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MORE CHARACTERIZATIONS OF S-MULTIPLICATION MODULES

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Abstract Recently, Anderson et al. in ([3], Comm. Alg. (2020)) introduced a generalization of multiplication modules called S-multiplication modules. In this article, the properties of S-prime submodules of S-multiplication modules are studied. Furthermore, we show that the concepts of simple modules, multiplication modules, and S-multiplication modules are equivalent in a torsion-free divisible module over an integral domain. Finally, we construct ring of matrices R' whose entries in a module M and demonstrate that M is S-multiplication R-module is equivalent to the existence of a specific ideal A of rings R' is S-multiplication ideal.

1 Introduction

Throughout the article, we shall assume, unless otherwise stated, that all rings R are regarded to be associative rings with identity and all modules are assumed to be nonzero unital right R-modules. A submodule Y of R-module X is denoted by $Y \leq_R X$ and an ideal I of R is denoted by $I \leq R$.

Let X be a right R-module. The annihilator of X, denoted by $Ann_R(X)$, is

$$Ann_R(X) = \{r \in R \mid Xr = 0\}. \tag{1.1}$$

If $Ann_R(X) = 0$, then X is called a *faithful module*. For any submodule Y of X, $Ann_R(X/Y)$ will be denoted by $(Y :_R X)$. Therefore $(Y :_R X) = \{r \in R \mid Xr \subseteq Y\}$ and may be written as $(Y : X)$ if ring R is understood.

For right R-modules X and Y, $Hom_R(X,Y)$ denotes the set of all R-homomorphisms from X to Y and $\text{End}_R(X)$ denotes the set of R-endomorphism of X. We refer the reader to [([3], [4], [5], [15])] for more basic concepts and other notations.

In 1981 ([8]), A. Barnard introduced the concepts of multiplication modules. A right R-module X is called *a multiplication module*, if for any submodule Y of X there exists an ideal I of R such that

$$Y = XI. \tag{1.2}$$

Moreover, He showed that distributive modules are characterized as modules for which every finitely generated submodule is a multiplication module. In 1988, ([9]) Patrick F. Smith showed that a right R-module X is a multiplication module if and only if $Y = X(Y :_R X)$ for all submodule Y of X. They also provided characterizations of multiplication modules over Noetherian rings and multiplication modules over Artinian rings. Currently, there is a significant amount of literature dedicated to the investigation of multiplication modules. (See [10], [1], [2], [14], [13]).

Recently, Anderson et al. in ([3], Comm. Alg. (2020)) introduced a generalization of multiplication modules called S-multiplication modules, by using a multiplicatively closed subset S of a ring R. A nonempty subset S of a ring R is said to be a *multiplicatively closed subset* (briefly, m.c.s.) of a ring R ([7]), if it contains multiplicative identity and closed under multiplication.

A right R-module X is said to be *S-multiplication module*, if for each submodule Y of X there exist $s \in S$ and an ideal I of R such that

$$Ys \subseteq XI \subseteq Y. \tag{1.3}$$

In this article, we shall examine the S-multiplication module and continue the work of Anderson et al. ([3]). We give some characterizations of S-multiplication modules (Proposition 2.3, Theorem 2.19) and show that the concept of S- multiplication module, simple module, and multiplication module are equivalent in a torsion-free divisible module over an integral domain (Theorem 2.13). Moreover, we study S-prime submodules of S-multiplication modules (Theorem 2.20, Proposition 2.24, and Corollary 2.25). Finally, we provide a characterization of a finite (external) sum of modules $\{M_i\}_{i=1}^{k}$ such that M_j is faithful for some $j \in \{1, \ldots, k\}$ (Theorem 2.28).

2 S-multiplication modules

Let S be a m.c.s. of a ring R. In this section, we provide an example and some properties of S-multiplication modules.

Definition 2.1. ([3], Anderson, 2020) Suppose that X is a right R-module and S is a m.c.s. of a ring R. Then X is called an *S-multiplication module*, if for each submodule Y of X there exist $s \in S$ and an ideal I of R such that $Ys \subseteq XI \subseteq Y$.

An ideal A of R is an *S-multiplication ideal*, if for every ideal B of A there exist $s \in S$ and ideal I of R such that $Bs \subseteq AI \subseteq B$.

From now, we assume that $0 \notin S$. One can see that for every R-module X if $S \cap Ann_R(X) \neq \emptyset$ then X is trivially an S-multiplication module. Moreover, by the definition, we see that every multiplication module is an S-multiplication module, but the converse is not true in general. As shown the following example.

Example 2.2. Consider $X = \mathbb{Z}_2 \times \mathbb{Z}_2$ as a right $\mathbb{Z}$-module. Let $S = \{2^n \mid n \in \mathbb{N}_0\}$, where $\mathbb{N}_0$ is the set of all non-negative integers. Then

(i) X is an $S-$multiplication module.

