Palestine Journal of Mathematics

Vol 13(Special Issue II)(2024) , 81–86

Essential *M*-Noetherian Modules and Rings

H. Chakraborty, R. K. Singh and M. K. Patel

Communicated by Mohammad Ashraf

Dedicated to Prof. B. M. Pandeya on his 78th birthday

MSC 2020 Classification: 16P40, 16P60, 16P70.

Keywords and phrases: Essential submodules, essential M-Noetherian modules, essential R-Noetherian rings.

Acknowledgement: M. K. Patel wishes to thank NBHM, DAE with file no.: 02211/3/2019 NBHM (R.P.) RD-II/1439, for financial assistantship.

Abstract The concept of an e-M-Noetherian module is a generalisation of Noetherian and e-Noetherian modules which is defined as every ascending chain of essential M-cyclic submodules gets terminated. In this paper, we proved that if N is an e-N-Noetherian R-module and M be simple R-module, then the direct sum of N and M is e-$(N \oplus M)$-Noetherian. Let R be a ring then we define an e-R-Noetherian ring, if it is an e-M-Noetherian module, where $M{=}_RR$. Also we proved that for a principal ideal ring R, the notions of an e-Noetherian and e-R-Noetherian rings are equivalent.

1 Introduction

Suppose M and N are left R-modules, then N is known as M-cyclic if $N \cong M/K$ for some submodule K of M. In [4], authors have discussed about the concept of chain conditions on M-cyclic submodules and also studied about chain conditions on R-cyclic left ideals of rings. With this motivation, we introduce the chain conditions on essential M-cyclic submodules and essential R-cyclic ideals in details. A submodule K of a module M is called an essential if for every submodule L of M which is nonzero and has non-trivial intersection with K. A module M is known as Noetherian, if every ascending chain of submodules of M gets terminated, for example any finite abelian groups over integers. A module M is defined to be an e-Noetherian, if for any chain $L_1 \subseteq_e L_2 \subseteq_e L_3 \subseteq_e \cdots \subseteq_e L_n \subseteq_e \cdots$ of essential submodules of M gets stabilized, i.e. there exists some positive integer r, such that $L_r = L_{r+1}$. For example if we consider $\mathbb{Z}_{32}$, then the ascending chain of essential submodules $16\mathbb{Z}_{32} \subseteq_e 8\mathbb{Z}_{32} \subseteq_e 4\mathbb{Z}_{32} \subseteq_e 2\mathbb{Z}_{32} \subseteq_e \mathbb{Z}_{32}$ is stationary. We recall that a module L is called an uniform, if every non-zero submodules is essential in L.

In this paper we introduce the notion of e-M-Noetherian modules and rings and then we study about the characterizations of these modules and rings. The paper itself is split into three sections, where first section is devoted to introduction including motivation and preliminary ideas related to the article. In the second section we discuss about several fundamental properties of e-M-Noetherian modules, where we extend few results of Noetherian and e-Noetherian modules and also establish some new statements. We proved that an e-Noetherian module is always an e-M-Noetherian but the converse is true only if $End(M)$ is a division ring. We then proved that if N is an e-N-Noetherian R-module and M is a simple R-module then the direct sum of N and M is e-$(N \oplus M)$-Noetherian (Theorem 2.12). We also showed that if each M_i are e-M_i-Noetherian then $\sum_{i=1}^{r} \oplus M_i$ is e-$(\sum_{i=1}^{r} \oplus M_i)$-Noetherian and conversely (Proposition 2.9).

In the third section we discuss about e-R-Noetherian rings. We proved that every principal ideal ring R with ascending chain of essential left R-cyclic ideals is e-R-Noetherian (Proposition 3.1) and a ring R is an e-R-Noetherian if and only if any direct sum of injective modules is injective (Theorem 3.3). Besides, we showed that for a commutative ring R if it is an e-R-Noetherian, then every descending chain on annihilators stabilizes and if it is self-injective, then R is an e-

R-Noetherian (Lemma 3.6).

Throughout this paper, we consider R is an associative ring with unit element and all modules are unitary left R-modules.

