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ON k-Γ-HYPERIDEALS IN Γ-SEMIHYPERRING

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Abstract The notions of k-Γ-hyperideal, k-Γ-hyperideal of type-I, k-prime-Γ-hyperideal, 2-prime-Γ-hyperideal, weakly prime-k-Γ-hyperideal, maximal Γ-hyperideal and m-k-Γ-hyperideal in Γ-semihyperring are introduced and studied with some examples. We have proved some fundamental results of these Γ-hyperideals and investigate their relationship.

1 Introduction

An algebraic hyperstructure theory emerged from the theory of classical algebraic structure. In an algebraic hyperstructure the composition of two elements gives us a non-empty set instead of an element. The study of hyperstructures was initiated by a French mathematician Marty [17] in the year 1934. Later, many researchers across the globe are studying hyperstructures and contributing in the form of books and journal articles. Readers are requested to refer [4, 5, 6, 7] for basic and fundamental concepts of hyperstructures. Where Corsini provided many examples stating that how hyperstructures are useful in many different areas of mathematics including rough sets, probability, automata, lattice theory, and coding theory. In the year 2007 a book on hyperrings was published by Davvaz and Leoreanu-Fotea [8] that gives us a detailed overview of the theory. Then in 2016, a book [9] on semihypergroups was written by Davvaz that is incredibly helpful for beginners to understand the fundamental concepts of semihypergroups.

In 1964, [18] the concept of a Γ-ring was first introduced by Nobusawa. Then in [23] Rao introduced the notion of a Γ-semiring, the generalization of a Γ-ring and semiring. Vougiouklis [26] introduced the notion of a semihyperring as a generalization of a semiring in which both operations are hyperoperations. Later, the concepts of a Γ-semihyperring as a generalization of a semiring, a Γ-semiring, and a semihyperring are introduced and studied by Dehkori and Davvaz [10, 11, 12]. The concept of ideal may differ in different algebraic structures. Though the semiring is a generalization of ring, ideal of semiring do not coincide with ideal of ring. For example, an ideal of a semiring need not be the kernel of a semiring homomorphism. This problem was solved by Henriksen [14] by defining a new class of ideals called as k-ideals in semirings by which one can obtain analogous results that of ring. The notion of a weakly prime ideal of an associative ring with unity was given by Anderson and Smith [2]. Later, Dubey [13] worked on the concepts of prime and weakly prime ideals in semirings. Ameri and Hedayati [1, 15, 16] studied the concept of a k-hyperideal in a semihyperring, where the addition is a hyperoperation. Whereas the concept of k-hyperideal on an ordered semihyperring was introduced by Omidi and Davvaz [20].

Here we prove some results on prime, weakly prime k-ideal, maximal ideal for Γ-semihyperring. Further, we introduce k-Γ-hyperideal and k-Γ-hyperideal of type-I in Γ-semihyperring. Then some fundamental results of ring homomorphism such as the mapping of k-Γ-hyperideals under ring homomorphism are proved and their relationships are studied.

2 Preliminaries

We begin this section by recalling the notion of Γ-semihyperring and some definitions and properties from [3, 21] which are necessary for the rest of the article.

Definition 2.1. [3] Let H be a nonempty set and a hyperoperation on H is a map $\circ : H \times H \to \wp^*(H)$, where $\wp^*(H)$ is a collection of all non-empty subsets of H. The pair $(H, \circ)$ is called a hypergroupoid. For any two non-empty subsets A and B of H and $x \in H$,

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \ A \circ \{x\} = A \circ x \text{ and } \{x\} \circ A = x \circ A.$$

Definition 2.2. [3] A hypergroupoid $(H, \circ)$ is called a semihypergroup if for all $a, b, c \in H$ we have, $(a \circ b) \circ c = a \circ (b \circ c)$, which means that

$$\bigcup_{u \in a \circ b} u \circ c = \bigcup_{v \in b \circ c} a \circ v.$$

In addition if for every $a \in H$, $a \circ H = H = H \circ a$, then $(H, \circ)$ is called a hypergroup.

Definition 2.3. [21] Let R be a commutative semihypergroup and Γ be a commutative group. Then R is called a Γ-semihyperring if there is a map $R \times \Gamma \times R \to \wp^*(R)$ (images to be denoted by $a\alpha b$ for all $a, b \in R$, $\alpha \in \Gamma$ and $\wp^*(R)$ is the set of all non-empty subsets of R that satisfy the following conditions:

(i) $a\alpha(b + c) = a\alpha b + a\alpha c$

(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$

(iii) $a(\alpha + \beta)c = a\alpha c + a\beta c$

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$; for all $a, b, c \in R$ and for all $\alpha, \beta \in \Gamma$.

