3), ..., Ł_m^\delta(v_{k+1}).$$

□

Theorem 3.12. *Let δ-$\mathfrak{LF}$ set in Q satisfying the condition (3.8). Then $Ł_m^\delta$ be an δ-$\mathfrak{LF}$ ideal of Q.*

Proof. Note that $(...((0 * \underbrace{q) * q) * ...) * q}_{n\ times} = 0$. It follows from (3.8) that $Ł^{\delta}_{\dot{m}}(0) \geq Ł^{\delta}_{\dot{m}}(q)$. Let $q, v, e \in Q$ be such that $q * v \geq c$. Then

$$0 = (q * v) * e = (...(((q * v) * e) * \underbrace{0) * ...) * 0}_{n-2\ times},$$

and so

$$Ł^{\delta}_{\dot{m}}(q) \geq min\{Ł^{\delta}_{\dot{m}}(v), Ł^{\delta}_{\dot{m}}(e), Ł^{\delta}_{\dot{m}}(0)\} = min\{Ł^{\delta}_{\dot{m}}(v), Ł^{\delta}_{\dot{m}}(e)\}.$$

Hence, by Proposition 3.10, we conclude that $Ł^{\delta}_{\dot{m}}$ be an δ-$\mathfrak{LF}$ ideal of Q. □

Proposition 3.13. *Let $\dot{m}$ be a $\mathfrak{FI}$ of Q, and $Ł^{\delta}_{\dot{m}}$ be an δ-$\mathfrak{LF}$ deal of Q. If the inequality holds $q \leq v$ in X, then $Ł^{\delta}_{\dot{m}}$ satisfies:*

$$(\forall q \in Q)\ (Ł^{\delta}_{\dot{m}}(q) \geq Ł^{\delta}_{\dot{m}}(v)). \tag{3.9}$$

Proof. $Ł^{\delta}_{\dot{m}}$ be an δ-$\mathfrak{LF}$ ideal of Q, and let $q, v, e \in Q$ be such that $q * v \leq e$. Then $(q * v) * e = 0$,

$$\begin{aligned} Ł^{\delta}_{\dot{m}}(q) &\geq min\{Ł^{\delta}_{\dot{m}}(q * v), Ł^{\delta}_{\dot{m}}(q)\} \\ &= min\{min\{Ł^{\delta}_{\dot{m}}((q * v) * e), Ł^{\delta}_{\dot{m}}(e)\}, Ł^{\delta}_{\dot{m}}(v)\} \\ &= min\{min\{Ł^{\delta}_{\dot{m}}(0), Ł^{\delta}_{\dot{m}}(e)\}, Ł^{\delta}_{\dot{m}}(v)\} \\ &= min\{Ł^{\delta}_{\dot{m}}(v), Ł^{\delta}_{\dot{m}}(e)\}. \end{aligned}$$

□

Theorem 3.14. *If $\dot{m}$ is a $\mathfrak{FI}$ of Q, then its δ-$\mathfrak{LF}$ set $Ł^{\delta}_{\dot{m}}$ in Q is an δ-$\mathfrak{LF}$ ideal of Q.*

Proof. Considering that $\dot{m}$ is a $\mathfrak{FI}$ of Q. Let $q, v \in Q$ and $s_{\dot{a}}, s_{\dot{b}} \in (0, 1]$ be $\ni [q * v/s_{\dot{a}}] \in Ł^{\delta}_{\dot{m}}$ and $[v/s_{\dot{b}}] \in Ł^{\delta}_{\dot{m}}$. Then $Ł^{\delta}_{\dot{m}}(q * v) \geq s_{\dot{a}}$ and $Ł^{\delta}_{\dot{m}}(v) \geq s_{\dot{b}}$. Thus

$$\begin{aligned} Ł^{\delta}_{\dot{m}}(q) &= \max\{0,\ \dot{m}(q) + \delta - 1\} \\ &\geq \max\{0,\ \min\{\dot{m}(q * v),\ \dot{m}(v)\} + \delta - 1\} \\ &= \max\{0, min\{\dot{m}(q * v) + \delta - 1, \dot{m}(v) + \delta - 1\}\} \\ &= \min\{\max\{0, \dot{m}(q * v) + \delta - 1\}, max\{0, \dot{m}(v) + \delta - 1\}\} \\ &= \min\{Ł^{\delta}_{\dot{m}}(q * v),\ Ł^{\delta}_{\dot{m}}(v)\} \\ &\geq \min\{s_{\dot{a}},\ s_{\dot{b}}\}. \end{aligned}$$

So $[q/\min\{s_{\dot{a}},\ s_{\dot{b}}\}] \in Ł^{\delta}_{\dot{m}}$. Hence $Ł^{\delta}_{\dot{m}}$ is an δ-$\mathfrak{LF}$ ideal of Q. □

The next example shows that the contrary to Theorem 3.14 is true.

