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On the capitulation problem of some pure metacyclic fields of degree 20

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Abstract Let $\Gamma = \mathbb{Q}(\sqrt[5]{n})$ be a pure quintic field, where n is a 5^{th} power-free natural number, $k_0 = \mathbb{Q}(\zeta_5)$ be the cyclotomic field generated by a primitive 5^{th} root of unity ζ_5, and $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$ be the normal closure of Γ. Let $k_5^{(1)}$ be the Hilbert 5-class field of k, $C_{k,5}$ the 5-ideal class group of k, and $C_{k,5}^{(\sigma)}$ the group of ambiguous classes under the action of $Gal(k/k_0) = \langle\sigma\rangle$. When $C_{k,5}$ is of type $(5,5)$ and rank $C_{k,5}^{(\sigma)} = 1$, we treat the capitulation problem of the 5-ideal classes of $C_{k,5}$ in the six intermediate extensions of $k_5^{(1)}/k$.

1 Introduction

Let K be a number field, and F be an unramified abelian extension of K. We say that an ideal $\mathcal{I}$ (or the ideal class of $\mathcal{I}$) of K capitulates in F, if $\mathcal{I}$ becomes principal by extending it to an ideal of F. Let p be a prime number and $C_{K,p}$ be the p-ideal class group of K such that $C_{K,p}$ is of type (p,p). Let $K_p^{(1)}$ be the Hilbert p-class field of K, i.e the maximal abelian unramified p-extension of K. The most important result on capitulation is the "Principal Ideal Theorem" conjectured by D. Hilbert in [5] and proved by his student P. Furtwängler in [2]. The theorem asserts that the class group of a given number field capitulates completely in its Hilbert class field. Having established the "Principal Ideal Theorem", the question remains, which ideal classes of K capitulate in a field which lies between K and its Hilbert class field. A number of researchers have studied this question, Taussky and Scholz in [12] were the first who treated the capitulation in unramified cyclic degree-3-extensions of several imaginary quadratic fields. Tannaka and Terada proved that for a cyclic extension K/K_0 with Galois group G, the G-invariant ideal classes in K capitulate in the genus field of K/K_0. Hilbert in his celebrated Zahlbericht, proved his Theorem 94 which states that in an unramified cyclic extension L/K, there are non-principal ideals which capitulate in L. Miyake in [10] gave a generalization of Hilbert's theorem 94, by proving that in an unramified abelian extension F of K, at least $[F:K]$ classes of K capitulate in F.
In this paper we consider $p = 5$ and $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$, ($\zeta_5 = e^{\frac{2i\pi}{5}}$), which is the normal closure of the pure quintic field $\Gamma = \mathbb{Q}(\sqrt[5]{n})$. We suppose that the 5-class group $C_{k,5}$ of k is of type $(5,5)$, then the Hilbert 5-class field $k_5^{(1)}$ of k has degree 25 over k, and the extension $k_5^{(1)}/k$ contains six intermediate extensions. By $k_0 = \mathbb{Q}(\zeta_5)$ we denote the 5^{th} cyclotomic field, where ζ_5 is a primitive 5^{th} root of unity, and $\lambda = 1 - \zeta_5$ is the unique prime of k_0 above 5. We note that the congruence $n \not\equiv \pm1, \pm7 \pmod{25}$ is equivalent to λ being ramified in k/k_0. Let $C_{k,5}^{(\sigma)}$ be the group of ambiguous ideal classes of k under the action of $Gal(k/k_0) = \langle\sigma\rangle$, p and q are primes such that $p \equiv -1 \pmod 5$ and $q \equiv \pm2 \pmod 5$. According to [[1], Theorem 1.1], if $C_{k,5}$ is of type $(5,5)$ and rank $C_{k,5}^{(\sigma)} = 1$ we have three forms of the radicand n:
- $n = 5^e p \not\equiv \pm1, \pm7 \pmod{25}$ with $e \in \{0,1,2,3,4\}$ and $p \not\equiv -1 \pmod{25}$.
- $n = p^e q \equiv \pm1, \pm7 \pmod{25}$ with $e \in \{1,2,3,4\}$, $p \not\equiv -1 \pmod{25}$ and $q \not\equiv \pm7 \pmod{25}$.

- $n = p^e$ with $e \in \{1, 2, 3, 4\}$ and $p \equiv -1 \pmod{25}$.

We will study the capitulation of $C_{k,5}$ in the six intermediate extensions of $k_5^{(1)}/k$ in these cases. The paper can be viewed as the continuation of the research of Taussky in [15]. The theoretical results are underpinned by numerical examples obtained with the computational system PARI/GP [16].

Notations.

We use the following notations around the paper:

- $\Gamma = \mathbb{Q}(\sqrt[5]{n})$: a pure quintic field, where $n \neq 1$ is a 5^{th} power-free natural number.

- $k_0 = \mathbb{Q}(\zeta_5)$: the 5^{th} cyclotomic field.

- $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$: the normal closure of Γ, a quintic Kummer extension of k_0.

- Γ_1, Γ_2, Γ_3, Γ_4: the four conjugate quintic fields of Γ, contained in k.