(ii) X is not a multiplication module.

Proof. (i) Let $N \leq_R X$. Choose $2 \in S$. Since $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$ and $0 = N2 = X(2\mathbb{Z}) \subseteq N$, X is an S-multiplication module.
(ii) One can show that submodule $\mathbb{Z}_2 \times 0$ of X can not written as the form XA for all ideal A of $\mathbb{Z}$. Suppose to the contrary that $\mathbb{Z}_2 \times 0 = XA$. Since A is an ideal of $\mathbb{Z}$, A can be written of the form $n\mathbb{Z}$ for some $n \in \mathbb{N}_0$.
Case 1. n is an even number. Let $(m_1, m_2) \in X$ and $r \in n\mathbb{Z}$. Since n is an even number, $(m_1, m_2)r = (m_1 r, m_2 r) = (0, 0)$. Then $XA = 0$.
Case 2. n is a odd number. Let $(m_1, m_2) \in X$. Since n is a odd number, $(m_1, m_2) = (m_1 n, m_2 n) = (m_1, m_2)n \in XA$. Then $X = XA$.
From Case 1. and Case 2., this contradict to the fact that $\mathbb{Z}_2 \times 0 \neq XA$. □

Proposition 2.3. *Let M be a right R-module. Then The following are equivalences:*

(i) M is an S-multiplication module.

(ii) For any submodule H of M there exists $s \in S$ such that $Hs \subseteq M(H : M) \subseteq H$.

(iii) For any submodules H and L of M such that $(H : M) = (L : M)$ there exist $s, t \in S$ such that $Hs \subseteq L$ and $Lt \subseteq H$.

Proof. (i) $\Longleftrightarrow$ (ii) It is easy.
(ii) $\Longrightarrow$ (iii) Let H and L be submodules of M such that $(H : M) = (L : M)$. By assumption, there exist $s, t \in S$ which $Hs \subseteq M(H :_R M) \subseteq H$ and $Lt \subseteq M(L : M) \subseteq L$. Therefore $Hs \subseteq M(H : M) = M(L : M) \subseteq L$. Similarly, $Lt \subseteq M(L : M) = M(H : M) \subseteq H$.

(iii) $\Longrightarrow$ (ii) Let $H \leq_R M$ and $L := M(H:M)$. Since $L \subseteq H$, $(L:M) \subseteq (H:M)$. Conversely, for $r \in (H:M)$, we get $Mr \subseteq M(H:M) = L$ and hence $r \in (L:M)$. Therefore $(H:M) = (L:M)$. By assumption, there exist $s, t \in S$ which $Hs \subseteq L$ and $Lt \subseteq H$. Since $M(H:M) \subseteq H$ and thus $Hs \subseteq L = M(H:M) \subseteq H$. □

Proposition 2.4. *Suppose that X is an R-module and R' a subring of R. If S is m.c.s. of R' and X is an S-multiplication R'-module, then X is an S-multiplication R-module.*

Proof. Let N be a submodule of R-module X. Hence N is a submodule of $X_{R'}$. By assumption, $Ns \subseteq XI_{R'} \subseteq N$ for some ideal $I_{R'} \leq R^{'}$ and element $s \in S$ since $X_{R'}$ is an S-multiplication module. Consider $I = RI_{R'}$. Observe that I is an ideal of R such that $Ns \subseteq XI_{R'} = (XR)I_{R'} = X(RI_{R'}) \subseteq N$. Therefore, X is an S-multiplication R-module. □

Recall from ([5]), a submodule $Y \leq_R X$ is called a *pure submodule* of X if for each $A \leq R$, we have $YA = Y \cap XA$.

Proposition 2.5. *Every pure submodule of an S-multiplication module is an S-multiplication module.*

Proof. Let X be an S-multiplication module and N a pure submodule of X. Let $L \leq_R N$. By assumption, $Lt \subseteq XI \subseteq L$ for some $t \in S$ and $I \leq R$ since X is an S-multiplication module. But N is a pure submodule of X, $NI = N \cap XI \subseteq N \cap L = L$. Let $lt \in Lt$. Since $L \leq_R N$, $lt \in Nt \subseteq N$. So $Lt \subseteq N \cap (Lt)$ and thus $N \cap (Lt) = Lt$. Then $Lt = N \cap (Lt) \subseteq N \cap (XI) = NI \subseteq L$. Therefore, N is an S-multiplication module. □

Corollary 2.6. *Every direct summand of an S-multiplication module is S-multiplication.*

Proposition 2.7. *Let X be an S-multiplication module and π a canonical projection map from R to $R/Ann_R(X)$. Suppose that E is a right ideal of R such that $Ann_R(X) \subseteq E$ and $\pi(E)$ is an essential right ideal of $\pi(R)$. Then for any submodule H of X such that $XE \cap H = 0$, we have $Hs = 0$ for some $s \in S$.*

