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Study of $*$-prime rings with a pair of derivations

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Abstract The main objective of this paper is to investigate the commutativity of $*$-prime rings with second kind involution $*$, which requires a pair of derivations that satisfy certain differential identities. Lastly, we give few examples to show that the assumptions made for our findings are not superfluous.

1 INTRODUCTION

All through this paper, $\mathfrak{R}$ will be used to describe an associative ring, and $\mathfrak{J}_Z$ is the centre of $\mathfrak{R}$. For any $t_1, t_2 \in \mathfrak{R}$, the notation $[t_1, t_2]$ illustrates the commutator $t_1t_2 - t_2t_1$, and $t_1 \circ t_2$ illustrates the anti-commutator $t_1t_2 + t_2t_1$. $\mathfrak{R}$ is called 2-torsion free if $2t_1 = 0 \implies t_1 = 0$. We use the basic identities $[t_1t_2, t_3] = t_1[t_2, t_3] + [t_1, t_3]t_2$ and $[t_1, t_2t_3] = [t_1, t_2]t_3 + t_2[t_1, t_3]$ for all $t_1, t_2, t_3 \in \mathfrak{R}$ very frequent. Recall that an involution is an order 2 anti-automorphism. A ring $\mathfrak{R}$ is called $*$-prime if $a\mathfrak{R}b = a\mathfrak{R}b^* = (0)$ or $a^*\mathfrak{R}b = a\mathfrak{R}b = (0)$ implies $a = 0$ or $b = 0$. Every prime ring is a $*$-prime ring but converse is not true in general; for instance let $S = \mathfrak{R} \times \mathfrak{R}^0$, where $\mathfrak{R}^0$ is the opposite ring of $\mathfrak{R}$. The mapping $*$ on S as $(t_1, t_2)^* = (t_2, t_1)$. Thus S is a $*$-prime ring but S is not a prime ring. We define " an element t_1 in $\mathfrak{R}$ is said to be hermitian if $t_1^* = t_1$ and skew-hermitian if $t_1^* = -t_1$." Where $\mathfrak{J}_H$ denotes the set of hermitian elements and $\mathfrak{J}_S$ denotes the set of skew-hermitian elements of $\mathfrak{R}$. If $char(\mathfrak{R}) \neq 2$ then every $t_1 \in \mathfrak{R}$ can be uniquely expressed as $2t_1 = h + k$ where $h \in \mathfrak{J}_H$ and $k \in \mathfrak{J}_S$. If $\mathfrak{J}_Z \subseteq \mathfrak{J}_H$, then $*$ is said to be of first kind otherwise, it is called second kind and in this case $\mathfrak{J}_S \cap \mathfrak{J}_Z \neq (0)$. Any $t_1 \in \mathfrak{R}$ is called normal, if its commutes with its image under involution $*$, and $\mathfrak{R}$ is called normal if every elements of $\mathfrak{R}$ is normal. See in[5].

A mapping ψ on $\mathfrak{R}$ is termed as derivation on $\mathfrak{R}$ if $\psi(t_1 + t_2) = \psi(t_1) + \psi(t_2)$ and $\psi(t_1t_2) = \psi(t_1)t_2 + t_1\psi(t_2)$ $\ forall\ \ t_1, t_2 \ \ \in \mathfrak{R}$. Let $b \in \mathfrak{R}$ be a fixed element of $\mathfrak{R}$, then the mapping ψ on $\mathfrak{R}$ defined by $\psi(t_1) = [b, t_1] = bt_1 - t_1b$ for all $t_1 \in \mathfrak{R}$ is called inner derivation induced by b. A map $f : \mathfrak{R} \to \mathfrak{R}$ is called centralizing on $\mathfrak{R}$ if $[f(t_1), t_1] \in \mathfrak{J}_Z$ holds for all $t_1 \in \mathfrak{R}$. In particular, if $[f(t_1), t_1] = 0$ holds for all $t_1 \in \mathfrak{R}$, then it is called commuting. Stimulated by the description of centralizing map, a map f from $\mathfrak{R}$ into itself is called $*$-centralizing if $[f(t_1), t_1^*] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$ and is called $*$-commuting if $[f(t_1), t_1^*] = 0$ for all $t_1 \in \mathfrak{R}$. The narrative of centralising and commuting maps dates back to 1955, when Divinsky proved that if a simple artinian ring has commuting non-trivial automorphisms, then it is commutative. After few years, Posner [14] established that the presence of a nonzero centralizing derivation on a prime ring implies commutativity of rings. The study of centralizing (resp. commuting) derivations and various generalizations of concept of a centralizing (resp. commuting) maps are the main concepts emerging directly from Posner's result, with many applications in various areas. Recently, a number of algebraists demonstrated the commutativity theorem for prime and semi-prime rings with or without involution, accepting identities on automorphism, derivations, left centralizers and generalized derivations (for example)[1, 2, 4, 7, 8, 10].