- Gal$(k/\Gamma) = \langle\tau\rangle$ such that τ is identity on Γ, and sends ζ_5 to its square. τ has order 4.

- Gal$(k/k_0) = \langle\sigma\rangle$ such that σ is identity on k_0, and sends $\sqrt[5]{n}$ to $\zeta_5\sqrt[5]{n}$. σ has order 5.

- The commuting relation between σ and τ is $\tau\sigma = \sigma^3\tau$.

- Let L a number field, denote by:

  - $L_5^{(1)}$, L^*: the Hilbert 5-class field of L, and the absolute genus field of L.

  - C_L, h_L, $C_{L,5}$: the class group, class number and 5-class group of L.

  - $[\mathcal{I}]$: the class of a fractional ideal $\mathcal{I}$.

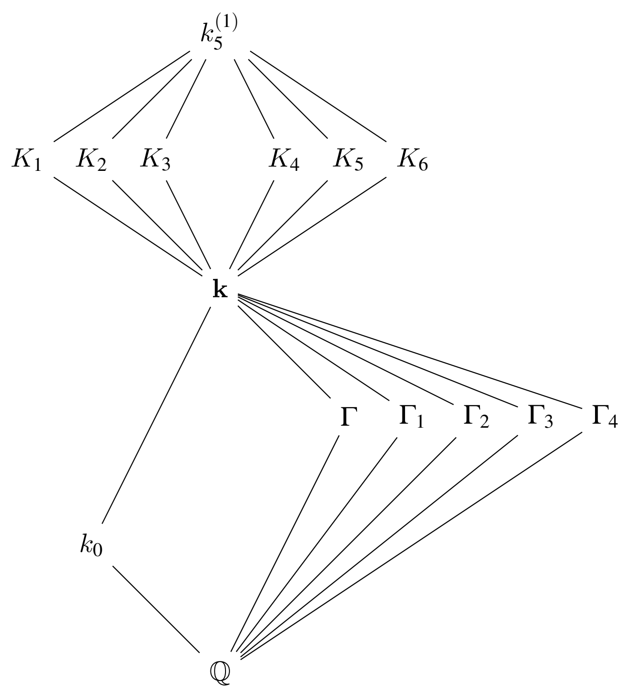

Figure 1: Sub-extensions of $k_5^{(1)}/\mathbb{Q}$

2 Construction of intermediate extensions

2.1 Absolute genus field

In chapter 7 of [6], Ishida has explicitly given the genus field of any pure field. For $\Gamma = \mathbb{Q}(\sqrt[5]{n})$ where n is a 5^{th} power-free natural number, we have the following theorem.

Theorem 2.1. *Let $\Gamma = \mathbb{Q}(\sqrt[5]{n})$ be a pure quintic field, where $n \geq 2$ is a 5^{th} power-free natural number, and let $p_1, , , , p_r$ be all prime divisors of n such that $p_i \equiv 1 \pmod 5$. Let Γ^* be the absolute genus field of Γ, then*

$$\Gamma^* = \prod_{i=1}^{r} M(p_i).\Gamma$$

where $M(p_i)$ denotes the unique subfield of degree 5 of the cyclotomic field $\mathbb{Q}(\zeta_{p_i})$. The genus number of Γ is given by, $g_\Gamma = [\Gamma^* : \Gamma] = 5^r$.

Proof. See [[6], Theorem 7]. □

Remark 2.2.

(1) If no prime $p \equiv 1 \pmod 5$ divides n, i.e $r = 0$, then $\Gamma^* = \Gamma$.

(2) For any $r \geq 0$, Γ^* is contained in the Hilbert 5-class field $\Gamma_5^{(1)}$ of Γ.

Corollary 2.3. *Let h_Γ denote the class number of Γ. Then*

(1) *If 5 divides exactly the class number h_Γ, then there is at most one prime $p \equiv 1 \pmod 5$ that divides the natural n.*

(2) *If 5 divides exactly h_Γ, and if a prime $p \equiv 1 \pmod 5$ divides n, then $\Gamma^* = \Gamma_5^{(1)}$, $(\Gamma_1)^* = (\Gamma_1)_5^{(1)}$, $(\Gamma_2)^* = (\Gamma_2)_5^{(1)}$, $(\Gamma_3)^* = (\Gamma_3)_5^{(1)}$ and $(\Gamma_4)^* = (\Gamma_4)_5^{(1)}$. Furthermore, $k.\Gamma_5^{(1)} = k.(\Gamma_1)_5^{(1)} = k.(\Gamma_2)_5^{(1)} = k.(\Gamma_3)_5^{(1)} = k.(\Gamma_4)_5^{(1)}$*

Proof.
(1) If $p_1, , , , p_r$ are all prime divisors of n that are congruent to 1 (mod 5), then $5^r | h_\Gamma$ therfore if 5 divides exactly h_Γ then $r \leq 1$, so one prime $p \equiv 1 \pmod 5$ divides n exist, or no one.