Example 2.4. [21] Let $(R, +, \cdot)$ be a semihyperring such that $x \cdot y = x \cdot y + x \cdot y$ and Γ be a commutative group. We define $x\alpha y \to x \cdot y$ for every $x, y \in R$, and $\alpha \in \Gamma$, then R is a Γ-semiyperring.

Definition 2.5. [21] A Γ-semihyperring R is said to be commutative if $a\alpha b = b\alpha a$, for all $a, b \in R$ and $\alpha \in \Gamma$.

Definition 2.6. [21] A Γ-semihyperring R is said to be with zero if there exists $0 \in R$ such that $a \in a + 0$ and $0 \in 0\alpha a$, $0 \in a\alpha 0$ for all $a \in R$ and $\alpha \in \Gamma$.

Let A and B be two non-empty subsets of a Γ-semihyperring R and $x \in R$, then

$$A + B = \{x \mid x \in a + b, a \in A, b \in B\}$$
$$A\Gamma B = \{x \mid x \in a\alpha b, a \in A, b \in B, \alpha \in \Gamma\}.$$

Definition 2.7. [21] A non-empty subset R_1 of a Γ-semihyperring R is called a Γ-subsemihyperring if it is closed with respect to the multiplication and addition, that is $R_1 + R_1 \subseteq R_1$ and $R_1\Gamma R_1 \subseteq R_1$.

Definition 2.8. [21] A right (left) ideal I of a Γ-semihyperring R is an additive sub semihyperrring of $(R, +)$ such that $I\Gamma R \subseteq I(R\Gamma I \subseteq I)$. If I is a both right and left ideal of R, then we say that I is a two-sided ideal or simply an ideal of R.

3 k-Γ-hyperideals and m-k-Γ-hyperideals

Sen and Adhikari discussed the characterization of k-ideals in semiring [24, 25]. In semihyperring k-hyperideal was studied by Ameri and Hedayati [1]. In this section, we study k-Γ-hyperideals and m-k-Γ-hyperideals in Γ-semihyperring.

Definition 3.1. Let R be a Γ-semihyperring. A non-empty subset I of R is said to be a left (right) k-Γ-hyperideal of R, if

(i) I is a left (right) Γ-hyperideal of R.

(ii) for any $a \in I$, $x \in R$ and $(a+x) \cap I \neq \phi$ $((x+a) \cap I \neq \phi)$ it implies that $x \in I$.

A two-sided k-Γ-hyperideal or simply a k-Γ-hyperideal if it is both left and right k-Γ-hyperideal of R.

Remark 3.2. Every k-Γ-hyperideal of a Γ-semihypering R is a Γ-hyperideal of R. But converse need not be true.

Example 3.3. Let $R = \{0, 1, 2, 3\} = \mathbb{Z}_4$ be a commutative semihypergroup with hyperoperation $\oplus$ and $\Gamma = \{\alpha, \beta, \gamma\}$ is commutative group with respect to addition. Then R is a Γ-semihyperring using following operations:

$\oplus$	0	1	2	3
0	$\{0\}$	$\{1\}$	$\{2,3\}$	$\{3\}$
1	$\{1\}$	$\{2,3\}$	$\{2,3\}$	$\{2,3\}$
2	$\{2,3\}$	$\{2,3\}$	$\{2,3\}$	$\{2,3\}$
3	$\{3\}$	$\{2,3\}$	$\{2,3\}$	$\{2,3\}$

α	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
3	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$

β	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$	$\{2,3\}$	$\{2,3\}$
2	$\{0\}$	$\{2,3\}$	$\{2,3\}$	$\{2,3\}$
3	$\{0\}$	$\{2,3\}$	$\{2,3\}$	$\{2,3\}$

γ	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{2,3\}$	$\{2,3\}$	$\{2,3\}$
2	$\{0\}$	$\{2,3\}$	$\{2,3\}$	$\{2,3\}$
3	$\{0\}$	$\{2,3\}$	$\{2,3\}$	$\{2,3\}$

Now $J = \{0, 2, 3\}$ is a Γ-hyperideal of R, but it is not a k-Γ-hyperideal of R, because $2 \in J$, $1 \in R$ with $2 \oplus 1 = \{2, 3\}$ and $\{2, 3\} \cap \{0, 2, 3\} \neq \phi$ but $1 \notin \{0, 2, 3\}$.