Example 3.15. Consider a BCI-algebra $Q = \{0,\ \dot{a}_1,\ \dot{a}_2,\ \dot{a}_3,\ \dot{a}_4\}$ with a binary operation " $*$ " stated by Table 2.

Table 2. Cayley table for binary " $*$ "

$*$	0	$\dot{a}_1$	$\dot{a}_2$	$\dot{a}_3$	$\dot{a}_4$
0	0	0	$\dot{a}_2$	$\dot{a}_3$	$\dot{a}_4$
$\dot{a}_1$	$\dot{a}_1$	0	$\dot{a}_2$	$\dot{a}_3$	$\dot{a}_4$
$\dot{a}_2$	$\dot{a}_2$	$\dot{a}_2$	0	$\dot{a}_4$	$\dot{a}_3$
$\dot{a}_3$	$\dot{a}_3$	$\dot{a}_3$	$\dot{a}_4$	0	$\dot{a}_2$
$\dot{a}_4$	$\dot{a}_4$	$\dot{a}_4$	$\dot{a}_3$	$\dot{a}_2$	0

Define a $\mathfrak{FS}$ $\dot{m}$ in Q as follows:

$$f : Q \to [0, 1], q \mapsto \begin{cases} 0.70 \text{ if } q = 0, \\ 0.66 \text{ if } q = \dot{a}_1, \\ 0.59 \text{ if } q = \dot{a}_2, \\ 0.55 \text{ if } q = \dot{a}_3, \\ 0.37 \text{ if } q = \dot{a}_4. \end{cases}$$

Given $\delta := 0.42$, the δ-$\mathfrak{LF}$ set $Ł^{\delta}_{\dot{m}}$ of $\dot{m}$ in Q is given as follows:

$$Ł^{\delta}_{\dot{m}} : Q \to [0,1], q \mapsto \begin{cases} 0.12 & \text{if } q = 0, \\ 0.08 & \text{if } q = \dot{a}_1, \\ 0.01 & \text{if } q = \dot{a}_2, \\ 0 & \text{if } q = \dot{a}_3, \\ 0 & \text{if } q = \dot{a}_4. \end{cases}$$

It is routine to verify that $Ł^{\delta}_{\dot{m}}$ is an δ-$\mathfrak{LF}$ ideal of Q. Also, $\dot{m}$ is a $\mathfrak{FI}$ of Q.
We consider a characterization of δ-$\mathfrak{LF}$ ideal.

Theorem 3.16. *Let $\dot{m}$ be a $\mathfrak{FS}$ in Q. Then, at the point, its δ-$\mathfrak{LF}$ set $Ł^{\delta}_{\dot{m}}$ in Q is an δ-$\mathfrak{LF}$ ideal of $Q \Leftrightarrow$ it satisfies:*

$$(\forall q,\ v \in Q)(Ł^{\delta}_{\dot{m}}(q) \geq \min\{Ł^{\delta}_{\dot{m}}(q * v),\ Ł^{\delta}_{\dot{m}}(v)\}). \tag{3.10}$$

Proof. Suppose $Ł^{\delta}_{\dot{m}}$ is an δ-$\mathfrak{LF}$ ideal of Q. Let $q, v \in Q$. It is clear that $[q * v/Ł^{\delta}_{\dot{m}}(q)] \in Ł^{\delta}_{\dot{m}}$ and $[v/Ł^{\delta}_{\dot{m}}(y)] \in Ł^{\delta}_{\dot{m}}$. Then

$$[q/\min\{Ł^{\delta}_{\dot{m}}(q * v),\ Ł^{\delta}_{\dot{m}}(v)\}] \in Ł^{\delta}_{\dot{m}}$$

by (3.10), which implies that $Ł^{\delta}_{\dot{m}}(q) \geq \min\{Ł^{\delta}_{\dot{m}}(q * v),\ Ł^{\delta}_{\dot{m}}(v)\}$.