Remark 1.1. Every essential L-cyclic submodules of an essential M-cyclic submodule L is again an essential M-cyclic.

2 Essential M-Noetherian Modules

A module M is said to be an essential M-Noetherian (in short, e-M-Noetherian) if any ascending chain $g_1(M) \subseteq_e g_2(M) \subseteq_e g_3(M) \subseteq_e \cdots \subseteq_e g_n(M) \subseteq_e \cdots$ of essential M-cyclic submodules of M is stationary, i.e. there exists $r \in \mathbb{N}$ such that $g_r(M) = g_{r+1}(M)$ where $g_i \in End(M)$ and $i \in \mathbb{N}$. Clearly every Noetherian, simple, uniform, semisimple modules are e-M-Noetherian. An example of an e-M-Noetherian module will be $\mathbb{Z}_{27}$. Consider the terminating chain $9\mathbb{Z}_{27} \subseteq_e 3\mathbb{Z}_{27} \subseteq_e \mathbb{Z}_{27}$, here every submodule in this ascending chain is essential and hence $\mathbb{Z}_{27}$ is an e-$\mathbb{Z}_{27}$-Noetherian. Clearly if a module is an uniform and Noetherian then it is an e-M-Noetherian.

Theorem 2.1. *Suppose M is an R-module, then*

(i) M is an e-M-Noetherian if and only if every family of non-empty essential M-cyclic submodules of M has a maximal element.

(ii) M is an e-M-Noetherian if and only if every essential M-cyclic submodules of M is finitely generated.

Proof. (i) Assume that M is an e-M-Noetherian module and suppose Ω be a non-empty family of essential M-cyclic submodules of M. If there is no maximal element in Ω then we can select continously a chain of essential M-cyclic submodules $g_0(M) \subseteq_e g_1(M) \subseteq_e \cdots \subseteq_e \cdots$ from Ω, which arrive at contradiction to our assumption that M is e-M-Noetherian. Hence, we get that the family of essential M-cyclic submodules of M has a maximal element.
Conversely, consider a chain $g_0(M) \subseteq_e g_1(M) \subseteq_e \cdots \subseteq_e \cdots$ of essential M-cyclic submodules of M. By assumption, the set of essential M-cyclic submodules of M as $\{g_n(M) : n \in \mathbb{N}\}$ contains a maximal element $g_i(M)$. Nonetheless, after that we have $g_k(M) = g_i(M)$ for all $k \geq i$, hence M is an e-M-Noetherian.

(ii) Let M be an e-M-Noetherian and consider an essential M-cyclic submodule P of M. We denote Ω the collection of all finitely generated essential P-cyclic submodules of P, then by Remark 1.1, Ω is the family of essential M-cyclic submodule of M. Because M is e-M-Noetherian, then Ω will have a maximal member $g_0(P)$ by part 1. Put $g_0(P) = Rg(p_1) + Rg(p_2) + Rg(p_3) + \cdots + Rg(p_n)$ and suppose that $g_0(P) \neq P$. So there exists $g(p) \in P$ such that $g(p) \notin g_0(P)$, but then $g_0(P) + Rg(p) = Rg(p_1) + Rg(p_2) + Rg(p_3) + \cdots + Rg(p_n) + Rg(p)$ is a member of Ω strictly containing $g_0(P)$, which contradict the maximality of $g_0(P)$ so we have $g_0(P) = P$ and thus P is finitely generated.
Conversely, now suppose that every essential M-cyclic submodule of M is finitely generated and assume an ascending chain $g_1(M) \subseteq_e g_2(M) \subseteq_e g_3(M) \subseteq_e \cdots \subseteq_e g_n(M) \subseteq_e \cdots$ of essential M-cyclic submodules of M. Put $g(M) = \cup_{i=1}^{\infty} g_i(M)$, then $g(M)$ is an essential M-cyclic submodule of M and hence $g(M)$ is finitely generated. Let $g(M) = Rg(m_1) + Rg(m_2) + Rg(m_3) + \cdots + Rg(m_r)$, now each $g(m_j)$ belongs to one of the $g_i(M)$'s, so there exists m such that $g(m_1), g(m_2), \cdots, g(m_r)$ belong to $g_m(M)$. But then $g(M) = g_m(M)$ and $g_n(M) = g_m(M)\ \forall n \geq m$, therefore M is an e-M-Noetherian.