Example 3.4. Consider $\mathbb{Z}$ with hyperoperations defined as follows $x + y = \{x, y\}$ and $x\gamma y$=$x \cdot y$, for all $x, y \in \mathbb{Z}$, and $\gamma \in \Gamma$, where Γ is a commutative group. Then $\mathbb{Z}$ is a Γ-semihyperrring. Here J=$< 2 >$=$\{2n \mid n \in \mathbb{Z}\}$ is a Γ-hyperideal of $\mathbb{Z}$ but not a k-Γ-hyperideal of $\mathbb{Z}$, because $5 \in R, 2 \in J$ and $5 + 2$ =$\{5, 2\}$ with $\{5, 2\} \cap J \neq \phi$, but $5 \notin J$.

Definition 3.5. A Γ-hyperideal J of a Γ-semihyperring R is said to be a m-k-Γ-hyperideal of R if $a \in J$, $1 \neq b \in R$ and $b\Gamma a \in J$ implies that $b \in J$.

Theorem 3.6. *Every m-k-Γ-hyperideal of a Γ-semihyperring R is a k-Γ-hyperideal of R.*

Proof. Let R be a Γ-semihyperring and J be a m-k-Γ-hyperideal of R. Suppose $a + b \in J$, where $a \in J$ and $b \in R$. Since J is a Γ-hyperideal of R, we have $(a + b)\Gamma b \in J$. This implies that $b \in J$, hence J is a k-Γ-hyperideal of R. □

Remark 3.7. Converse of the Theorem 3.6 need not be true. That is every k-Γ-hyperideal of a Γ-semihypering need not be a m-k-Γ-hyperideal of R.

Example 3.8. Let $R = \{a, b, c, d\}$ be a commutative semihypergroup with hyperoperation $\oplus$ and $\otimes$ on R defined as follows:

$\oplus$	a	b	c	d
a	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
c	$\{c\}$	$\{b\}$	$\{a,c\}$	$\{a,c,d\}$
d	$\{d\}$	$\{b\}$	$\{a,c,d\}$	$\{a,d\}$

$\otimes$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{b\}$	$\{a,c\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
d	$\{a\}$	$\{a,d\}$	$\{a\}$	$\{a\}$

Define a mapping $R \times \Gamma \times R \to \wp^*(R)$ by $x\gamma y = x \otimes y$, for every $x, y \in R$, and $\gamma \in \Gamma$. Then R is a Γ-semihyperring and $J = \{a, c, d\}$ be a k-Γ-hyperideal of R. Observed that $b\Gamma c = \{a, c\} \subseteq J$, $c \in J$, but $b \notin J$. Hence J is not an m-k-Γ-hyperideal of R.

Theorem 3.9. *Let R be a Γ-semihyperring. If k-Γ-hyperideal J of R is union of the other two k-Γ-hyperideal X and Y of R, then J coincides with exactly one of them.*

Proof. Let R be a Γ-semihyperring and X, Y and J are k-Γ-hyperideals of R such that $J = X \cup Y$. It follows that $J \subseteq X$ or $J \subseteq Y$. If $J \neq X$ and $J \neq Y$ then there exist $x, y \in J$ such that $x \in X$ but $x \notin Y$ and $y \in Y$ but $y \notin X$. As J is a Γ -hyperideal of R it follows that $x + y \subseteq J = X \cup Y$. This implies that $(x + y) \cap X \neq \phi$ or $(x + y) \cap Y \neq \phi$. Thus we obtain $y \in X$ or $x \in Y$, a contradiction. Hence either J=X or J=Y. □

Theorem 3.10. *Let R be a Γ-semihyperring and $\{J_k/k \in \Delta\}$ be a family of a k-Γ-hyperideals in R, where Δ is an indexing set. Then $\bigcap\limits_{k\in\Delta} J_k$ is a k-Γ-hyperideal of R.*

Theorem 3.11. *Let R be Γ-semihyperring. Then union of two k-Γ-hyperideals of R is a k-Γ-hyperideal of R if and only if one is contained in the other.*

We can generalized statement of Theorem 3.11 to union of n ideals as follows.

Theorem 3.12. *Let J_n be a family of k-Γ-hyperideals of a Γ-semihyperring R, with $J_p \subseteq J_q$ or $J_q \subseteq J_p$ for all $p, q, n \in \Delta$. Then $\bigcup\limits_{n\in\Delta} J_n$ is a k-Γ-hyperideal of R, where $|\Delta| \geq 2$.*

Proof. The proof is by mathematical induction on n. □

Definition 3.13. Let R and R' be Γ-semihyperrings. For all $x, y \in R$, $\alpha \in \Gamma$, $f \colon \Gamma \to \Gamma'$, a mapping $\phi : R \to R'$ is said to be

(i) Γ-homomorphism, if

a. $\phi(x + y) = \{\phi(t) \mid t \in x + y\} \subseteq \phi(x) + \phi(y)$
b. $\phi(x\alpha y) = \{\phi(t) \mid t \in x\alpha y\} \subseteq \phi(x)f(\alpha)\phi(y)$
c. $f(x + y) = f(x) + f(y)$.