(2) If h_Γ is exactly divisible by 5 and $p \equiv 1 \pmod 5$ divides n, then $g_\Gamma = 5$ so, $\Gamma^* = \Gamma_5^{(1)} = \Gamma.M(p)$, we have $g_\Gamma = 5$ so, $(\Gamma_1)^* = \Gamma_1 \prod_{i=1}^{r} .M(p_i)$ where $p_1, , , , p_r$ are primes congruent to $1 \pmod 5$ and divide n, if 5 divides exactly h_Γ then 5 divides exactly h_{Γ_1}, because Γ and Γ_1 are isomorphic, so we obtain $(\Gamma_1)^* = (\Gamma_1)_5^{(1)}$. For the other equalities we use the same reasoning. Furthermore,

$$\begin{cases} k.\Gamma_5^{(1)} = k.\Gamma.M(p) = k.M(p) \\ k.(\Gamma_1)_5^{(1)} = k.\Gamma_1.M(p) = k.M(p) \\ k.(\Gamma_2)_5^{(1)} = k.\Gamma_2.M(p) = k.M(p) \\ k.(\Gamma_3)_5^{(1)} = k.\Gamma_3.M(p) = k.M(p) \\ k.(\Gamma_4)_5^{(1)} = k.\Gamma_4.M(p) = k.M(p) \end{cases} \tag{2.1}$$

Hence, $k.\Gamma_5^{(1)} = k.(\Gamma_1)_5^{(1)} = k.(\Gamma_2)_5^{(1)} = k.(\Gamma_3)_5^{(1)} = k.(\Gamma_4)_5^{(1)}$ □

2.2 Relative genus field $(k/k_0)^*$ of k over k_0

Let Γ, k_0 and k be as before. The relative genus field $(k/k_0)^*$ of k over k_0 is the maximal abelian extension of k_0 which is contained in the Hilbert 5-class field $k_5^{(1)}$ of k. A class $\mathcal{A} \in C_{k,5}$ is called ambiguous class relatively to k/k_0 if $\mathcal{A}^\sigma = \mathcal{A}$ where σ is a chosen generator of $Gal(k/k_0)$. We note the group of ambiguous classes by $C_{k,5}^{(\sigma)}$.
The following proposition summarize some important results of [[7], section 5]. Since k is quintic Kummer extension of k_0, we can write $k = k_0(\sqrt[5]{n})$ such that $n = \mu\lambda^{e_\lambda}\pi_1^{e_1}....\pi_g^{e_g}$, where μ is unity of $\mathcal{O}_{k_0}$, $\lambda = 1 - \zeta_5$ the unique prime above 5 in k_0, π_i $(1 \le i \le g)$ are prime elements of k_0 such that $\pi_i \equiv a \pmod{5\mathcal{O}_{k_0}}$ with $a, e_i \in \{1,2,3,4\}$ and $e_\lambda, \in \{0,1,2,3,4\}$. We note by d the number of ramified prime in k/k_0.

Proposition 2.4. *Using the same notations as above.*

(1) $C_k^{(\sigma)} = \{\mathcal{A} \in C_k \mid \mathcal{A}^\sigma = \mathcal{A}\}$ *the subgroup of ambiguous classes coincides with the* 5*-class group* $C_{k,5}^{(\sigma)} = \{\mathcal{A} \in C_{k,5} \mid \mathcal{A}^\sigma = \mathcal{A}\}$.

(2) *rank* $C_{k,5}^{(\sigma)} = d - 3 + q^*$.

(3) $q^* = \begin{cases} 2 & \text{if } \zeta_5, \zeta_5 + 1 \text{ are norm of element in } k - \{0\}. \\ 1 & \text{if } \zeta_5^i(\zeta_5+1)^j \text{ is the norm of an element in } k - \{0\} \text{ for some exponents i and j.} \\ 0 & \text{if for no exponents i, j, the element } \zeta_5^i(\zeta_5+1)^j \text{ is a norm from } k - \{0\}. \end{cases}$

(4) *Let* $k = k_0(\sqrt[5]{n})$, *writing* $n = \mu\lambda^{e_\lambda}\pi_1^{e_1}....\pi_f^{e_f}\pi_{f+1}^{e_{f+1}}.....\pi_g^{e_g}$, *where each* $\pi_i \equiv \pm1, \pm7 \pmod{\lambda^5}$ *for* $1 \le i \le f$ *and* $\pi_j \not\equiv \pm1, \pm7 \pmod{\lambda^5}$ *for* $f+1 \le j \le g$. *Then we have:*

  (i) *there exists* $h_i \in \{1, .., 4\}$ *such that* $\pi_{f+1}\pi_i^{h_i} \equiv \pm1, \pm7 \pmod{\lambda^5}$, *for* $f+2 \le i \le g$.