□

Corollary 2.2. *Every submodule N of an e-M-Noetherian module M is also an e-M-Noetherian.*

Theorem 2.3. *M is an e-M-Noetherian module, when M be an e-Noetherian.*

Proof. Suppose $g_1(M) \subseteq_e g_2(M) \subseteq_e \cdots \subseteq_e g_n(M) \subseteq_e g_{n+1} \subseteq_e \cdots$ be an ascending chain of essential M-cyclic submodules of M. Because M is an e-Noetherian [9, Theorem 2] holds. Hence the above chain stabilizes. □

Proposition 2.4. *If End(M) is a division ring, then an e-M-Noetherian module is an e-Noetherian.*

Proof. Suppose M is an e-M-Noetherian module. So it has an ascending chain of essential M-cyclic submodules which terminates i.e. $h_1(M) \subseteq_e h_2(M) \subseteq_e \cdots h_n(M) = h_{n+1}(M) \cdots$. Choosing $h_i(M) = h(M_i)$ as in [4], we have the above chain as $h(M_1) \subseteq_e h(M_2) \subseteq_e \cdots \subseteq_e h(M_n) = h(M_{n+1}) \cdots$, where $h \in End(M)$. Since $End(M)$ is a division ring and so $h^{-1} \in End(M)$. If we operate it in the above chain we get a chain of essential submodules which is ascending and terminates i.e. $M_1 \subseteq_e M_2 \subseteq_e \cdots \subseteq_e M_n = M_{n+1} \cdots$. Thus M is an e-Noetherian. □

Proposition 2.5. *An epi-retractable module M is an e-M-Noetherian if and only if it is an e-Noetherian.*

Proof. Suppose M is an epi-retractable module which is an e-M-Noetherian. As every submodule in an epi-retractable module is M-cyclic hence any ascending chain of essential M-cyclic submodules turns into ascending chain of essential submodules of M that terminates, since M is an e-M-Noetherian. Converse part is obvious by Theorem 2.3. □

A sequence of the type $\cdots \to A_{n-1} \xrightarrow{\pi_{n-1}} A_n \xrightarrow{\pi_n} A_{n+1} \to \cdots$ of R homomorphisms is called an exact sequence if $Img(\pi_{n-1}) = ker(\pi_n)$ for all $n \in \mathbb{N}$. An exact sequence of the type $0 \to P \xrightarrow{\pi} Q \xrightarrow{\psi} R \to 0$ is known as short exact sequence, where π and ψ is one-one and onto homomorphisms respectively.

Theorem 2.6. *Suppose $0 \to M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2 \to 0$ be a short exact sequence of R-modules. Then M is an e-M-Noetherian if and only if both M_1 and M_2 are e-M_1-Noetherian and e-M_2-Noetherian respectively.*