(ii) Γ-reverse homomorphism, if

a. $\phi(x + y)=\{\phi(t) \mid t \in x + y\} \supseteq \phi(x) + \phi(y)$
b. $\phi(x\alpha y)=\{\phi(t) \mid t \in x\alpha y\}\supseteq \phi(x)f(\alpha)\phi(y)$
c. $f(x + y) = f(x) + f(y)$.

(iii) Γ-strong or good homeomorphism, if

a. $\phi(x + y)= \phi(x) + \phi(y)$
b. $\phi(x\alpha y)=\phi(x)f(\alpha)\phi(y)$
c. $f(x + y) = f(x) + f(y)$.

A map ϕ is called an Γ-epimorphism if $\phi\colon R \to R'$ and $f\colon \Gamma \to \Gamma'$ are surjective. A map ϕ is called an Γ-isomorphism if $\phi\colon R \to R'$ and $f\colon \Gamma \to \Gamma'$ are bijective.

Theorem 3.14. *Let R and R' be two Γ-semihyperrings.*

(i) If $\phi\colon R \to R'$ is a reverse epimorphism and J is a k-Γ-hyperideal of R, then $\phi(J)$ is a k-Γ-hyperideal of R'.

(ii) If $\varphi\colon R \to R'$ is a homomorphism and K is a k-Γ-hyperideal of R', then $\varphi^{-1}(K)$ is a k-Γ-hyperideal of R.

Proof. (i) Let R and R' be Γ-semihyperrings and J be a k-Γ-hyperideal of R. It is easy to prove that $\phi(J)$ is a Γ-hyperideal of R'. Only to prove that $\phi(J)$ is a k-Γ-hyperideal of R. Let $x \in R'$, $a \in \phi(J)$ and $(x{+}a){\cap}\phi(J) \neq \phi$ then there exist some $r \in R$ and $j \in J$ such that $x = \phi(r)$ and $a = \phi(j)$. By hypothesis, ϕ is a reverse epimorphism $\phi(r + j) \supseteq \phi(r) + \phi(j)$ and $\phi(r + j) \cap \phi(J) \neq \phi$. Then there exists $t \in J$ such that $\phi(t) \subseteq \phi(r + j) \cap \phi(J)$. It follows that $t \in (r + j) \cap J$ and $j \in J$ hence $r \in J$. Thus $x = \phi(r) \in \phi(J)$ and $\phi(J)$ is a k-Γ-hyperideal of R'.

(ii) To prove that $\varphi^{-1}(K)$ is a Γ-hyperideal of R trivially. Only to prove that $\varphi^{-1}(K)$ is a k-Γ-hyperideal of R. Let an element $a \in R$, $x \in \varphi^{-1}(K)$ and $(a+x) \cap \varphi^{-1}(K) \neq \phi$. This implies that $\varphi(a), \varphi(x) \in K$ and $\varphi(a+x) \cap \varphi(\varphi^{-1}(K) \neq \phi$. Moreover $\varphi(a+x) \subseteq \varphi(a)+\varphi(x) \cap K \neq \phi$, since $\varphi(\varphi^{-1}(K)) = K$. This implies that $\varphi(a) \in K$ and hence $\varphi^{-1}(K)$ is a k-Γ-hyperideal of R.

□

Remark 3.15. If R is a Γ-hyperring, then every Γ-hyperideal is k-Γ-hyperideal.

Proposition 3.16. *If R is a Γ-semihyperring with zero and J is a k-Γ-hyperideal of R, then zero belongs to J.*

Theorem 3.17. *Let R and R' be Γ-semihyperrings with zero and a mapping $\phi\colon R \to R'$ is a Γ-homomorphism. Then $ker\phi$=$\{x \in R \mid \phi(x) = 0\}$ is a k-Γ-hyperideal of R.*

Proof. Let R and R' be Γ-semihyperrings with zero and a mapping $\phi\colon R \to R'$ is a Γ-homomorphism. It is easy to prove a $ker\phi$ is Γ-hyperideal of R. Next to prove that $ker\phi$ is k-Γ-hyperideal of R. Let $a \in R$, $x \in ker\phi$ and $(a+x) \cap ker\phi \neq \phi$. Then there exists $z \in (a+x) \cap ker\phi$ such that $\phi(z) \subseteq \phi(a+x) \subseteq \phi(a)+\phi(x)$=$\phi(a)$, thus $\phi(a) = \phi(z) = 0$. Then $a \in ker\phi$ and hence $ker\phi$ is a k-Γ-hyperideal of R. □