Then again, assume that $Ł^{\delta}_{\dot{m}}$ fulfills the condition (3.10). Let $q, v \in Q$ and $s_{\dot{a}}, s_{\dot{b}} \in (0,1]$ be $\ni [q * v/s_{\dot{a}}] \in Ł^{\delta}_{\dot{m}}$ and $[v/s_{\dot{b}}] \in Ł^{\delta}_{\dot{m}}$. Then $Ł^{\delta}_{\dot{m}}(q * v) \geq s_{\dot{a}}$ and $Ł^{\delta}_{\dot{m}}(v) \geq s_{\dot{b}}$, which implies from (3.10) that

$$Ł^{\delta}_{\dot{m}}(q) \geq \min\{Ł^{\delta}_{\dot{m}}(q * v),\ Ł^{\delta}_{\dot{m}}(v)\} \geq \min\{s_{\dot{a}},\ s_{\dot{b}}\}.$$

Thus $[q_{\min\{s_{\dot{b}}, s_{\dot{b}}\}}] \in Ł^{\delta}_{\dot{m}}$. So $Ł^{\delta}_{\dot{m}}$ is an δ-$\mathfrak{LF}$ ideal of Q. □

Proposition 3.17. *If $\dot{m}$ is a $\mathfrak{FI}$ of Q, then its δ-$\mathfrak{LF}$ set $Ł^{\delta}_{\dot{m}}$ satisfies:*

$$(\forall q,\ v \in Q)\ (Ł^{\delta}_{\dot{m}}(q * v) = Ł^{\delta}_{\dot{m}}(0) \Rightarrow Ł^{\delta}_{\dot{m}}(q) \geq Ł^{\delta}_{\dot{m}}(v)). \tag{3.11}$$

Proof. Assume that $Ł^{\delta}_{\dot{m}}(q * v) = Ł^{\delta}_{\dot{m}}(0)$ for all $q \in Q$. Then

$$\begin{aligned} Ł^{\delta}_{\dot{m}}(q) &\geq \min\{Ł^{\delta}_{\dot{m}}(q * v),\ Ł^{\delta}_{\dot{m}}(v)\} \\ &= \min\{Ł^{\delta}_{\dot{m}}(0),\ Ł^{\delta}_{\dot{m}}(v)\} \\ &= Ł^{\delta}_{\dot{m}}(v) \end{aligned}$$

for all $q, v \in Q$. □

Proposition 3.18. *If $\dot{m}$ is a $\mathfrak{FI}$ of a BCI-algebra Q, then its $\delta-\mathfrak{LF}$ set $Ł^{\delta}_{\dot{m}}$ satisfies:*

$$(\forall q \in Q)(Ł^{\delta}_{\dot{m}}(0 * q) \geq Ł^{\delta}_{\dot{m}}(q)). \tag{3.12}$$

Proof. Let $\dot{m}$ be a $\mathfrak{FI}$ of a BCI-algebra Q, then

$$\dot{m}(0 * q) \geq \min\{\dot{m}((0 * q) * q),\ \dot{m}(q)\} = \min\{\dot{m}((0 * q),\ \dot{m}(q)\} = \min\{\dot{m}(0),\ \dot{m}(q)\} = \dot{m}(q)$$

for all $q \in Q$. As a result of (3.2) that $Ł^{\delta}_{\dot{m}}(0 * q) \geq Ł^{\delta}_{\dot{m}}(q)$ for all $q \in Q$. □

Proposition 3.19. *If $\dot{m}$ is a $\mathfrak{FI}$ of a BCI-algebra Q, then its $\delta-\mathfrak{LF}$ set $Ł^{\delta}_{\dot{m}}$ satisfies:*

$$[q * v/s_{\dot{a}}] \in Ł^{\delta}_{\dot{m}},\ [v/s_{\dot{b}}] \in Ł^{\delta}_{\dot{m}} \Rightarrow [q/\min\{s_{\dot{a}},\ s_{\dot{b}}\}] \in Ł^{\delta}_{\dot{m}} \tag{3.13}$$

for all $q, v \in Q$ and $s_{\dot{a}}, s_{\dot{b}} \in (0,1]$.