Proof. Suppose that M_1 be a submodule of M and that $M/M_1 \cong M_2$. Assume that M is an e-M-Noetherian. Since M_1-cyclic submodule of M_1 are also M-cyclic submodule of M, so we have M_1 is an e-M_1-Noetherian by Corollary 2.2. For next part, consider a chain of essential M_2-cyclic submodules of M_2 i.e. $g_1(M_2) \subseteq_e g_2(M_2) \subseteq_e g_3(M_2) \subseteq_e \cdots \subseteq_e g_n(M_2) \subseteq_e \cdots$ of M/M_1 where $g_i \in End(M_2)$ and corresponding to this chain by [12, Proposition 1.9], we have $h_1(M) \subseteq_e h_2(M) \subseteq_e h_3(M) \subseteq_e \cdots \subseteq_e h_n(M) \subseteq_e \cdots$ of essential M-cyclic submodules of M which contains M_1 and there exists n such that $h_r(M) = h_n(M)$ for all $r \geq n$, as M is an e-M-Noetherian. Returning to M/M_1, we see that $g_r(M_2) = g_n(M_2)\ \forall\ r \geq n$. Hence $M/M_1 \cong M_2$ is an e-M_2-Noetherian module.
Conversely, now suppose that M_1 and $M/M_1 \cong M_2$ are e-M_1-Noetherian and e-M_2-Noetherian respectively. Now consider the ascending chain of essential M-cyclic submodules i.e. $g_1(M) \subseteq_e g_2(M) \subseteq_e g_3(M) \cdots \subseteq_e g_n(M) \subseteq_e \cdots$ of M. Claim that $g_n(M) = g_m(M)$ for all $n \geq m$. This gives ascending chains $g_1(M) \cap M_1 \subseteq_e g_2(M) \cap M_1 \subseteq_e g_3(M) \cap M_1 \subseteq_e \cdots \subseteq_e g_r(M) \cap M_1 \subseteq_e \cdots$ and $(g_1(M) + M_1)/M_1 \subseteq_e (g_2(M) + M_1)/M_1 \subseteq_e (g_3(M) + M_1)/M_1 \subseteq_e \cdots \subseteq_e (g_r(M) + M_1)/M_1 \subseteq_e \cdots$. of M_1 and $M/M_1 \cong M_2$ respectively. Clearly both of these chains terminates, so there exists $m \in \mathbb{N}$ such that $(g_n(M) \cap M_1) = (g_m(M) \cap M_1)$ and $(g_n(M) + M_1)/M_1 = (g_m(M) + M_1)/M_1$ for all $n \geq m$. Now we proceed to the final step.
Since $g_n(M) = g_n(M) \cap (g_n(M) + M_1)$
= $g_n(M) \cap (g_m(M) + M_1)$
= $g_m(M) + (g_n(M) \cap M_1)$ (by modular property)
= $g_m(M) + (g_m(M) \cap M_1)$ (since $g_n(M) \cap M_1 = g_m(M) \cap M_1$)
= $g_m(M) \cap (g_m(M) + M_1)$ (by modular property)
=$g_m(M)$
hence $g_n(M) = g_m(M)\ \forall\ n \geq m$. Hence M is an e-M-Noetherian module. □

Corollary 2.7. *A module M is an e-M-Noetherian if and only if its submodule K and quotient module M/K are e-K-Noetherian and e-M/K-Noetherian respectively.*

Proof. If we assume an exact sequence $0 \to K \xrightarrow{\phi} M \xrightarrow{\psi} M/K \to 0$, where ϕ is one-one homomorphism and ψ an onto homomorphism. Clearly the proof holds from above result. □

Corollary 2.8. *(i) Every submodules and quotient modules of an uniform e-Noetherian modules are e-M-Noetherian.*

(ii) If M is an uniform module such that it contains a submodule K with K as e-K-Noetherian and M/K as e-M/K-Noetherian then M is an e-M-Noetherian.

Proof. The above theorem leads directly to proof. □

Proposition 2.9. *Let $M_1, M_2, M_3, \cdots, M_r$ are any R-modules. Then $\oplus_{i=1}^{n} M_i$ is e-$\oplus_{i=1}^{r} M_i$-Noetherian if and only if each M_i is an e-M_i-Noetherian.*

Proof. Let $\oplus_{i=1}^{r} M_i$ is e-$\oplus_{i=1}^{r} M_i$-Noetherian. It is obvious that the result is true for $r = 1$. Consider for $r \geq 2$, an exact sequence $0 \to \oplus_{i=1}^{r-1} M_i \xrightarrow{\phi} \oplus_{i=1}^{r} M_i \xrightarrow{\pi} M_r \to 0$ where the mapping ϕ and π are the injection and projection mapping respectively. Using Theorem 2.6, we have $\oplus_{i=1}^{r-1} M_i$ and M_r are e-$\oplus_{i=1}^{r-1} M_i$-Noetherian and e-M_r-Noetherian respectively. Taking $i = r$ then clearly we get M_i is an e-M_i-Noetherian.
Conversely, assume that each M_i is an e-M_i-Noetherian. By the principle of mathematical induction, suppose $\oplus_{i=1}^{r-1} M_i$ is e-$\oplus_{i=1}^{r-1} M_i$-Noetherian and M_r is an e-M_r-Noetherian respectively. Taking the exact sequence $0 \to \oplus_{i=1}^{r-1} M_i \xrightarrow{\phi} \oplus_{i=1}^{r} M_i \xrightarrow{\pi} M_r \to 0$, where the mapping ϕ and π are the injection and projection mapping respectively and so by Theorem 2.6, we have $\oplus_{i=1}^{r} M_i$ is e-$\oplus_{i=1}^{r} M_i$-Noetherian. □