Theorem 3.18. *Let R be a Γ-semihyperring with zero, J be a Γ-hyperideal of R and $\frac{R}{J} = \{a+J \mid a \in R\}$. Then $\frac{R}{J}$ is a Γ-semihyperring with hyperoperations defined as follows:*

$$(a+J) \oplus (b+J) = \{z+J \mid z \in a+b+J\}$$
$$(a+J)\gamma(b+J) = a\gamma b + J,$$

for all $a, b \in R$ and $\gamma \in \Gamma$.

Proof. Let R be a Γ-semihyperring with zero and J is a Γ-hyperideal of R. Both these hyperoperations are well defined. It is easy to prove that $(\frac{R}{J}, \oplus)$ is a commutative semihypergroup. Now we only need to prove following:

(i) $(a+J)\alpha((b+J) \oplus (c+J)) = (a\alpha b+J) \oplus (a\alpha c+J)$

(ii) $((a+J) \oplus (b+J))\alpha c+J) = (a\alpha c+J) \oplus (b\alpha c+J)$

(iii) $(a+J)(\alpha+\beta)(b+J) = (a\alpha b+J) \oplus (a\beta b+J)$

(iv) $(a+J)\alpha(b\beta c+J) = (a\alpha b+J)\beta(c+J)$.

To prove (1) let $X = (a+J)\alpha((b+J) \oplus (c+J))$ and $Y = (a\alpha b+J) \oplus (a\alpha c+J)$, to prove that $X = Y$. Let $x+J \in X$, then there exists $t+J \in ((b+J) \oplus (c+J))$ such that

$$\begin{aligned} x+J &= (a+J)\alpha(t+J) \\ &= (a\alpha t+J)\, for\ some\ \ t \in b+J+c+J. \end{aligned}$$

Then $a\alpha t \in a\alpha b + a\alpha J + a\alpha c + a\alpha J \subseteq a\alpha b + J + a\alpha c + J$, hence $x+J \subseteq (a\alpha b+J) \oplus (a+\alpha c+J) = Y$. Therefore, $X \subseteq Y$. Conversely, suppose that $x+J \in Y$ then $x \in a\alpha b+J+a\alpha c+J$=$a\alpha b+a\alpha c+J$=$a\alpha(b+c)+J$. Thus

$$\begin{aligned} x+J &\subseteq a\alpha(b+c)+J \\ &= (a+J)\alpha(b+c)+J \\ &= (a+J)\alpha(b+J+c+J) \\ &= (a+J)\alpha((b+J) \oplus (c+J)). \end{aligned}$$

Hence, $x+J \in (a+J)\alpha((b+J) \oplus (c+J)) = X$, therefore $X = Y$.
Similarly (2), (3) and (4) can be proved. □

Theorem 3.19. *Let R and $\frac{R}{J}$ be commutative Γ-semihyperrings with zero and J be a Γ-hyperideal of R. Then T is a k-Γ-hyperideal of $\frac{R}{J}$ if and only if $T = \frac{J'}{J}$, where J' is a k-Γ-hyperideal of R and $J \subseteq J'$.*

Proof. Let R and $\frac{R}{J}$ be commutative Γ-semihyperrings with zero and J is a Γ-hyperideal of R. Define $J' = \{a \in R \mid a + J \subseteq T\}$, it is clear that $T = \frac{J'}{J}$. To prove J' is a k-Γ-hyperideal of R. Let $a, b \in J'$. Then $a + J,\ b + J \subseteq T = \frac{J'}{J}$ hence $(a + J) \oplus (b + J) \subseteq T$ which means that $a + b \subseteq J'$. Let $x \in a + b \subseteq a + b + J = a + J + b + J$ then $x + J \subseteq (a + J) \oplus (b + J) \subseteq T$ and $x \in J'$. Suppose $a \in R$ and $x \in J'$. Then $a + x \in \frac{R}{J}$ and $a + x \in T$. By hypothesis $a\gamma x + J = (a + J)\gamma(x + J) \in T$. Hence, $a\gamma x \subseteq J'$. Similarly, $x\gamma a \subseteq J'$, therefore, J' is a Γ-hyperideal of R. Now to prove that J' is a k-Γ-hyperideal of R. For this take $a \in R$, $x \in J'$ and $(a+x)\cap J' \neq \phi$. Then $((a+J)\oplus(x+J))\cap T \neq \phi$. Since $a+x \subseteq a+x+J = a+J+x+J$ and $z \in (a + x) \cap J$ implies that $z \in a + J + x + J$. Thus $z + J \in (a + J) \oplus (x + J) \cap T$ therefore, $a + J \in T$. Hence, $a \in J'$ and J' is a k-Γ-hyperideal of R. Now to prove that $J \subseteq J'$. Let $j \in J$. Then $j + J = J \in T = \frac{J'}{J}$. Since T is a k-Γ-hyperideal of $\frac{R}{J}$ and $\frac{R}{J}$ has a zero element implies that $T = \frac{J'}{J}$ has the same zero element as that of $\frac{R}{J}$. Thus $j \in J'$ and $J \subseteq J'$.