Proof. Let $q, v \in Q$ and $s_{\dot{a}}, s_{\dot{b}} \in (0, 1]$ be such that $[q * v/s_{\dot{a}}] \in Ł^{\delta}_{\dot{m}}$ and $[v/s_{\dot{b}}] \in Ł^{\delta}_{\dot{m}}$. Then $Ł^{\delta}_{\dot{m}}(q * v) \geq s_{\dot{a}}$ and $Ł^{\delta}_{\dot{m}}(v) \geq s_{\dot{b}}$. Thus

$$\begin{aligned} Ł^{\delta}_{\dot{m}}(q) &= \max\{0,\ \dot{m}(q) + \delta - 1\} \\ &\geq \max\{0, min\{\dot{m}(q * (0 * v)), \dot{m}(0 * v)\} + \delta - 1\} \\ &\geq \max\{0,\ \min\{\dot{m}((q * v) * 0),\ \min\{\dot{m}(0),\ \dot{m}(v)\}\} + \delta - 1\} \\ &= \max\{0,\ \min\{\dot{m}(q * v),\ \dot{m}(v)\} + \delta - 1\} \\ &= \max\{0, min \dot{m}(q * v) + \delta - 1, \dot{m}(v) + \delta - 1\}\} \\ &= \min\{\max\{0, \dot{m}(q * v) + \delta - 1\}, max\{0, \dot{m}(v) + \delta - 1\}\} \\ &= \min\{Ł^{\delta}_{\dot{m}}(q * v),\ Ł^{\delta}_{\dot{m}}(v)\} \\ &\geq \min\{s_{\dot{a}},\ s_{\dot{b}}\}. \end{aligned}$$

So $[q/\min\{s_{\dot{a}},\ s_{\dot{b}}\}] \in Ł^{\delta}_{\dot{m}}$. □

We set some prerequisites for a $\mathfrak{LF}$ set to be a $\mathfrak{LF}$ ideal.

Theorem 3.20. *Let $\dot{m}$ be a $\mathfrak{FS}$ in Q. If δ-$\mathfrak{LF}$ set $Ł^{\delta}_{\dot{m}}$ of $\dot{m}$ in Q satisfies:*

$$[q * v/s_{\dot{b}}] \in Ł^{\delta}_{\dot{m}},\ [v/s_{\dot{e}}] \in Ł^{\delta}_{\dot{m}} \Rightarrow [q/\min\{s_{\dot{b}},\ s_{\dot{c}}\}] \in Ł^{\delta}_{\dot{m}} \tag{3.14}$$

for all $s_{\dot{b}}, s_{\dot{c}} \in (0, 1]$ and $q, v, e \in Q$ with $e \leq q$. Then $Ł^{\delta}_{\dot{m}}$ is an δ-$\mathfrak{LF}$ ideal of Q.

Proof. Let $q, v \in Q$ and $s_{\dot{a}}, s_{\dot{b}} \in (0, 1]$ be such that $[q * v/s_{\dot{a}}] \in Ł^{\delta}_{\dot{m}}$ and $[v/s_{\dot{b}}] \in Ł^{\delta}_{\dot{m}}$. Since $q \leq q\ \forall\ q \in Q$, As a result of (3.14) that $[q/\min\{s_{\dot{a}},\ s_{\dot{b}}\}] \in Ł^{\delta}_{\dot{m}}$. Hence $Ł^{\delta}_{\dot{m}}$ is an δ-$\mathfrak{LF}$ ideal of Q. □

Proposition 3.21. *If $\dot{m}$ is a $\mathfrak{FS}$ in a BCI-algebra X, then $\delta-\mathfrak{LF}$ ideal $Ł^{\delta}_{\dot{m}}$ of Q satisfies:*

$$[(x * (0 * y))/s_{\dot{a}}] \in Ł^{\delta}_{\dot{m}},\ [v/s_b] \in Ł^{\delta}_{\dot{m}} \Rightarrow [x/\min\{s_a, s_b\}] \in Ł^{\delta}_{\dot{m}} \tag{3.15}$$

for all $q, v \in Q$ and $s_{\dot{a}}, s_{\dot{b}} \in (0, 1]$.

Proof. Let $q, v \in Q$ and $s_{\dot{a}}, s_{\dot{b}} \in (0, 1]$ be such that $[(q * (0 * v))/s_{\dot{a}}] \in Ł^{\delta}_{\dot{m}}$ and $[v/s_{\dot{b}}] \in Ł^{\delta}_{\dot{m}}$. Then $Ł^{\delta}_{\dot{m}}(q * (0 * v)) \geq s_{\dot{a}}$ and $Ł^{\delta}_{\dot{m}}(v) \geq s_{\dot{b}}$. It follows from Theorem 3.16 that

$$\begin{aligned} Ł^{\delta}_{\dot{m}}(q) &\geq \min\{Ł^{\delta}_{\dot{m}}(q * (0 * v)),\ Ł^{\delta}_{\dot{m}}(0 * v)\} \\ &\geq \min\{Ł^{\delta}_{\dot{m}}(q * (0 * v)),\ \min\{Ł^{\delta}_{\dot{m}}(0),\ Ł^{\delta}_{\dot{m}}(v)\}\} \\ &= \{Ł^{\delta}_{\dot{m}}(q * (0 * v)),\ Ł^{\delta}_{\dot{m}}(v)\} \geq \min\{s_{\dot{a}},\ s_{\dot{b}}\}, \end{aligned}$$

i.e., $[q/\min\{s_{\dot{a}},\ s_{\dot{b}}\}] \in Ł^{\delta}_{\dot{m}}$. □

4 $\in$-set and q-set of Lukasiewicz fuzzy ideals

We explore how the $\in$-set and q-set of $\mathfrak{LF}$ together can be ideals.