  (ii) *if* $n \not\equiv \pm1 \pm 7 \pmod{\lambda^5}$ *and* $q^* = 1$, *then the genus field* $(k/k_0)^*$ *is given as:*

$$(k/k_0)^* = k(\sqrt[5]{\pi_1}, ...\sqrt[5]{\pi_f}, \sqrt[5]{\pi_{f+1}\pi_{f+2}^{h_{f+2}}}, ...\sqrt[5]{\pi_{f+1}\pi_g^{h_g}})$$

  where h_i *is chosen as in (i).*

  (iii) *in the other cases of* q^* *and congruence of* n, $(k/k_0)^*$ *is given by deleting an appropriate number of* 5^{th} *root from the right side of (ii).*

Let $C_{k,5}$ and $C_{k,5}^{(\sigma)} = 1$ as before. As mentioned above, we have three possible forms of the radicand n, and the genus field $(k/k_0)^*$ is given explicitly by:

Lemma 2.5. *Let* q *and* p *prime numbers such that* $q \equiv \pm2 \pmod 5$ *and* $p \equiv -1 \pmod 5$. *Let* π_1 *and* π_2 *primes of* k_0 *such that* $p = \pi_1\pi_2$. *If* $C_{k,5}$ *is of type* $(5,5)$ *and rank* $C_{k,5}^{(\sigma)} = 1$, *then:*

(1) $n = 5^e p \not\equiv \pm1, \pm7 \pmod{25}$, *where* $e \in \{0,1,2,3,4\}$ *and* $p \not\equiv -1 \pmod{25}$. *Then* $(k/k_0)^* = k(\sqrt[5]{\pi_1^{\alpha_1}\pi_2^{\alpha_2}})$, *with* $\alpha_1, \alpha_2 \in \{1,2,3,4\}$.

(2) $n = p^e q \equiv \pm 1 \pm 7 \pmod{25}$, *where* $e \in \{1,2,3,4\}$, $p \not\equiv -1 \pmod{25}$ *and* $q \not\equiv \pm 7 \pmod{25}$. *Then* $(k/k_0)^* = k(\sqrt[5]{q\pi_i^{\alpha_i}})$, *with* $i = 1$ *or* 2 *and* $\alpha \in \{1,2,3,4\}$.

(3) $n = p^e$, *where* $e \in \{1,2,3,4\}$, $p \equiv -1 \pmod{25}$. *Then* $(k/k_0)^* = k(\sqrt[5]{\pi_1^{\alpha_1}\pi_2^{\alpha_2}})$, *with* $\alpha_1, \alpha_2 \in \{1,2,3,4\}$.

Proof. For the three forms of n, we refer the reader to [[1], Theorem 1.1]. For the relative genus field $(k/k_0)^*$ of each case, its follows from (4) proposition 2.4 and [[7], Theorem 5.15]. □

2.3 Intermediate extensions of $k_5^{(1)}/k$

Let Γ, k and $C_{k,5}$ be as above. When $C_{k,5}$ is of type $(5,5)$, it has 6 subgroups of order 5, denoted by H_i, $1 \leq i \leq 6$. Let K_i be the intermediate extension of $k_5^{(1)}/k$, corresponding by class field theory to H_i.
We begin by some results on the 5-class group $C_{k,5}$.

Lemma 2.6. *Let* $\mathrm{Gal}(k/\Gamma) = \langle\tau\rangle \simeq \mathbb{Z}/4\mathbb{Z}$ *and* $C_{k,5}$ *be the 5-class group of* k. *Let* $C_{k,5}^+ = \{\mathcal{A} \in C_{k,5} \mid \mathcal{A}^{\tau^2} = \mathcal{A}\}$ *and* $C_{k,5}^- = \{\mathcal{A} \in C_{k,5} \mid \mathcal{A}^{\tau^2} = \mathcal{A}^{-1}\}$. *Then*

$$C_{k,5} \simeq C_{k,5}^+ \times C_{k,5}^-.$$

Proof. Let $\mathcal{A} \in C_{k,5}$. Then $\mathcal{A} = \mathcal{A}^{\frac{1+\tau^2}{2}}.\mathcal{A}^{\frac{1-\tau^2}{2}}$. (Note that $\mathcal{A}^{\frac{1+\tau^2}{2}}$ and $\mathcal{A}^{\frac{1-\tau^2}{2}}$ are well defined since $C_{k,5}$ is a $\mathbb{Z}_5[\langle\tau\rangle]$-module.) Hence every element in $C_{k,5}$ can be written as product of an elemnet in $C_{k,5}^+$ and an elemnet in $C_{k,5}^-$. Now let $\mathcal{A} \in C_{k,5}^+ \cap C_{k,5}^-$. Then $\mathcal{A} = \mathcal{A}^{\tau^2} = \mathcal{A}^{-1}$ that is $\mathcal{A}^2 = 1$, thus $\mathcal{A} = 1$ since its a 5-class. Hence $C_{k,5} \cong C_{k,5}^+ \times C_{k,5}^-$. □

Now let u be the index of subgroup E_0 generated by the units of intermediate fields of the extension $k/\mathbb{Q}$ in the units group of k. In [[11], Theorems I, II], C.Parry proved that u is a divisor of 5^6, and he presented the relation formula between the class numbers of k and Γ as follows: $h_k = (\frac{u}{5})(\frac{h_\Gamma}{5})^4$. We have the following lemma:

Lemma 2.7. *If the 5-class group* $C_{k,5}$ *of* k *is of type* $(5,5)$, *then we have* $C_{\Gamma,5} \simeq C_{k,5}^+$.