Converse follows easily. □

4 k-Γ-hyperideal of type-I, Prime Γ-hyperideal, and Maximal Γ-hyperideal

Sen and Adhikari [24] studied the characterizations of a k-ideal in semirings. Our work in this section is inspired by their research work. In this section, we introduce the notions of k-Γ-hyperideal of type-I and k-maximal-Γ-hyperideal in Γ-semihyperrings. We explored the relations between k-Γ-hyperideal and k-Γ-hyperideal of type-I.

Definition 4.1. Let R be a Γ-semihyperring. A non-empty subset J of R is called a left(right) k-Γ-hyperideal of type-I of R if

(i) J is a left(right) Γ-hyperideal of R and

(ii) for any $a \in J$, $x \in R$ with $a + x \subseteq J(x + a \subseteq J)$ implies that $x \in J$.

A two-sided k-Γ-hyperideal of type-I or simply a k-Γ-hyperideal of type-I if it is both left and right k-Γ-hyperideal of type-I of R.

Now we discuss some relations between k-Γ-hyperideals and k-Γ-hyperideals of type-I.

Remark 4.2. Every k-Γ-hyperideal of a Γ-semihyperring is a k-Γ-hyperideal of type-I, but the converse need not be true.

Example 4.3. Let $R = \{a, b, c, d\}$. Then R is a commutative semihypergroup and hyperoperation $\oplus$, $\otimes$ defined on R as follows:

$\oplus$	a	b	c	d
a	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{d\}$
c	$\{c\}$	$\{b\}$	$\{a, c\}$	$\{a, c, d\}$
d	$\{d\}$	$\{d\}$	$\{a, c, d\}$	$\{a, d\}$

$\otimes$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{b\}$	$\{a, c\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
d	$\{a\}$	$\{a, d\}$	$\{a\}$	$\{a\}$

Define a mapping $R\times\Gamma\times R \to \wp^*(R)$ by $x\gamma y = x\otimes y$ for every $x, y \in R$, and $\gamma \in \Gamma$. Then R is a Γ-semihyperring. Observe that $J = \{a, c\}$ is a k-Γ-hyperideal of type-I but not a k-Γ-hyperideal. Since $a \oplus x \subseteq J$ and $c \oplus x \subseteq J$ imply that $x \in J$. Hence, J is a k-Γ-hyperideal of type-I but is not a k-Γ-hyperideal of R since $c \oplus d = \{a, c, d\} \cap J \neq \phi$, but $d \notin J$.

In semiring, 2-prime ideals were first introduced by Nanda Kumar in 2010 [19]. Here we introduce 2-prime-Γ-hyperideals (2-prime-Γ-hyperideal of type-I) in Γ-semihyperring.

Definition 4.4. [22] A proper Γ-hyperideal P of a Γ-semihyperring R is said to be a prime-Γ-hyperideal of R if Γ-hyperideal A and B of R satisfying $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Example 4.5. In Example 3.8 $\{a, c, d\}$ is a prime-Γ-hyperideal of R but $\{a\}, \{a, c\}$ & $\{a, d\}$ are not prime-Γ-hyperideals of R. As $\{a, c\}\Gamma\{a, d\} = \{a\}$, but $\{a, c\} \nsubseteq \{a\}$ and $\{a, d\} \nsubseteq \{a\}$ also $\{a, c, d\}\Gamma\{a, c, d\} = \{a\} \subseteq \{a, c\}$ but $\{a, c, d\} \nsubseteq \{a, c\}$.

Definition 4.6. A proper Γ-hyperideal P of R is called a 2-prime (a 2-prime type-I) if for any two k-Γ-hyperideals (k-Γ-hyperideals of type-I) I and J with $I\Gamma J \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$.