Proof. Suppose that X is an S-multiplication module. Let $A := Ann_R(X)$ and E a right ideal of R such that $A \subseteq E$. Let H be a submodule of X such that $XE \cap H = 0$. Thus $Hs \subseteq XF \subseteq H$ for some ideal F of R and $s \in S$. Then $X(E \cap F) \subseteq XE \cap XF \subseteq XE \cap H = 0$ and we have $E \cap F \subseteq A$. Thus $\pi(E \cap F) = \bar{0}$. Let $x \in \pi(E) \cap \pi(F)$. Then $x = \pi(e) = \pi(f)$ for some $e \in E$ and $f \in F$. Since $x = \pi(e) = e + A$ and $x = \pi(f) = f + A$, $f - e \in A$. But $A \subseteq E$, we have $f = f - e + e \in E$. Thus $x = \pi(f) \in \pi(E \cap F)$. We have $\pi(E) \cap \pi(F) \subseteq \pi(E \cap F) = \bar{0}$. So $\pi(E) \cap \pi(F) = \bar{0}$. Since $\pi(E)$ is an essential ideal in $\pi(R)$, $\pi(F) = \bar{0}$. It follows that $F \subseteq A = Ann_R(X)$. So $XF = 0$. But $Hs \subseteq XF = 0$, $Hs = 0$. □

Lemma 2.8. *Every R-module Z is faithful $\mathrm{End}_R(Z)$-module.*

Proof. Assume that Z is a right R-module. Define $\mathrm{End}_R(Z) \times Z \to Z$ by $\varphi \cdot z = \varphi(z)$ for all $z \in Z$ and $\varphi \in \mathrm{End}_R(Z)$. Hence Z is a left $\mathrm{End}_R(Z)$-module. Let $\varphi \in Ann_{\mathrm{End}_R(Z)}(Z)$. Then $\varphi Z = 0$ and we have $\varphi = 0$. So $Ann_{\mathrm{End}_R(Z)}(Z) = 0$. Hence Z is a left faithful $\mathrm{End}_R(Z)$-module. □

Proposition 2.9. *Every right S-multiplication R-module is a left faithful S-multiplication* $\mathrm{End}_R(X)$*-module.*

Proof. Let $\mathcal{R}'$ denote the $\mathrm{End}_R(X)$ and N be a submodule of an $\mathcal{R}'$-module X. For $r \in R$, we define $\varphi_r : X \to X$ by $\varphi_r(m) = mr$ for all $m \in X$. Obviously, we got $\varphi_r \in \mathcal{R}'$. Let $r \in R$ and $n \in N$. Since $nr = \varphi_r(n) = \varphi_r \cdot n \in N$, N is a submodule of R-module X. By assumption, there exist $I \leq R$ and element $s \in S$ such that $Ns \subseteq XI \subseteq N$. Let $\Phi_I = \{\varphi_r \in \mathcal{R}' \mid r \in I\}$. Hence $\mathcal{R}'\Phi_I\mathcal{R}' \leq \mathcal{R}'$ and we have $\mathcal{R}'\Phi_I\mathcal{R}' \cdot X = (\mathcal{R}'\Phi_I\mathcal{R}')X = \mathcal{R}'\Phi_I(\mathcal{R}'(X)) = (\mathcal{R}'\Phi_I)(X) = \mathcal{R}'(\Phi_I(X)) =$

$\mathcal{R}'(XI) = (\mathcal{R}'(X))I = XI$. So $Ns \subseteq XI = \mathcal{R}'\Phi_I\mathcal{R}' \cdot X \subseteq N$. Then X is an S-multiplication module over $\mathcal{R}'$ and by Lemma 2.8, we have X is a faithful S-multiplication $\text{End}_R(X)$-module. □

Proposition 2.10. *An R-module M is an S-multiplication module if and only if for every submodule $N \leq_R M$, $Ns \subseteq \sum_{a \in I} Im\varphi_a \subseteq N$ for some element $s \in S$ and ideal $I \leq R$.*

Proof. Let N be submodule of an S-multiplication R-module M. Then there exist $s \in S$ and an ideal $I \leq R$ such that $Ns \subseteq MI \subseteq R$. For each $a \in I$, define the map $\varphi_a : M \to M$ by $m \mapsto ma$. From the definition of S-multiplication module, it is enough to show that $\sum_{a \in I} \text{Im}\varphi_a = MI$.
($\subseteq$) Let $x \in \sum_{a \in I} \text{Im}\varphi_a$. Then x can be written as a finite sum of elements of $\text{Im}\varphi_a$, where $a \in I$. That is, there exist $k \in \mathbb{N}_0$ and finite set $\{a_1, \ldots, a_k\}$ of I such that $x = \sum_{i=1}^{k} x_i$, where each $x_i \in \text{Im}\varphi_{a_i}$ for all $i \in \{1, \ldots, k\}$. Let $i \in \{1, \ldots, k\}$. Since $x_i \in \text{Im}\varphi_{a_i}$, there exist $m_i \in M$ such that $x_i = \varphi_{a_i}(m_i) = m_i a_i \in MI$ and hence $x = \sum_{i=1}^{k} x_i \in MI$.
($\supseteq$) The converse is similar. □

From now on until corollary 2.15, R is assume to be an integral domain.