Proof. Let $C_{k,5}$ is of type $(5,5)$, then $|C_{k,5}| = 25$. According to [11], $|C_{k,5}| = (\frac{u}{5})\frac{|C_{\Gamma,5}|^4}{5^4} = 25$, namely $uh_\Gamma^4 = 5^7$. Let $n = v_5(u)$ and $n' = v_5(h_\Gamma)$ where v_5 is the 5-valuation, so we have that $n + 4n' = 7$. The unique values which verify the equality is $n = 3$ and $n' = 1$. Therefore we get that $u = 5^3$ and h_Γ is exactly divisible by 5, which means that $C_{\Gamma,5}$ is of order 5 .
According to Lemma 2.6, we have that $C_{k,5} \simeq C_{k,5}^+ \times C_{k,5}^-$. Since $C_{k,5}$ is of type $(5,5)$ then $C_{k,5}^+$ is of order 5, otherwise $C_{k,5}^+$ is of order 25, then all classes are fixed by τ^2, which is not true for $\mathcal{A}^{\frac{1-\tau^2}{2}}$, so $C_{k,5}^+$ is of order 5. There is a natural inclusion $C_{\Gamma,5} \hookrightarrow C_{k,5}$ since 5 relatively prime to $[k:\Gamma] = 4$. Furthermore, $C_{\Gamma,5} \hookrightarrow C_{k,5}^+$ as $\mathcal{A}^{\tau^2} = \mathcal{A}$ for all $\mathcal{A} \in C_{\Gamma,5}$. Thus $C_{\Gamma,5} \simeq C_{k,5}^+$. □

The following theorem allows us to determine the six intermediate extensions of $k_5^{(1)}/k$, using the action of $Gal(k/\mathbb{Q})$ on $C_{k,5}$.

Theorem 2.8. *Using the same notations as above.*

(1) *rank* $C_{k,5}^{(\sigma)} \geq 1$.

(2) *If rank* $C_{k,5}^{(\sigma)} = 1$ *then* $C_{k,5}^{(\sigma)} = C_{k,5}^+ = C_{k,5}^{1-\sigma}$ *with* $C_{k,5}^{1-\sigma}$ *is the principal genus.*

(3) *Let* $\Gamma_5^{(1)}, (\Gamma_1)_5^{(1)}, (\Gamma_2)_5^{(1)}, (\Gamma_3)_5^{(1)}$ *and* $(\Gamma_4)_5^{(1)}$ *be respectively the Hilbert 5-class fields of* Γ, Γ_1, Γ_2, Γ_3 *and* Γ_4. *If rank* $C_{k,5}^{(\sigma)} = 1$, *then* $k\Gamma_5^{(1)}, k(\Gamma_1)_5^{(1)}, k(\Gamma_2)_5^{(1)}, k(\Gamma_3)_5^{(1)}, k(\Gamma_4)_5^{(1)}$ *and* $(k/k_0)^*$ *are the six intermediate extensions of* $k_5^{(1)}/k$. *Furthermore* σ *permutes the fields* $k\Gamma_5^{(1)}, k(\Gamma_1)_5^{(1)}, k(\Gamma_2)_5^{(1)}, k(\Gamma_3)_5^{(1)}$ *and* $k(\Gamma_4)_5^{(1)}$

Proof.

(1) Since $Gal(k/k_0) = \langle\sigma\rangle$ is a cyclic group of order 5, which acts on $C_{k,5}$, then the orbits of $C_{k,5}$ under $\langle\sigma\rangle$ must all have order 1 or 5. The identity element is its own orbit, so there must be other orbits of order 5. This shows that the ambiguous subgroup $C_{k,5}^{(\sigma)}$ is non-trivial. Hence rank $C_{k,5}^{(\sigma)} \geq 1$.