Definition 4.1. Let $\dot{m}$ be a $\mathfrak{FS}$ in Q. For an δ-$\mathfrak{LF}$ set $Ł^{\delta}_{\dot{m}}$ of $\dot{m}$ in Q and $s \in (0, 1]$, consider the sets

$$(Ł^{\delta}_{\dot{m}},\ s)_{\in} := \{q \in Q \mid [q/s] \in Ł^{\delta}_{\dot{m}}\},$$

which is called the $\in$-*set*, respectively, of $Ł^{\delta}_{\dot{m}}$ (with value s).

Theorem 4.2. *Let $Ł^{\delta}_{\dot{m}}$ be an δ-$\mathfrak{LF}$ set of a $\mathfrak{FS}$ $\dot{m}$ in Q. Then $\in$-set $(Ł^{\delta}_{\dot{m}}, s)_{\in}$ of $Ł^{\delta}_{\dot{m}}$ with value $s \in (0.5, 1]$ is a ideal of Q if and only if the following affirmation is true.*

$$(\forall q, v \in Q)\ (\min\{Ł^{\delta}_{\dot{m}}(q * v),\ Ł^{\delta}_{\dot{m}}(v)\} \leq \max\{Ł^{\delta}_{\dot{m}}(q),\ 0.5\}). \tag{4.1}$$

Proof. Presume that $\in$-set $(Ł^{\delta}_{\dot{m}}, s)_{\in}$ of $Ł^{\delta}_{\dot{m}}$ with a value of $s \in (0.5, 1]$ is an ideal of Q. If this is the case (4.1) is invalid, then $\exists\, l, m \in Q \ni$

$$\min\{Ł^{\delta}_{\dot{m}}(l * m),\ Ł^{\delta}_{\dot{m}}(m)\} > \max\{Ł^{\delta}_{\dot{m}}(l),\ 0.5\}.$$

If we take $s := \min\{Ł^{\delta}_{\dot{m}}(l * m),\ Ł^{\delta}_{\dot{m}}(m)\}$, then $s \in (0.5, 1]$ and $[l * m/s], [m/s] \in Ł^{\delta}_{\dot{m}}$, i.e., $l, m \in (Ł^{\delta}_{\dot{m}}, s)_{\in}$. Since $(Ł^{\delta}_{\dot{m}}, s)_{\in}$ is a ideal of Q, we have $l \in (Ł^{\delta}_{\dot{m}}, s)_{\in}$. But $[l/s] \notin Ł^{\delta}_{\dot{m}}$ implies $l \notin (Ł^{\delta}_{\dot{m}}, s)_{\in}$, a contradiction. Thus we have

$$\min\{Ł^{\delta}_{\dot{m}}(q * v),\ Ł^{\delta}_{\dot{m}}(v)\} \le \max\{Ł^{\delta}_{\dot{m}}(q),\ 0.5\}.$$

At the other end, suppose that $Ł^{\delta}_{\dot{m}}$ satisfies (4.1). Let $s \in (0.5, 1]$ and $q, v \in Q$ be $\ni q * v \in (Ł^{\delta}_{\dot{m}}, s)_{\in}$ and $v \in (Ł^{\delta}_{\dot{m}}, s)_{\in}$. Then $Ł^{\delta}_{\dot{m}}(q * v) \ge s$ and $Ł^{\delta}_{\dot{m}}(v) \ge s$, which imply from (4.1) that

$$0.5 < s \le \min\{Ł^{\delta}_{\dot{m}}(q * v),\ Ł^{\delta}_{\dot{m}}(v)\} \le \max\{Ł^{\delta}_{\dot{m}}(q),\ 0.5\}.$$

Thus $[q/t] \in Ł^{\delta}_{\dot{m}}$, i.e., $q \in (Ł^{\delta}_{\dot{m}}, s)_{\in}$. So $(Ł^{\delta}_{\dot{m}}, s)_{\in}$ is an ideal of Q for $s \in (0.5, 1]$. □

Definition 4.3. Let $\dot{m}$ be a $\mathfrak{F}\mathfrak{S}$ in Q. For an δ-$\mathfrak{L}\mathfrak{F}$ set $Ł^{\delta}_{\dot{m}}$ of $\dot{m}$ in Q and $s \in (0, 1]$, consider the sets

$$(Ł^{\delta}_{\dot{m}}, s)_q := \{q \in Q \mid [q/s]qŁ^{\delta}_{\dot{m}}\},$$

are referred to as the q-*set*, respectively, of $Ł^{\delta}_{\dot{m}}$ (with value s).