Proposition 2.11. *Let R be a ring and J be a non-zero ideal of R such that J_R an S-multiplication ideal. For any $\varphi, \psi \in \text{End}_R(J)$ if $\varphi \circ \psi = 0$, then $\varphi = 0$ or $\psi = 0$.*

Proof. Let $\varphi, \psi \in \text{End}_R(J)$ such that $\varphi \circ \psi = 0$. Since J_R is an S-multiplication ideal, $\varphi(J)r_1 \subseteq JB \subseteq \varphi(J)$ and $\psi(J)r_2 \subseteq JC \subseteq \psi(J)$ for some $B, C \leq R$ and $r_1, r_2 \in S$. Then $J(BC) = (JB)C \subseteq \varphi(J)C = \varphi(JC) \subseteq \varphi(\psi(J)) = (\varphi \circ \psi)(J) = 0$ and thus $J(BC) = 0$. But $J \neq 0$ and R is an integral domain. Therefore, either $B = 0$ or $C = 0$.
If $B = 0$. Since $\varphi(J)r_1 \subseteq JB = 0$, $\varphi(J)r_1 = 0$. But $r_1 \neq 0$, then $\varphi(J) = 0$ so $\varphi = 0$.
Otherwise, if $C = 0$. Since $\psi(J)r_2 \subseteq JC = 0$, $\psi(J)r_2 = 0$. But $r_2 \neq 0$, then $\psi(J) = 0$ so $\psi = 0$. □

Corollary 2.12. *Let R a ring. For any $\varphi, \psi \in \text{End}_R(R_R)$ if $\varphi \circ \psi = 0$, then $\varphi = 0$ or $\psi = 0$.*

Proof. By Proposition 2.11 and R_R is an S-multiplication module. □

Recall from ([15]), a right module X is said to be *divisible*, if for every nonzero divisor $r \in R$ and every $m \in X$, we have $m = nr$ for some element $n \in X$.

Moreover, ([16]) Let X be R-modules. The set of torsion element is $T(X) := \{m \in X : mr = 0 \text{ for some } 0 \neq r \in R\}$. Recall that an R-module X is said to be *torsion-free* if the torsion subset $T(X) = 0$, and X is called a *torsion module* if $X = T(X)$.

Theorem 2.13. *Let R be an integral domain and X an torsion-free divisible module. Then the following are equivalent:*

(i) X is an S-multiplication module.

(ii) X is a simple module.

(iii) X is a multiplication module.

Proof. (i) $\Longrightarrow$ (ii) Suppose that Y is a nonzero submodule of R-module X. Then $Ys \subseteq XI \subseteq Y$ for some $I \leq R$ and element $s \in S$ since X is S-multiplication. Observe that $I \neq 0$. Otherwise, we have $Ys \subseteq XI = 0$. Let $y \in Y$, $ys \in Ys = 0$. Since $s \neq 0$ and $ys = 0$, this implies that $y \in T(X) = 0$. Hence $Y = 0$ contradiction. Let $m \in X$ and $a \in I$. Since X is divisible, therefore there is $m' \in X$ so that $m = m'a$. Hence $m = m'a \in XI \subseteq Y$ and hence

$m \in Y$. We have $X \subseteq Y$. So X is a simple module.
(ii) $\Longrightarrow$ (iii) Obvious.
(iii) $\Longrightarrow$ (i) Obvious. □

Proposition 2.14. *Let X be a faithful module over R. If X is an S-multiplication module, each $Y \leq_R X$ with $S \cap Ann_R(Y) = \varnothing$ is faithful.*

Proof. Let X be a faithful module and Y be a nonzero submodule of X with $S \cap Ann_R(Y) = \varnothing$. Since X is S-multiplication module, $Ys \subseteq XJ \subseteq Y$ for some element $s \in S$ and ideal $J \leq R$. But $S \cap Ann_R(Y) = \varnothing$, we have $Ys \neq 0$. Then $XJ \neq 0$, that is $J \neq 0$. Let $v \in R$ so that $Yv = 0$. Then $XJv \subseteq Yv = 0$. Since X is a faithful module, $Jv = 0$. But $J \neq 0$. Therefore, we have $v = 0$. Hence Y is a faithful module. □

One can view multiplication modules as $\{1\}$-multiplication modules and for each submodules N of X, $\{1\} \cap Ann_R(N) = \varnothing$. Then we got the following results.

Corollary 2.15 ([10]). *Every nonzero submodule of faithful multiplication module is faithful.*

In the next two propositions, R is considered to be a commutative ring.