(2) A class $\mathcal{A} = [\mathcal{P}]$ where $\mathcal{P}$ is a fractional ideal of k, is called strong ambiguous class, if $\mathcal{P}^\sigma = \mathcal{P}$. By $C_{k,s}^{(\sigma)}$ we denote the group of strong ambiguous classes. According to [[1], theorem 1.1] or [[8], Lemma 3.3], since rank $C_{k,5}^{(\sigma)} = 1$ then the radicand n is not divisible by any prime $p \equiv 1 \pmod 5$. We have that elements of $C_{k,s}^{(\sigma)}$ are fixed by τ^2 since τ^2 fixes all primes ramified in k/k_0 and according to [[3], Proposition 2], $C_{k,5}^{(\sigma)}/C_{k,s}^{(\sigma)}$ is generated by a class modulo $C_{k,s}^{(\sigma)}$ of element chosen in $(C_{k,5}^{(\sigma)})^+$. Thus all elements of $C_{k,5}^{(\sigma)}$ are fixed by τ^2, then $C_{k,5}^{(\sigma)} \subset C_{k,5}^+$. Therefore $C_{k,5}^{(\sigma)} = C_{k,5}^+$ because they are groups of order 5.
Since σ acts trivially on $C_{k,5}^+$, we have $1-\sigma$ annihilates $C_{k,5}^+$. Then $(C_{k,5}^+)^{1-\sigma} \subset C_{k,5}^+$. To prove that $C_{k,5}^{1-\sigma} = C_{k,5}^+$ it's sufficient to show that $(C_{k,5}^-)^{1-\sigma}$ is non-trivial and $(C_{k,5}^-)^{1-\sigma} \subset C_{k,5}^+$. If this is trivial then $C_{k,5}^{1-\sigma}$ should be trivial, which imply that $(k/k_0)^*$ will coincide with $k_5^{(1)}$, because by class field theory $(k/k_0)^*$ corresponds to $C_{k,5}^{1-\sigma}$, which is not true in our case. Hence $(C_{k,5}^-)^{1-\sigma}$ is non-trivial.
Let $\mathcal{X} \neq 1 \in (C_{k,5}^-)^{1-\sigma} = \{\mathcal{A}^{1-\sigma} \mid \mathcal{A} \in C_{k,5}^-\}$. We put $\mathcal{X} = \mathcal{A}^{1-\sigma}$ with $\mathcal{A} \in C_{k,5}^-$. We have $\mathcal{X}^{\tau^2} = (\mathcal{A}^{1-\sigma})^{\tau^2} = \mathcal{A}^{\tau^2}((\mathcal{A}^\sigma)^{-1})^{\tau^2} = \mathcal{A}^{-1}((\mathcal{A}^\sigma)^{\tau^2})^{-1} = \mathcal{A}^{-1}(\mathcal{A}^{\sigma\tau^2})^{-1}) = \mathcal{A}^{-1}(\mathcal{A}^{\tau^2\sigma^4})^{-1} = \mathcal{A}^{\sigma^4-1} = (\mathcal{A}^{-\sigma^3-\sigma^2-\sigma-1})^{1-\sigma}$. By the commuting relation between σ and τ we have $(\mathcal{A}^{-\sigma^3-\sigma^2-\sigma-1})^{\tau^2} = (\mathcal{A}^{-\sigma^3\tau^2-\sigma^2\tau^2-\sigma\tau^2-\tau^2}) = (\mathcal{A}^{-\sigma^3-\sigma^2-\sigma-1})^{-1}$, so we get that $\mathcal{A}^{-\sigma^3-\sigma^2-\sigma-1} \in C_{k,5}^-$ then we deduce that $((C_{k,5}^-)^{1-\sigma})^{\tau^2} \subset (C_{k,5}^-)^{1-\sigma}$, so $(C_{k,5}^-)^{1-\sigma} \subset C_{k,5}^+$ which means that $(C_{k,5}^-)^{1-\sigma} = C_{k,5}^+$. Hence $C_{k,5}^{1-\sigma} = C_{k,5}^+$.

(3) It's sufficient to see that the six extensions are quintic cyclic and unramified extensions of k, then they are all contained in $k_5^{(1)}$. We need to show that they are distinct. We have $\langle\tau^2\rangle \subset Gal(k/\Gamma)$, $Gal(k/\Gamma_1) = \langle\tau\sigma^2\rangle$, $Gal(k/\Gamma_2) = \langle\tau\sigma^4\rangle$, $Gal(k/\Gamma_3) = \langle\tau\sigma\rangle$ and $Gal(k/\Gamma_4) = \langle\tau\sigma^3\rangle$. By class field theory, the six subgroups of $C_{k,5}$, denoted H_i $(1 \leq i \leq 6)$, which correspond to the six extensions are as follows:
- $(k/k_0)^*$ corresponds to the principal genus $H_1 = C_{k,5}^{1-\sigma} = C_{k,5}^{(\sigma)} = C_{k,5}^+$
- By Lemma 2.7, we have $C_{k,5}^+ \simeq C_{\Gamma,5}$. Since $C_{k,5}/C_{k,5}^- \simeq C_{k,5}^+$ and $C_{\Gamma,5} \simeq Gal(\Gamma_5^{(1)}/\Gamma)$, we see that $C_{k,5}/C_{k,5}^- \simeq Gal(\Gamma_5^{(1)}/\Gamma) \simeq Gal(k\Gamma_5^{(1)}/k)$. Hence $k\Gamma_5^{(1)}$ corresponds to $H_6 = C_{k,5}^-$.
- $k(\Gamma_1)_5^{(1)}$ corresponds to $H_5 = \{\mathcal{X} \in C_{k,5} \mid \mathcal{X}^{(\tau\sigma^2)^2} = \mathcal{X}^{-1}\}$
- $k(\Gamma_2)_5^{(1)}$ corresponds to $H_4 = \{\mathcal{X} \in C_{k,5} \mid \mathcal{X}^{(\tau\sigma^4)^2} = \mathcal{X}^{-1}\}$
- $k(\Gamma_3)_5^{(1)}$ corresponds to $H_3 = \{\mathcal{X} \in C_{k,5} \mid \mathcal{X}^{(\tau\sigma)^2} = \mathcal{X}^{-1}\}$
- $k(\Gamma_4)_5^{(1)}$ corresponds to $H_2 = \{\mathcal{X} \in C_{k,5} \mid \mathcal{X}^{(\tau\sigma^3)^2} = \mathcal{X}^{-1}\}$
We have that $H_1 \neq H_6$. Suppose that $H_1 = H_5$, for $\mathcal{X} \neq 1 \in H_1 = H_5$ we have $\mathcal{X}^{\tau^2} = \mathcal{X}^\sigma = \mathcal{X}$ and $\mathcal{X}^{(\tau\sigma^2)^2} = \mathcal{X}^{\tau^2\sigma} = \mathcal{X}^{-1} = \mathcal{X}$, so we get that $\mathcal{X} = 1$ because its a 5-class. Hence we deduce that $H_1 \neq H_5$.
Now suppose that $H_5 = H_4$, for $\mathcal{X} \neq 1 \in H_5 = H_4$ we have $\mathcal{X}^{(\tau\sigma^2)^2} = \mathcal{X}^{\tau^2\sigma} = \mathcal{X}^{-1}$ and $\mathcal{X}^{(\tau\sigma^4)^2} = \mathcal{X}^{\tau^2\sigma^2} = \mathcal{X}^{-1}$, then $\mathcal{X}^{\tau^2\sigma^2} = \mathcal{X}^{\tau^2\sigma}$. By the commuting relation between σ and τ we have $\mathcal{X}^{\sigma^3\tau^2} = \mathcal{X}^{\sigma^4\tau^2}$, which imply that $\mathcal{X}^\sigma = \mathcal{X}$, so $\mathcal{X} \in H_1$, and since $H_1 \neq H_5$, then $\mathcal{X} = 1$ whence $H_5 \neq H_4$. By the same reasoning for the other cases, we prove that the subgroups H_i $(1 \leq i \leq 6)$ of $C_{k,5}$ are distinct, which means that the six quintic unramified extensions $(k/k_0)^*$, $k\Gamma_5^{(1)}, k(\Gamma_1)_5^{(1)}, k(\Gamma_2)_5^{(1)}, k(\Gamma_3)_5^{(1)}$ and $k(\Gamma_4)_5^{(1)}$ are distinct.
We denote by $K_1 = (k/k_0)^*$, $K_2 = k(\Gamma_4)_5^{(1)}$, $K_3 = k(\Gamma_3)_5^{(1)}$, $K_4 = k(\Gamma_2)_5^{(1)}$, $K_5 = k(\Gamma_1)_5^{(1)}$ and $K_6 = k(\Gamma)_5^{(1)}$. Since $K_2 = K_3^\sigma$, $K_3 = K_4^\sigma$, $K_4 = K_5^\sigma$, $K_5 = K_6^\sigma$ and $K_6 = K_2^\sigma$, then σ permutes the fields K_2, K_3, K_4, K_5 and K_6. □