Theorem 4.4. *Let $Ł^{\delta}_{\dot{m}}$ be an δ-$\mathfrak{L}\mathfrak{F}$ set of a $\mathfrak{F}\mathfrak{S}$ $\dot{m}$ in Q. If $\dot{m}$ is a $\mathfrak{F}\mathfrak{I}$ of Q, then q-set $(Ł^{\delta}_{\dot{m}}, s)_q$ of $Ł^{\delta}_{\dot{m}}$ with a value of $s \in (0, 1]$ is a ideal of Q.*

Proof. Let and $q,\ v \in (Ł^{\delta}_{\dot{m}}, s)_q$, $s \in (0, 1]$. Then $[q*v/t]qŁ^{\delta}_{\dot{m}}$ and $[v/s]qŁ^{\delta}_{\dot{m}}$, i.e., $Ł^{\delta}_{\dot{m}}(q*v)+s > 1$ and $Ł^{\delta}_{\dot{m}}(v) + s > 1$. It follows from Theorems 3.14 and 3.16 that

$$Ł^{\delta}_{\dot{m}}(q) + s \ge \min\{Ł^{\delta}_{\dot{m}}(q * v),\ Ł^{\delta}_{\dot{m}}(v)\} + s = \min\{Ł^{\delta}_{\dot{m}}(q * v) + s,\ Ł^{\delta}_{\dot{m}}(v) + s\} > 1.$$

Thus $[q/s]qŁ^{\delta}_{\dot{m}}$. So $q \in (Ł^{\delta}_{\dot{m}}, s)_q$. Hence $(Ł^{\delta}_{\dot{m}}, s)_q$ is a ideal of *Q*. □

Theorem 4.5. *Let $\dot{m}$ be a $\mathfrak{F}\mathfrak{S}$ in Q. For an δ-$\mathfrak{L}\mathfrak{F}$ set $Ł^{\delta}_{\dot{m}}$ of $\dot{m}$ in X, if the q-set $(Ł^{\delta}_{\dot{m}}, s)_q$ is a ideal of Q, then $Ł^{\delta}_{\dot{m}}$ satisfies:*

$$[q * v/s_{\dot{a}}]qŁ^{\delta}_{\dot{m}},\ [v/s_{\dot{b}}]qŁ^{\delta}_{\dot{m}} \Rightarrow [q/\max\{s_{\dot{a}},\ s_{\dot{b}}\}] \in Ł^{\delta}_{\dot{m}} \tag{4.2}$$

for all $q, v \in Q$ and $s_{\dot{a}}, s_{\dot{b}} \in (0, 0.5]$.

Proof. Let $q, v \in Q$ and $s_{\dot{a}}, s_{\dot{b}} \in (0, 0.5]$ be such that $[q * v/s_{\dot{a}}]qŁ^{\delta}_{\dot{m}}$ and $[v/s_{\dot{b}}]qŁ^{\delta}_{\dot{m}}$. Then $x \in (Ł^{\delta}_{\dot{m}}, s_{\dot{a}})_q \subseteq (Ł^{\delta}_{\dot{m}}, \max\{s_{\dot{a}}, s_{\dot{b}}\})_q$ and $y \in (Ł^{\delta}_{\dot{m}}, s_{\dot{b}})_q \subseteq (Ł^{\delta}_{\dot{m}}, \max\{s_{\dot{a}}, s_{\dot{b}}\})_q$. Thus $q \in (Ł^{\delta}_{\dot{m}}, \max\{s_{\dot{a}}, s_{\dot{b}}\})_q$. Since $\max\{s_{\dot{a}}, s_{\dot{b}}\} \le 0.5$,

$$Ł^{\delta}_{\dot{m}}(q) > 1 - \max\{s_{\dot{a}}, s_{\dot{b}}\} \ge \max\{s_{\dot{a}},\ s_{\dot{b}}\}.$$