Proposition 2.16. *Let R be a ring and X a right R-module. If X is a faithful S-multiplication torsion module then for any $m \in X$, $ms = 0$ for some $s \in S$.*

Proof. Let X be a faithful S-multiplication torsion module and $m \in X$. If $m = 0$, we can choose $1 \in S$ and $m1 = 0$. Next, suppose that $0 \neq m \in X$. Since X is torsion, there exists a nonzero divisor $c \in R$ such that $mc = 0$. Since X is S-multiplication module, $mRs \subseteq XI \subseteq mR$ for some an ideal $I \leq R$ and $s \in S$. Then $XIc \subseteq mRc = 0$ and hence $XIc = 0$. Therefore $Ic = 0$. But c is a nonzero divisor, $I = 0$. So $ms \in mRs \subseteq XI = 0$. Hence $ms = 0$. □

Proposition 2.17. *Let X be a torsion-free R-module. If X is an $(R\backslash P)$-multiplication module, then $X \neq XP$ for all prime ideal P of R.*

Proof. Let X be an $(R\backslash P)$-multiplication module. Assume that $X = XP$. For $0 \neq m \in X$, $mRs \subseteq XI \subseteq mR$ for some $s \notin P$ and some $I \leq R$. Then $(mR)s \subseteq XI = (XP)I \subseteq XIP \subseteq mRP \subseteq mP$ and thus $ms = mp$ for some $p \in P$. So $m(s-p) = 0$. But $s - p = 0$ since X is a torsion-free module. Therefore, we have $s = p \in P$. It is contradiction since $s \in (R\backslash P)$. □

Recall from ([11]), a right R-module X is said to be *S-Artinian module*, if each family of descending chain of submodules $\{N_i\}_{i \in I}$ of X there exist $s \in S$ and a positive integer k such that $N_k s \subseteq N_n$ for each $n \geq k$. Note that the concept of S-Artinian module is a generalization of Artinian module.

Theorem 2.18. *Let M be an S-multiplication right R-module. If $R/Ann_R(M)$ is an S-Artinian R-module then M is an S-Artinian module.*

Proof. Let $\mathcal{J} := Ann_R(M)$ and $Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \ldots \supseteq Y_n \supseteq ...$ be a descending chain of submodules of a right R-module M. Since $(Y_1 : M) \supseteq (Y_2 : M) \supseteq (Y_3 : M) \supseteq \ldots \supseteq (Y_n : M) \supseteq ...$ is a descending chain of ideal of R containing $Ann(M)$, $(Y_1 : M)/\mathcal{J} \supseteq (Y_2 : M)/\mathcal{J} \supseteq (Y_3 : M)/\mathcal{J} \supseteq \ldots \supseteq (Y_n : M)/\mathcal{J} \supseteq ...$ is a descending chain of ideal of $R/\mathcal{J}$. But $R/\mathcal{J}$ is an S-Artinian module, there exist $s \in S$ and a positive integer k such that $(Y_k : M)s/\mathcal{J} \subseteq (Y_n : M)/\mathcal{J}$ for all $n \geq k$, and hence $(Y_k : M)s \subseteq (Y_n : M)$. Since M is an S-multiplication module, there exists $s' \in S$ such that $Y_k s' \subseteq M(Y_k : M)$. So, $Y_k s' s \subseteq M(Y_k : M)s \subseteq M(Y_n : M) \subseteq Y_n$. So $Y_k s' s \subseteq Y_n$ for all $n \geq k$. □

Theorem 2.19. *Let X_R be a right module, $R' = \{\begin{bmatrix} a & m \\ 0 & a \end{bmatrix} | m \in X$ and $a \in R\}$ and $A = \{\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} | m \in X\}$. Then the following are equivalent:*

(i) R' is the ring under matrix addition and matrix mutiplication.

(ii) A is the ideal of R'.

(iii) X_R is an S-multiplication module if and only if $A_{R'}$ is an S-multiplication right ideal of $R^{'}$.