Corollary 2.10. *Let $M_1, M_2, M_3, \cdots, M_r$ are simple R-modules. Then $\oplus_{i=1}^{r} M_i$ is e-$\oplus_{i=1}^{r} M_i$-Noetherian.*

Corollary 2.11. *The homomorphic image of an e-M-Noetherian module is an e-M-Noetherian.*

Proof. Suppose M is an e-M-Noetherian and K be any submodule of M. If $g : M \to K$ is a R-module homomorphism then we have $M/kerg \cong g(M)$ by first isomorphism theorem. Hence $g(M)$ is an e-M-Noetherian by Corollary 2.7. □

Theorem 2.12. *Let N be an e-N-Noetherian module and M be a simple R-module, then $N \oplus M$ is e-$(N \oplus M)$-Noetherian.*

Proof. Let $f(M_1) \subseteq_e f(M_2) \subseteq_e \cdots$ be any chain of essential $(N \oplus M)$-cyclic submodules of $N \oplus M$. We discuss two cases.

1. If $f(M_i) \cap M = 0$ $\forall i$, then all $f(M_i) \cong K_i$, where K_i are any submodules of N. Given that N is e-N-Noetherian this implies that $f(M_n) = f(M_i)$ $\forall i \geq n$, proof is done.

2. We have $M \cap f(M_i) \neq 0$ since $f(M_i)$ is nonzero. Since M is simple so this implies that $f(M_i) \cap M = M$ and thus $M \subseteq f(M_i)$ $\forall i \geq p$. Applying modular property we get, $f(M_i) = f(M_i) \cap (N \oplus M) = (f(M_i) \cap N) \oplus M$, similarly $f(M_{i+1}) = (f(M_{i+1}) \cap N) \oplus M$. Suppose that $f(M_k) \cap N \subseteq_e f(M_{k+1}) \cap N \subseteq_e \cdots$ is a chain of ascending essential N-cyclic submodules of N. Because N is an e-N-Noetherian this follows that there exists $s \geq k$, $f(M_i) \cap N = f(M_{i+1}) \cap N$ $\forall i \geq s$. Thus $(f(M_i) \cap N) \oplus M = (f(M_{i+1}) \cap N) \oplus M$ $\forall i \geq s$ and hence $f(M_i) = f(M_{i+1})$ $\forall i \geq s$. Thus $N \oplus M$ is e-$(N \oplus M)$-Noetherian. □

Lemma 2.13. *If M be an e-M-Noetherian left R-module then M cannot have an infinite direct sum of nonzero submodules.*

Proof. Proof is obvious. □

3 Essential R-Noetherian rings

We call a ring R an essential R-Noetherian ring (in short, e-R-Noetherian) if it is e-M-Noetherian module, where M=$_RR$ for example, residue class of integers modulo n for $\forall n > 0$ and semisimple rings. An ideal I of R is finitely generated if there is a finite subset X of R such that I=$< X >$. If an ideal I is generated by one element then it is known as principal ideal. A ring R where every ideal is principal is known as a principal ideal ring. If R is also an integral domain then it is called a principal ideal domain.