Definition 4.7. Let R be a Γ-semihypering and P be a proper Γ-hyperideal of R. Then P is a 1-prime-Γ-hyperideal (1-prime- Γ-hyperideal of type-I) if I is a k-Γ-hyperideal (k-Γ-hyperideal of type-I) and J is a Γ-hyperideal of R such that $I\Gamma J \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$.

Remark 4.8. Every prime-Γ-hyperideal of a Γ-semihyperrring R is a 2-prime-Γ-hyperideal of R but the converse need not be true.

Example 4.9. Let $R = \{a, b, c, d\}$ be a commutative semihypergroup with hyperoperations $\oplus$ and $\otimes$ defined R as follows:

$\oplus$	a	b	c	d
a	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{b\}$	$\{b\}$	$\{a, b, c\}$	$\{a, b, d\}$
c	$\{c\}$	$\{a, b, c\}$	$\{a, c\}$	$\{a, c, d\}$
d	$\{d\}$	$\{a, b, d\}$	$\{a, c, d\}$	$\{a, d\}$

$\otimes$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a, b\}$
d	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, c\}$

Now $\{a\}$, $\{a, b\}$, $\{a, b, c, d\}$ are Γ-hyperideals of R and $\{a\}$ and $\{a, b, c, d\}$ are k-Γ-hypeideals of R. Here, Γ-hyperideal $\{a, b\}$ is a 2-prime-Γ-hyperideal of R but not a prime-Γ-hyperideal of R. Since $\{a, b, c\}\Gamma\{a, b, c\} = \{a\} \subseteq \{a, b\}$, but $\{a, b, c\} \nsubseteq \{a, b\}$.

Theorem 4.10. *Let R be a Γ-semihyperring with zero, J be a k-Γ-hyperideal of R and X be a non-empty subset of R. Then $(J \colon X) = \{a \in R \mid a\gamma x,\ x\gamma a \in J,\ \gamma \in \Gamma, \forall x \in X\}$ is a k-Γ-hyperideal of type-I of R.*

Proof. Let J be a k-Γ-hyperideal of a Γ-semihyperring R. It is easy to check that $(J : X)$ is a Γ-hyperideal of R. To see that $(J : X)$ is a k-Γ-hyperideal of type-I, let $a \in (J : X)$ and $a + z \subseteq (J : X)$ where $z \in R$. Then $a\Gamma x, x\Gamma a \subseteq J$ and $(a + z)\Gamma x \subseteq J$, for all $x \in X$. We obtained $a\Gamma x + z\Gamma x = (a + z)\Gamma x \subseteq J$. As J is a k-Γ-hyperideal of type-I of R and $a\Gamma x \subseteq J$, we have $z\Gamma x \subseteq J$. Similarly, $x\Gamma z \subseteq J$. This implies that $z \in (J : X)$ and hence $(J : X)$ is a k-Γ-hyperideal of R. □

Definition 4.11. Let X be a non-empty subset of a Γ-semihyperring R. Then the intersection of all Γ-hyperideals of R containing X is called a Γ-hyperideal of R generated by X, denoted by

$$< X >= \bigcap\{J | J \text{ is a } \Gamma\text{-hyperideal of } R \text{ and } X \in J\}.$$

The intersection of all k-Γ-hyperideals of R containing J is called a k-closure and denoted by $\bar{J}$.

Definition 4.12. A proper Γ-hyperideal M of R is said to be a maximal Γ-hyperideal (respectively k-maximal Γ-hyperideal) if I is a Γ-hyperideal (resp. k-Γ-hyperideal) in R such that $M \subseteq I$ then $I = R$.

Theorem 4.13. *Let R be a Γ-semihypering with zero and J be a Γ-hyperideal of R. Then $\bar{A} = \{r \in R \mid r + I \subseteq J, \exists\, I \subseteq J\}$ is a k-Γ-hyperideal of type-I of R such that $J \subseteq \bar{A}$.*

Proof. We only prove that $\bar{A}$ is a k-Γ-hyperideal of type-I. Let $a \in \bar{A}$ and $a + r \subseteq \bar{A}$ where $r \in R$. Then there exists I such that $a + I \subseteq J$. Let $x \in a + r \Rightarrow x \in \bar{A}$. Which gives for each $x \in a + r$, there exists I_x such that $x + I_x \subseteq J$. Therefore $x + I_x \subseteq a + r + \bigcup\limits_{x \in a+r} I_x \subseteq J$. Now

$$a + r + \bigcup_{x \in a+r} I_x + I \subseteq J + I \subseteq J \Rightarrow a + \bigcup_{x \in a+r} I_x + r + I \subseteq J.$$