Proof. (i) Clear.
(ii) Clear.
(iii) ($\Longrightarrow$) Suppose $B \leq A$. Define $N := \{n \in X| \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} \in B\}$. It implies $N \leq_R X$. Therefore, $Ns \subseteq XI \subseteq N$ for some ideal $I \leq R$ and element $s \in S$ since X is right S-multiplication module. Define $C = \{\begin{bmatrix} a & x \\ 0 & a \end{bmatrix} |a \in I \text{ and } x \in X\}$. Then $C \leq R^{'}$. Let $m \in N$. So $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} s \in Bs$.
Since $ms \in Ns \subseteq XI$, $ms \in XI$ and thus $ms = \sum_{l=1}^{k} m_l i_l$ for $k \in \mathbb{N}$ and $m_l \in X, i_l \in I$ for all $l(1 \leq l \leq k)$. But $\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} s = \begin{bmatrix} 0 & ms \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sum_{l=1}^{k} m_l i_l \\ 0 & 0 \end{bmatrix} = \sum_{l=1}^{k} \begin{bmatrix} 0 & m_l i_l \\ 0 & 0 \end{bmatrix} = \sum_{l=1}^{k} (\begin{bmatrix} 0 & m_l \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i_l & m_l \\ 0 & i_l \end{bmatrix}) \in AC$. This implies that $Bs \subseteq AC$. Let $\sum_{l=1}^{k} \begin{bmatrix} 0 & m_l \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i_l & m_l \\ 0 & i_l \end{bmatrix} \in AC$. Since $\sum_{l=1}^{k} \begin{bmatrix} 0 & m_l \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i_l & m_l \\ 0 & i_l \end{bmatrix} = \sum_{l=1}^{k} \begin{bmatrix} 0 & m_l i_l \\ 0 & 0 \end{bmatrix}$ and $m_l i_l \in XI \subseteq N$ for all $l(1 \leq l \leq k)$ and hence $\sum_{l=1}^{k} m_l i_l \in N$. Therefore $\sum_{l=1}^{k} \begin{bmatrix} 0 & m_l \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i_l & m_l \\ 0 & i_l \end{bmatrix} \in B$. So $Bs \subseteq AC \subseteq B$ and hence A is an S-multiplication ideal of $R^{'}$.

($\Longleftarrow$) Suppose that A is an S-multiplication right ideal of $R^{'}$. Let $N \leq_R X$. Define $B = \{\begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} |n \in N\}$. So $B \leq R^{'}$ and $B \subseteq A$. Since $A_{R'}$ is an S-multiplication right ideal, we have $Bs \subseteq AC \subseteq B$ for some ideal $C \leq R^{'}$ and element $s \in S$. Let $I = \{a \in R :$ the matrix $\begin{bmatrix} a & m \\ 0 & a \end{bmatrix} \in C$ for some $m \in X\}$. Since $C \leq R^{'}$, $I \leq R$. Choose $n \in N$. Therefore $\begin{bmatrix} 0 & ns \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix} s \in Bs \subseteq AC$. Then $\begin{bmatrix} 0 & ns \\ 0 & 0 \end{bmatrix} \in AC$ and thus $\begin{bmatrix} 0 & ns \\ 0 & 0 \end{bmatrix} = \sum_{l=1}^{k} \begin{bmatrix} 0 & m_l \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_l & m_l^{'} \\ 0 & r_l \end{bmatrix} = \sum_{l=1}^{k} \begin{bmatrix} 0 & m_l r_l \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sum_{l=1}^{k} m_l r_l \\ 0 & 0 \end{bmatrix}$. So $ns = \sum_{l=1}^{k} m_l r_l \in XI$.
Hence $Ns \subseteq XI$.............($*$).
Let $x \in XI$. Then $x = \sum_{l=1}^{k} m_l i_l$ for $k \in \mathbb{N}$ and $m_l \in X, i_l \in I$ for all $l(1 \leq l \leq k)$. For $l \in \{1, 2, ..., n\}$, There is $m_l^{'}$ for each i_l such that $\begin{bmatrix} i_l & m_l^{'} \\ 0 & i_l \end{bmatrix} \in C$. Since $\begin{bmatrix} 0 & m_l \\ 0 & 0 \end{bmatrix} \in A$, $\sum_{l=1}^{k} \begin{bmatrix} 0 & m_l \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i_l & m_l^{'} \\ 0 & i_l \end{bmatrix} = \sum_{l=1}^{k} \begin{bmatrix} 0 & m_l i_l \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sum_{l=1}^{k} m_l i_l \\ 0 & 0 \end{bmatrix} \in AC \subseteq B$. Then $\sum_{l=1}^{k} m_l i_l \in N$ and hence $XI \subseteq N$..............($**$).

From ($*$) and ($**$), we have $Ns \subseteq XI \subseteq N$, as desired. □

Recall from ([12]), a submodule $Q \leq_R X$ with $(Q :_R X) \cap S = \varnothing$ is said to be *S-prime submodule* of X, if there is an $s \in S$ such that $ma \in Q$ implies $as \in (Q :_R X)$ or $ms \in Q$ for each $a \in R$ and $m \in X$. Also $I \leq R$ is said to be *S-prime ideal*, if it is an S-prime submodule of the submodule R_R.