Proposition 3.1. *Every principal ideal ring R with ascending chain of essential left R-cyclic ideals $f(I_1) \subseteq_e f(I_2) \subseteq_e \cdots \subseteq_e f(I_n) \subseteq_e \cdots$ is e-R-Noetherian, where $f \in End(R)$.*

Proof. Consider the family of essential left R-cyclic ideals of R as $\{f(I_i) : i \in \mathbb{N}\}$ where $f \in End(R)$ such that $f(I_r) \subseteq f(I_{r+1})\ \forall r \in \mathbb{N}$. Then $f(I) = \bigcup_{i\in\mathbb{N}} f(I_i)$ is also essential principal left R-cyclic ideal of R, as every $f(I_i)$'s are essential in R. Let $f(I)$ be generated by an element $a \in f(I)$. Now since $a \in f(I)$, so there exists an index $k \in \mathbb{N}$ such that $a \in f(I_k)$. Claim that $f(I_k) = f(I_r)\ \forall\, r \geq k$. Suppose this is not true, then there exists $r > k$ such that $f(I_k) \subseteq f(I_r)$ and $f(I_k) \neq f(I_r)$ i.e. $f(I_r) \setminus f(I_k)$ is nonempty. Suppose $x \in f(I_r)$ but $x \notin f(I_k)$, then $x \in f(I) = \bigcup_{i\in\mathbb{N}} f(I_i)$ so $x = b.a$ for some $b \in R$ as $'a'$ generator of $f(I)$. Again $f(I_k)$ is left ideal and $a \in f(I_k)$, we have $b.a \in f(I_k)$ as $x = b.a$ implies that $x \in f(I_k)$ which arrive at contradiction to our supposition $x \notin f(I_k)$. Thus the given chain of essential principal left ideal will terminate and hence R is e-R-Noetherian. □

Corollary 3.2. *Every principal ideal domain R is an e-R-Noetherian.*

Proof. Proof is straight forward in light of above proposition. □

Theorem 3.3. *A ring R is left e-R-Noetherian if and only if any direct sum of injective left R-modules is injective.*

Proof. Let R is left e-R-Noetherian ring and assume $F = \oplus_{i\in I} F_i$ is a direct sum of injective left R-modules F_i. Suppose L is a left ideal of R and we consider the homomorphism $\phi : L \to F$. Since R is e-R-Noetherian, we have every ideal is finitely generated and suppose the generators be $x_1, x_2, \cdots, x_n$. Now each $\phi(x_k)$ consists of finitely many nonzero components in F which implies that $\phi(x_k) \in \oplus_{i\in I_k} F_i$ where I_k is finite subset of I. Again let $I^* = \bigcup_{i=1}^{n} I_i$ and $F^* = \oplus_{i\in I^*} F_i$ so each $\phi(x_k) \in F^*$ hence $\phi(L) \subseteq F^*$. Also I^* is finite so we get a homomorphism ${}_R R \to F^* \subseteq F$ because F^* is injective. Thus F is injective.
Conversely, we assume that the direct sum of injective left R-modules is injective. Let $f(K_1) \subseteq_e f(K_2) \subseteq_e \cdots$ be a chain of essential R-cyclic ideals of R. Consider $P = \bigcup_{i\in\mathbb{N}} f(K_i)$ which is a left essential R-cyclic ideal of R. Suppose J=$E(R/f(K_1)) \oplus E(R/f(K_2)) \oplus \cdots$ and we define a function $\psi : P \to J$ by

$$\psi(p) = (p + f(K_1), p + f(K_2) \cdots,) \qquad (p \in P)$$

Now if $p \in P$ then there exists $q \in \mathbb{N}$ such that $p \in f(K_q)$ so that $p + f(K_r) = 0\ \forall r \geq q$ and $\psi(p) \in J$. Clearly J is injective and we have $x \in J$ such that $\psi(p) = px \quad (p \in P)$. Also we have $l \in \mathbb{N}$ and $x_i \in E(R/f(K_i)),\ i \in \mathbb{N}$ such that $x = (x_1, x_2, \cdots)$ and $x_s = 0\ \forall s \geq l$. For $p \in P$ we have

$$(p + f(K_1), p + f(K_2) \cdots,) = \psi(p) = p(x_1, x_2, \cdots)$$

suggests that $p + f(K_l) = px_l = 0$ and so $p \in f(K_l)$. Hence $f(K_q) \subseteq_e f(K_l)$. Thus $f(K_l) = f(K_{l+1}) = \cdots$ and so R is left e-R-Noetherian. □

Theorem 3.4. *A ring R is an e-R-Noetherian if and only if every finitely generated R-module M is an e-M-Noetherian.*