3 Capitulation problem

Let Γ, k and $C_{k,5}$ be as above. $C_{k,5}$ is elementary abelian bicyclic of type $(5,5)$, it has 6 subgroups H_i $1 \le i \le 6$ of order 5. Let K_i be the intermediate extension of $k_5^{(1)}/k$, corresponding by class field theory to H_i. Since each K_i is cyclic of order 5 over k, there is at least one subgroup of order 5 of $C_{k,5}$, i.e, at least one H_l for some $l \in \{1,2,3,4,5,6\}$, which capitulates in K_i (by Hilbert's theorem 94).

Definition 3.1. Let $\mathcal{S}_j$ be a generator of H_j $(1 \le j \le 6)$ corresponding to K_j. For $1 \le j \le 6$, let $i_j \in \{0,1,2,3,4,5,6\}$. We say that the capitulation is of type $(i_1, i_2, i_3, i_4, i_5, i_6)$ to mean the following:

(1) when $i_j \in \{1,2,3,4,5,6\}$, then only the class $\mathcal{S}_{i_j}$ and its powers capitulate in K_j;
(2) when $i_j = 0$, then all the 5-classes capitulate in K_j.

By theorem 2.8,we have that $C_{k,5}^{(\sigma)} = C_{k,5}^{+}$. Let $C_{k,5}^{+} = \langle \mathcal{A} \rangle$ and $C_{k,5}^{-} = \langle \mathcal{X} \rangle$. According to Lemma 2.7, $C_{k,5}^{+} \simeq C_{\Gamma,5}$, we can consider $C_{\Gamma,5}$ as subgroup of $C_{k,5}$, and we may choose the class $\mathcal{A}$ as $\mathcal{A} = [j_{k/\Gamma}(\mathcal{B})]$, when $\mathcal{B}$ is ideal of Γ such that its class generates $C_{\Gamma,5}$.
We order the six intermediate fields K_i of $k_5^{(1)}/k$ $(1 \le i \le 6)$ as follows:

- $K_1 = (k/k_0)^*$ correspond by class field theory to the subgroup $H_1 = C_{k,5}^{1-\sigma} = \langle \mathcal{A} \rangle$.
- $K_6 = k\Gamma_5^{(1)}$ correspond to the subgroup $H_6 = C_{k,5}^{-} = \langle \mathcal{X} \rangle$.
- $K_2 = k(\Gamma_4)_5^{(1)}$, $K_3 = k(\Gamma_3)_5^{(1)}$, $K_4 = k(\Gamma_2)_5^{(1)}$ and $K_5 = k(\Gamma_1)_5^{(1)}$.