Hence we get $J + r + I \subseteq J \Rightarrow r + I \subseteq J$ and hence $r \in \bar{A}$. Therefore $\bar{A}$ is a k-Γ-hyperideal of type-I of a Γ-semihyperring R. Since $a + J \subseteq J$, for all $a \in J$, we get $J \subseteq \bar{A}$. □

Corollary 4.14. *Let R be a Γ-semihyperring. Then J is a k-Γ-hyperideal of R if and only if $J = \bar{J}$.*

Theorem 4.15. *If P is a k-Γ-hyperideal of a Γ-semihyperring R, then prime-Γ-hyperideal of R and 2-prime-Γ-hyperideal of type-I coincide.*

Proof. We know that every prime-Γ-hyperideal of a Γ-semihyperring R is a 2-prime-Γ-hyperideal of R. Conversely let P is a 2-prime-Γ-hyperideal of type-I such that $X\Gamma Y \subseteq P$ then $Y \in (P : X)_r$. Now by Theorem 4.10 $(P : X)$ is a k-Γ-hyperideal of type-I of R, then $\bar{Y} \subseteq (P : X)_r$. Hence $X\Gamma\bar{Y} \subseteq P$ and using this we get $X \subseteq (P : \bar{Y})_r$. We also know that $(P : \bar{Y})_r$ is a k-Γ-hyperideal of type-I of R. Hence $\bar{X} \in (P : \bar{Y})_r$, then $\bar{X}\Gamma\bar{Y} \subseteq P$. Since $\bar{X}$ and $\bar{Y}$ are k-Γ-hyperideals of type-I of R, we get $\bar{X} \subseteq P$ or $\bar{Y} \subseteq P$. Then, $X \subseteq P$ or $Y \subseteq P$, hence P is prime-Γ-hyperideal of R. □

Theorem 4.16. *Any proper Γ-hyperideal of R is a subset of maximal Γ-hyperideal of R.*

Lemma 4.17. *Let R be a Γ-semihyperring with unity $(\neq 0)$. Then R has at least one k-maximal-Γ-hyperideal.*

Definition 4.18. A Γ-hyperideal J of a Γ-semihyperring R is said to be an irreducible (strongly irreducible) Γ-hyperideal of R if for any Γ-hyperideals X and Y of R, if $X \cap Y = J(X \cap Y \subseteq J)$, then $X = J(X \subseteq J)$ or $Y = J(Y \subseteq J)$.

Theorem 4.19. *In a Γ-semihyperring R, following statements hold*

(i) Every maximal-Γ-hyperideal is an irreducible Γ-hyperideal.

(ii) Every prime- Γ-hyperideal is strongly irreducible Γ-hyperideal.

(iii) Every strongly irreducible Γ-hyperideal is an irreducible Γ-hyperideal.

(iv) Every prime-Γ-hyperideal is an irreducible Γ-hyperideal.

(v) Every m-k-Γ-hyperideal is a maximal Γ-hyperideal.

(vi) Every maximal m-k-Γ-hyperideal is an irreducible Γ-hyperideal.

Proof. (i) Let M be a maximal-Γ-hyperideal of R. Suppose M is not irreducible Γ-hyperideal of R and $M = I \cap J$. This implies that $M \neq I$ and $M \neq J$, then $M \subset I \subset R$ or $M \subset J \subset R$. Which is a contradiction as M is a maximal-Γ-hyperideal of R. Hence, M is an irreducible Γ-hyperideal of R.

(ii) Let P be a prime-Γ-hyperideal of R. Suppose A and B are Γ-hyperideals of R such that $A \cap B \subseteq P$. By definition of prime-Γ-hyperideal implies that $A \subseteq P$ or $B \subseteq P$. Hence P is strongly irreducible Γ -hyperirideal of R.

(iii) Suppose P is a strongly irreducible Γ-hyperideal of R. Let A and B are Γ-hyperideals of R such that $A \cap B = P$ then $A \cap B \subseteq P$. This implies that $A \subseteq P$ or $B \subseteq P$. Hence, $A = P$ or $B = P$. therefore, P is irreducible Γ-hyperideal of R.

(iv) This holds from (2) and (3).

(v) Let M be a m-k-Γ-hyperideal of R. Suppose that N is a Γ-hyperideal of R, such that $M \subseteq N \subseteq R$. Let $x \in N$ and $a \in M$ then $x\Gamma a \subseteq N$, since N is a Γ-hyperideal. Similarly $x\Gamma a \subseteq M$, since M is also Γ-hyperideal. By definition of m-k-Γ-hyperideal, $x \in M$. Therefore $N \subseteq M, \ \forall x \in N$ and hence $M = N$.

(vi) This holds by using (1) and (5).

□

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