Proof. We consider R is an e-R-Noetherian ring and M is a finitely generated R-module. Let F be any free module so there exists a surjective homomorphism $\psi : F \to M$. Because M is finitely generated so $F \cong \oplus_{i=1}^{n} R$ where $n \in \mathbb{N}$. Consider $\phi : \oplus_{i=1}^{n} R \to M$ is a onto R-homomorphism. Clearly $\oplus_{i=1}^{n} R$ is an R-module hence we have homomorphic image of ϕ i.e. M is also e-M-Noetherian. Thus M is e-M-Noetherian. Conversely let M be a finitely generated R-module which is an e-M-Noetherian. We know that for any finitely generated module over a Noetherian ring is Noetherian. Using this fact we can say R-module R is an e-R-Noetherian. □

Lemma 3.5. *Let R be an uniform e-R-Noetherian regular ring. Then R is semisimple.*

Proof. Let R be an e-R-Noetherian ring. Clearly every ideal of R is finitely generated. Since R is regular hence every finitely generated ideal is a direct summand of R which implies that R is a semisimple ring. □

Lemma 3.6. *Let R be a commutative ring.*

(i) If R is an e-R-Noetherian then it holds descending chain condition on annihilators.

(ii) If R is self-injective then R is an e-R-Noetherian.

Proof. (i) Consider the descending chain of essential R-cyclic ideals which are annihilators of R, i.e. $K_1 \supseteq_e K_2 \supseteq_e \cdots$. If we take annihilators on the above chain we attain an ascending chain i.e. $ann(K_1) \subseteq_e ann(K_2) \subseteq_e \cdots$. Because R is an e-R-Noetherian we have an index r, such that $ann(K_i) = ann(K_r)\ \forall i \geq r$. Again considering annihilators of these annihilators ideals we obtain $K_i = K_r\ \forall i \geq r$.

(ii) Let R be self injective, so R fulfills the descending chain condition on annihilators and by [2, Theorem 2], the ring R is Quasi Frobenius. Hence R is an e-Noetherian by [2, Theorem 2] and thus R is an e-R-Noetherian. □

References

[1] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Springer-Verlag, 1992.

[2] J. Clark, When is a self-injective semi perfect ring quasi-frobenius?, Journal of Algebra, 165: 531–542, 1994.

[3] A. Facchini and Z. Nazemian, On isonoetherian and isoartinian modules, Contemporary Mathematics, 730: 1–22, 2019.

[4] T. Gera, S. C. Gannamaneni and M. K. Patel, Chain conditions on M- cyclic submodules, 81–88, De Gruyter, Berlin, Boston, 2023, doi:10.1515/9783110785807-01.

[5] R. B. Warfield Jr. and K. R. Goodearl, An Introduction to Noncommutative Noetherian Rings, Cambridge University Press, 2004.

[6] T. Y. Lam, Lectures on Ring and Modules, Springer New York, 1998

[7] E. P. Armendariz, Rings with dcc on essential left ideals, Communications in Algebra, 8(3):299-308, 1980 doi:10.1080/00927878008822460

[8] A. Facchini and Z. Nazemian, Modules with chain conditions up to isomorphism, Journal of Algebra, 453: 578-601, 2016, doi:10.1016/j.jalgebra.2016.01.025

[9] P. Nath, H. Chakraborty and G. Bardhan, e-Noetherian modules and rings, 101–108, De Gruyter, Berlin, Boston, 2023, doi:10.1515/9783110785807-013.

[10] B. L. Osofsky, Chain conditions on essential submodules, Proceedings of the American Mathematical Society, 114(1):11–19, 1992.

[11] S. Prakash and A. K. Chaturvedi, Iso-Noetherian rings and modules, Communications in Algebra, 47(2): 676-683, 2019.

[12] D. W. Sharpe and P. Vamos, Injective modules, Cambridge University Press, 2008.

Author information

H. Chakraborty, R. K. Singh and M. K. Patel, Department of Science and Humanities (Mathematics), National Institute of Technology Nagaland, Dimapur-797103, India.
E-mail: himanchakra@gmail.com, nitranjitbhu@gmail.com, mkpitb@gmail.com