So $[q/\max\{s_{\dot{a}},\ s_{\dot{b}}\}] \in Ł^{\delta}_{\dot{m}}$. □

5 O-set of Lukasiewicz fuzzy ideals

Definition 5.1. Let $\dot{m}$ be a $\mathfrak{F}\mathfrak{S}$ in Q. For an δ-$\mathfrak{L}\mathfrak{F}$ set $Ł^{\delta}_{\dot{m}}$ of $\dot{m}$ in Q, consider a set:

$$\mathrm{O}(Ł^{\delta}_{\dot{m}})\ := \{q \in Q \mid Ł^{\delta}_{\dot{m}}(q) > 0\} \tag{5.1}$$

are referred to as an O-*set* of $Ł^{\delta}_{\dot{m}}$. It has been noted that

$$\mathrm{O}(Ł^{\delta}_{\dot{m}}) = \{q \in Q \mid \dot{m}(q) + \delta - 1 > 0\}.$$

Theorem 5.2. *Let $Ł^{\delta}_{\dot{m}}$ be an δ-$\mathfrak{L}\mathfrak{F}$ set of a $\mathfrak{F}\mathfrak{S}$ $\dot{m}$ in Q. If $\dot{m}$ is a $\mathfrak{F}\mathfrak{I}$ of Q, then O-set $O(Ł^{\delta}_{\dot{m}})$ of $Ł^{\delta}_{\dot{m}}$ is an ideal of Q.*

Proof. Let $q, v \in \text{O}(\text{Ł}^{\delta}_{\dot{m}})$. Then $\dot{m}(q * v) + \delta - 1 > 0$ and $\dot{m}(v) + \delta - 1 > 0$. Suppose $\dot{m}$ is a $\mathfrak{FI}$ of Q. Then $\text{Ł}^{\delta}_{\dot{m}}$ is an δ-$\mathfrak{LF}$ ideal of Q. It follows from Theorem 3.16 that

$$\text{Ł}^{\delta}_{\dot{m}}(q) \geq \min\{\text{Ł}^{\delta}_{\dot{m}}(q * v),\ \text{Ł}^{\delta}_{\dot{m}}(v)\} = \min\{\dot{m}(q * v) + \delta - 1,\ \dot{m}(v) + \delta - 1\} > 0$$

Thus $q \in \text{O}(\text{Ł}^{\delta}_{\dot{m}})$. So $\text{O}(\text{Ł}^{\delta}_{\dot{m}})$ is a ideal of Q. □

Theorem 5.3. *If $\dot{m}$ is a $\mathfrak{FS}$ in Q. If an δ-$\mathfrak{LF}$ set $\text{Ł}^{\delta}_{\dot{m}}$ of $\dot{m}$ in Q satisfies:*

$$[q * v/s_{\dot{a}}] \in \text{Ł}^{\delta}_{\dot{m}},\ [v/s_{\dot{b}}] \in \text{Ł}^{\delta}_{\dot{m}} \Rightarrow [q/\max\{s_{\dot{a}}, s_{\dot{b}}\}]q\text{Ł}^{\delta}_{\dot{m}} \tag{5.2}$$

for all $q, v \in Q$ and $s_{\dot{a}}, s_{\dot{b}} \in (0, 1]$, then O*-set* $\text{O}(\text{Ł}^{\delta}_{\dot{m}})$ *of* $\text{Ł}^{\delta}_{\dot{m}}$ *is an ideal of Q.*

Proof. Let's assume $\text{Ł}^{\delta}_{\dot{m}}$ fulfills from (5.2), $\forall\, q, v \in Q$ and $s_{\dot{a}}, s_{\dot{b}} \in (0, 1]$. Let $q * v,\ v \in \text{O}(\text{Ł}^{\delta}_{\dot{m}})$. Then $\dot{m}(q * v) + \delta - 1 > 0$, $\dot{m}(v) + \delta - 1 > 0$. Since $[x/\text{Ł}^{\delta}_{\dot{m}}(q * v)] \in \text{Ł}^{\delta}_{\dot{m}}$ and $[v/\text{Ł}^{\delta}_{\dot{m}}(v)] \in \text{Ł}^{\delta}_{\dot{m}}$. it follows from (5.2) that

$$[q/\max\{\text{Ł}^{\delta}_{\dot{m}}(q * v),\ \text{Ł}^{\delta}_{\dot{m}}(v)\}]q\text{Ł}^{\delta}_{\dot{m}}. \tag{5.3}$$