Proposition 2.20. *Let X be an S-multiplication R-module and Q be an S-prime submodule of X. If K and L are submodules of X such that $K \cap L \subseteq Q$, then $Ls^{'} \subseteq Q$ or $Ks^{'} \subseteq Q$ for some $s^{'} \in S$.*

Proof. Suppose Q is an S-prime submodule of X. Let K and $L \leq_R X$ such that $K \cap L \subseteq Q$. Since Q is an S-prime submodule of X, there exists $s \in S$ such that for each $r \in R$ and $m \in X$, $mr \in Q$ implies $rs \in (Q :_R X)$ or $ms \in Q$. Assume that $Ls \nsubseteq Q$. There exists $m \in L$ such that $ms \in Ls$ but $ms \notin Q$. Let $a \in (K :_R X)$. Since $ma \in L(K :_R X) \subseteq L \cap K \subseteq Q$ and Q is an S-prime submodule of X, $as \in (Q :_R X)$ or $ms \in Q$. But $ms \notin Q$, we have $as \in (Q :_R X)$ so thus $(K : X)s \subseteq (Q : X)$. By assumption, there exists $t \in S$ such that $Kt \subseteq X(K :_R X)$ because X is an S-multiplication module. So

$$K(ts) = (Kt)s \subseteq X(K :_R X)s \subseteq X(Q :_R X) \subseteq Q.$$

Hence $Ks^{'} \subseteq Q$ where $s^{'} = ts$. □

Theorem 2.21. *Let X be an S-multiplication module and Q a submodule of X. Then Q is an S-prime submodule of X if and only if $(Q :_R X)$ is an S-prime ideal of R.*

Proof. By ([3], Proposition 4). □

Proposition 2.22. *Let S be multiplicatively closed subset of R and Q an ideal of R such that $Q \cap S = \varnothing$. Then Q is an S-prime ideal of R if and only if there exists $s \in S$ such that for each ideal I, J of R with $IJ \subseteq Q$, so either $Is \subseteq Q$ or $Js \subseteq Q$.*

Proof. By ([12], Corollary 2.6). □

Now, we will provided proposition 2.22 on the version of submodules of M. First, we will start with the definition of product of two submodules.

Definition 2.23 ([1]). Let X be an $R-$module and K, L submodules of X. The product of K and L is defined as $KL = X(K :_R X)(L :_R X)$.

If R is a commutative ring, one can show that $KL = LK$. In the next two proposition, R is considered to be commutative.

Proposition 2.24. *Let Z be an S-multiplication module over R and Q an S-prime submodule of Z. If K,$L \leq Z$ such that $KL \subseteq Q$, then $Ks \subseteq Q$ or $Ls \subseteq Q$ for some element $s \in S$.*

Proof. Suppose that $KL \subseteq Q$. By assumption, $Kt \subseteq Z(K : Z)$ and $Lu \subseteq Z(L : Z)$ for some $t, u \in S$. Since Q is an S-prime submodule of Z and by Theorem 2.21, $(Q : Z)$ is an S-prime ideal of R. Since $KL = Z(K : Z)(L : Z) \subseteq Q$, $(K : Z)(L : Z) \subseteq (Q : Z)$. By ([12], Corollary 2.6), $(K : Z)s^{'} \subseteq (Q : Z)$ or $(L : Z)s^{'} \subseteq (Q : Z)$ for some $s' \in S$
Case 1. If $(K : Z)s^{'} \subseteq (Q : Z)$. Since $Kt \subseteq Z(K : Z)$, $Kts^{'} \subseteq Z(K : Z)s^{'} \subseteq Z(Q : Z) \subseteq Q$.
Case 2. If $(L : Z)s^{'} \subseteq (Q : Z)$. Since $Lu \subseteq Z(L : Z)$, $Lus^{'} \subseteq Z(L : Z)s^{'} \subseteq Z(Q : Z) \subseteq Q$.
From Case 1. and Case 2., we can choose $s = s^{'}tu$ and $Ks = Ks^{'}tu = Kts^{'} \subseteq Qu \subseteq Q$ or $Ls = Ls^{'}tu = Lus^{'}t \subseteq Qt \subseteq Q$. □

Corollary 2.25. *Let X be an S-multiplication module and Q be a submodule of X_R satisfying $(Q :_R X) \cap S = \varnothing$. If there is an $u \in S$ such that any submodules L and N of X with $LN \subseteq Q$, so either $Lu \subseteq Q$ or $Nu \subseteq Q$, then Q is an S-prime submodule of X.*

Proof. Let B and C be an ideals of R such that $BC \subseteq (Q : X)$. Since $(XB)(XC) = X(BC) \subseteq X(Q : X) \subseteq Q$. By assumption, there exist $u \in S$ such that $(XB)u \subseteq Q$ or $(XC)u \subseteq Q$.

Case i). $(XB)u \subseteq Q$. Then $Bu \subseteq (Q : X)$.
Case ii). $(XC)u \subseteq Q$. Then $Cu \subseteq (Q : X)$.
By proposition 2.22, we have $(Q :_R X)$ is an S-prime ideal of R. By Theorem 2.21, Q is an S-prime submodule of X. □

Now, we prove that the converse of Corollary 2.25 also hold if S satisfy maximal multiple condition. Recall from ([3]), a m.c.s. S of R is said to satisfy the maximal multiple condition, if there exists an $s \in S$ such that $t \mid s$ for each $t \in S$.