Our principal result on capitulation can be stated as follows:

Theorem 3.2. *Using the same notations as above.*

(1) *The class $\mathcal{A}$ capitulates in the six quintic extensions K_i, $i = 1,2,3,4,5,6$.*
(2) *The same number of classes capitulate in K_2, K_3, K_4, K_5 and K_6. More precisely, the possible capitulation types are $(0,0,0,0,0,0)$, $(1,0,0,0,0,0)$, $(0,1,1,1,1,1)$ or $(1,1,1,1,1,1)$. Possible Taussky types are (A,A,A,A,A,A) or (A,B,B,B,B,B)*

Proof.

(1) Let $C_{k,5}^{(\sigma)} = C_{k,5}^{+} = \langle \mathcal{A} \rangle$, where $\mathcal{A} = [j_{k/\Gamma}(\mathcal{B})]$ such that $\mathcal{B}$ is ideal of Γ. Since $\Gamma_5^{(1)}$ is Hilbert 5-class field of Γ, then $\mathcal{B}$ becomes principal in $\Gamma_5^{(1)}$. Thus when $\mathcal{B}$ considered as ideal of $k\Gamma_5^{(1)}$, $\mathcal{B}$ becomes principal in $k\Gamma_5^{(1)}$. Then we get that $\mathcal{A}$ capitulates in $k\Gamma_5^{(1)} = K_6$. Since $K_2 = K_3^{\sigma}$, $K_3 = K_4^{\sigma}$, $K_4 = K_5^{\sigma}$, $K_5 = K_6^{\sigma}$, $K_6 = K_2^{\sigma}$ and $\mathcal{A}^{\sigma} = \mathcal{A}$, we deduce that $\mathcal{A}$ capitulates in K_2, K_3, K_4, K_5 and K_6. To prove that $\mathcal{A}$ capitulates in K_1, we use Tanaka-Terada's theorem. Since $K_1 = (k/k_0)^*$ is the genus field of k/k_0, and $\mathcal{A}$ is ambiguous class, then $\mathcal{A}$ capitulates in K_1.

(2) We have σ permute the fields K_6, K_5, K_4, K_3, and K_2, and $C_{k,5}^{\sigma} = C_{k,5}$. Since each element of $C_{k,5}^{(\sigma)} = \langle \mathcal{A} \rangle$ capitulates in K_6, K_5, K_4, K_3, and K_2, then we have two possibilities: Only $\mathcal{A}$ and its powers capitulate in the extensions K_6, K_5, K_4, K_3, and K_2, or all the 5-classes capitulate in K_6, K_5, K_4, K_3, and K_2. This give us as type of capitulation, $(i_1,0,0,0,0,0)$ or $(i_1,1,1,1,1,1)$ with $i_1 \in \{0,1,2,3,4,5,6\}$. Since $\mathcal{A}$ capitulates in K_1, then if $i_1 \neq 0$, so i_1 is necessary 1. Thus we prove that the possible types of capitulation are $(0,0,0,0,0,0)$, $(0,1,1,1,1,1)$, $(1,0,0,0,0,0)$ or $(1,1,1,1,1,1)$, then the possible Taussky types are (A,A,A,A,A,A) or (A,B,B,B,B,B) □

4 Numerical examples

The task to determine the capitulation in a cyclic quintic extension of a base field of degree 20, that is, in a field of absolute degree 100, is definitely far beyond the reach of computational algebra systems like MAGMA and Pari/GP. For this reason we give examples of a pure metacyclic fields $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$ with n takes values from the three forms mentioned above. We have $C_{k,5}$ is of type $(5,5)$ and rank$C_{k,5}^{(\sigma)} = 1$, hence by theorem 3.2 one of capitulation types occur.

Table 1: $k = \mathbb{Q}(\sqrt[5]{n}, \zeta_5)$ with $C_{k,5}$ is of type $(5,5)$ and rank$C_{k,5}^{(\sigma)} = 1$.

n	$h_{k,5}$	$C_{k,5}$	rank $(C_{k,5}^{(\sigma)})$		n	$h_{k,5}$	$C_{k,5}$	rank $(C_{k,5}^{(\sigma)})$
118	25	$(5,5)$	1		1999	25	$(5,5)$	1
145	25	$(5,5)$	1		2007	25	$(5,5)$	1
449	25	$(5,5)$	1		2507	25	$(5,5)$	1
475	25	$(5,5)$	1		2725	25	$(5,5)$	1
559	25	$(5,5)$	1		6725	25	$(5,5)$	1
718	25	$(5,5)$	1		7375	25	$(5,5)$	1
818	25	$(5,5)$	1		7493	25	$(5,5)$	1
1018	25	$(5,5)$	1		28625	25	$(5,5)$	1
1195	25	$(5,5)$	1		55625	25	$(5,5)$	1
1249	25	$(5,5)$	1		168125	25	$(5,5)$	1
1499	25	$(5,5)$	1		149^2	25	$(5,5)$	1
1945	25	$(5,5)$	1		199^3	25	$(5,5)$	1
1945	25	$(5,5)$	1		349^3	25	$(5,5)$	1

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