If $q \notin \text{O}(\text{Ł}^{\delta}_{\dot{m}})$, then $\text{Ł}^{\delta}_{\dot{m}}(q) = 0$. Thus we get

$$\begin{aligned}
\text{Ł}^{\delta}_{\dot{m}}(q) + \max\{\text{Ł}^{\delta}_{\dot{m}}(q * v),\ \text{Ł}^{\delta}_{\dot{m}}(v)\} &= \max\{\text{Ł}^{\delta}_{\dot{m}}(q * v),\ \text{Ł}^{\delta}_{\dot{m}}(v)\} \\
&= \max\{\max\{0, \dot{m}(q * v) + \delta - 1\$, max\{, \dot{m}(v) + \delta - 1\}\} \\
&= \max\{\dot{m}(q * v) + \delta - 1,\ \dot{m}(v) + \delta - 1\} \\
&= \max\{\dot{m}(q * v),\ \dot{m}(v)\} + \delta - 1 \\
&\leq 1 + \delta - 1 \\
&= \delta \leq 1,
\end{aligned}$$

which states that (5.3) is invalid. This is a contradiction. So $q \in O(\text{Ł}^{\delta}_{\dot{m}})$. Because of that, $\text{O}(\text{Ł}^{\delta}_{\dot{m}})$ is an ideal of Q. □

Theorem 5.4. *Let $\dot{m}$ be a $\mathfrak{FS}$ in Q. If $\text{Ł}^{\delta}_{\dot{m}}$ of $\dot{m}$ in Q the criteria (4.2) for all $q, v \in Q$ and $s_{\dot{a}}, s_{\dot{b}} \in (0, 1]$, then* O*-set* $\text{O}(\text{Ł}^{\delta}_{\dot{m}})$ *of* $\text{Ł}^{\delta}_{\dot{m}}$ *is an ideal of Q.*

Proof. Let $q, v \in \text{O}(\text{Ł}^{\delta}_{\dot{m}})$. Then $\dot{m}(q * v) + \delta - 1 > 0$ and $\dot{m}(v) + \delta - 1 > 0$. Hence

$$\begin{aligned}
\text{Ł}^{\delta}_{\dot{m}}(q * v) + 1 &= \max\{0,\ \dot{m}(q * v) + \delta - 1\} + 1 \\
&= \dot{m}(q * v) + \delta - 1 + 1 \\
&= \dot{m}(q * v) + \delta > 1
\end{aligned}$$

and

$$\begin{aligned}
\text{Ł}^{\delta}_{\dot{m}}(v) + 1 &= \max\{0,\ \dot{m}(v) + \delta - 1\} + 1 \\
&= \dot{m}(v) + \delta - 1 + 1 \\
&= \dot{m}(v) + \delta > 1,
\end{aligned}$$

i.e., $[q * v/1]q\text{Ł}^{\delta}_{\dot{m}}$ and $[v/1]q\text{Ł}^{\delta}_{\dot{m}}$. This arises from (4.2) that

$$[q/1] = [q/\max\{1, 1\}] \in \text{Ł}^{\delta}_{\dot{m}}. \tag{5.4}$$

If $q \notin \text{O}(\text{Ł}^{\delta}_{\dot{m}})$, then $\text{Ł}^{\delta}_{\dot{m}}(q) = 0 < 1$ and so (5.4) is invalid. This is a contradiction. Thus $q \in \text{O}(\text{Ł}^{\delta}_{\dot{f}})$. So $\text{O}(\text{Ł}^{\delta}_{\dot{m}})$ is a ideal of Q. □

6 Conclusion

Based on Lukasiewicz t-standard, Jun et al. [12] tended to supposed a $\mathfrak{LF}$ set and applied it to BCI/BCK-algebras. In this paper, we managed the idea of $\mathfrak{LF}$ ideals in BCK-algebras and looking for some properties. We thought about portrayal of a $\mathfrak{LF}$ ideal. We gave a condition to a $\mathfrak{LF}$ ideal to be a $\mathfrak{LF}$ ideal. We additionally gave conditions to the $\in$-set, q-set and O-set to be ideals. Utilizing the thoughts and consequences of this paper, we will concentrate on different sub-structures in a few logarithmic frameworks, for instance, BCC-algebras, BCH-algebras, equality algebras, EQ-algebras, hoop algebras, BE-algebras, GE-algebras, and so on, later on. We will likewise investigate Lukasiewicz bipolar fuzzy sets, Lukasiewicz Pythagorean fuzzy sets, Lukasiewicz picture fuzzy sets, and so forth as the speculation of $\mathfrak{LF}$ sets.

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