Corollary 2.26. *Let S be a m.c.s. of R satisfy maximal multiple condition and X an S-multiplication module. If Q is an S-prime submodule of X, then there exists $s \in S$ such that for all submodule L and N of X with $LN \subseteq Q$, so either $Ls \subseteq Q$ or $Ns \subseteq Q$.*

Proof. Let $S \subseteq R$ with $t \in S$ such that $s|t$ for all $s \in S$ and Q is an S-prime submodule of X. Suppose L and N are submodules of R-module X with $LN \subseteq Q$. By Proposition 2.24, $Ls^{'} \subseteq Q$ or $Ns' \subseteq Q$ for some $s' \in S$. By maximally element of t, there exist $v \in R$ such that $t = s'v$. Then $Lt = Ls'v \subseteq Qv \subseteq Q$ or $Nt = Ns'v \subseteq Qv \subseteq Q$. □

Recall from ([6]), a right R-module X is called *codomain*, whenever $W_R(X) = 0$, where $W_R(X) = \{c \in R \mid X \xrightarrow{c} X \text{ is not surjective }\}$.

Theorem 2.27. *Suppose X is a codomain. Then X is an S-multiplication module if and only if $S \cap Ann_R(N) \neq \varnothing$ for all nonzero proper submodule N of X.*

Proof. ($\Longrightarrow$) Suppose X is an S-multiplication module. Let $0 \neq N$ be a proper submodule of X. Since X is S-multiplication module, $Ns \subseteq XI \subseteq N$ for some element $s \in S$ and $I \leq R$. We have $I \subseteq W_R(X)$. Since X is a codomain, $W_R(X) = 0$. So $I = 0$ and thus $Ns = 0$. Then $s \in Ann_R(N)$ and hence $s \in S \cap Ann_R(N)$.
($\Longleftarrow$) Clear. □

Theorem 2.28. *Let $\{X_i\}_{i=1}^k$ be a finite set of a right R-modules and $X = \prod_{i=1}^{k} X_i$.*
If X_i is a faithful module for some $i \in \{1, 2, ..., k\}$, then X is an S-multiplication module if and only if it satisfies the following condition:

(i) X_i is an S-multiplication module.

(ii) There exist $s \in S$ and $I \leq R$ so that $X_i s \subseteq X_i I$ with $I \subseteq Ann_R(X_j)$ for all $j \neq i$.

(iii) There exist an elements $t \in S$ with $t \in Ann_R(X_j)$ for all $j \neq i$.

Proof. Let X_i be a faithful module for some i.
($\Longrightarrow$) Assuming that X is S-multiplication module.
(i) Let H be a submodules of X_i. Since $(0 \times \ldots \times H \times \ldots \times 0)$ is a submodule of X, $(0 \times \ldots \times H \times \ldots \times 0)s \subseteq XI \subseteq (0 \times \ldots \times H \times \ldots \times 0)$ for some $I \leq R$ and element $s \in S$. Then $Hs \subseteq (X_i)I \subseteq H$. So X_i is $S-$multiplication module.
(ii) Since $\chi_i := (0 \times ... \times X_i \times ... \times 0)$ is a submodule of X, therefore $\chi_i s \subseteq XI \subseteq \chi_i$ for some element $s \in S$ and $I \leq R$. Then $X_i s \subseteq (X_i)I \subseteq X_i$ and $X_j I = 0$ for all $j \neq i$. Then $I \subseteq Ann_R(X_j)$ for all $j \neq i$.
(iii) Consider $H = \prod_{j \neq i} X_j$. Since H is a submodule of X, it follow that $Ht \subseteq XJ \subseteq H$ for some element $t \in S$ and $J \leq R$. Then $X_i J = 0$. Since X_i is faithful module, $J = 0$ and thus $X_j t \subseteq X_j J = 0$ for all $j \neq i$. So $X_j t = 0$ and hence $t \in Ann_R(X_j)$.
($\Longleftarrow$) Let H be a submodule of X and $\pi_j : X \to X_j$ a projection map from X to X_j for all $j \in \{1, 2, ..., k\}$. Set $H_j = \pi_j(H)$ for all $j \in \{1, 2, ..., k\}$. Since X_i is an S-multiplication, this implies that $H_i r \subseteq X_i A \subseteq H_i$ for some $A \leq R$ and element $r \in S$. By iii) there exists $t \in Ann_R(X_j)$ for all $j \neq i$. Then

$$H_i rt \subseteq X_i At \subseteq X_i A \subseteq H_i.$$

and $X_j At \subseteq X_j t = 0$. So

$$Hrt \subseteq (0 \times ... \times H_i rt \times ... \times 0) \subseteq (0 \times ... \times X_i At \times ... \times 0) = XAt \subseteq (0 \times ... \times H_i \times ... \times 0) \subseteq H.$$

Hence X is an S-multiplication module. □

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