Very Recently, Ali and Dar, [2] starts the study of $*$-centralizing derivation in prime rings with involution and proved $*$-version of classical results of Posner [14], and they proved that "Let $\mathfrak{R}$ be a prime ring with involution $*$ such that $char(\mathfrak{R}) \neq 2$. Let ψ be a nonzero derivation of $\mathfrak{R}$ such that $[\psi(t_1), t_1^*] \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$ and $\psi(\mathfrak{J}_S \cap \mathfrak{J}_Z) \neq \{0\}$. Then $\mathfrak{R}$ is commutative". Further, this result was extended by Najjar et al. [9] for the second kind involution instead of condition $\psi(\mathfrak{J}_S \cap \mathfrak{J}_Z) \neq \{0\}$. Recently Alahmadi et al. [1] generalized above result for generalized derivation and they prove that "Let $\mathfrak{R}$ be a prime ring with involution $*$ of the second kind such that $char(\mathfrak{R}) \neq 2$, if $\mathfrak{R}$ admits a nonzero generalized derivation F associated with a derivation d such that $[F(t), t^*] \in \mathfrak{J}_Z$ for all $t \in \mathfrak{R}$. then $\mathfrak{R}$ is commutative". In this direction a lots of work have been done in the recent years (See for reference [3, 6, 11] where further references can be found).

The main target of our paper is to investigate the commutativity of $*$-prime rings that satisfy some central identities involving pair of derivation. Our motivation for this manuscript comes from the types of identities studied by Mamouni et al. in [9] and motivated by these types of identities. To prove our main results, we need some lemmas as well as some facts, so we start with the proof of these lemmas and facts.

2 MAIN RESULTS

Lemma 2.1. *Let $\mathfrak{R}$ be a $*$-prime ring of $char(\mathfrak{R}) \neq 2$, then $\mathfrak{R}$ is 2-torsion free.*

Proof. Let, $u \in \mathfrak{R}$ and $2u = 0$ suggests, $2u(vw) = 0$ for all $v, w \in \mathfrak{R}$ and $u\mathfrak{R}(2w) = 0$ for all $w \in \mathfrak{R}$. Since $char(\mathfrak{R}) \neq 2$ and $\mathfrak{R} \neq (0)$ then there exist $0 \neq p \in \mathfrak{R}$ such that $2p \neq 0$, forces $u\mathfrak{R}(2p) = (0) = u\mathfrak{R}(2p)^*$, by the definition of $*$-prime rings we have, either $u = 0$ or $2p = 0$ second case is not possible by the assumption and first case implies $\mathfrak{R}$ is 2-torsion free. □

Lemma 2.2. *In $*$-prime ring, $\mathfrak{J}_Z \cap \mathfrak{J}_H$ and $\mathfrak{J}_Z \cap \mathfrak{J}_S$ are free from zero-divisor.*

Proof. Let $a, b \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, such that $ab = 0$, implies $abu = 0$ for all $u \in \mathfrak{R}$ provide us $a\mathfrak{R}b = (0) = a\mathfrak{R}b^*$. So by definition of $*$-prime ring, we have either $a = 0$ or $b = 0$. □

Lemma 2.3. *Let $\mathfrak{R}$ be a 2-torsion free $*$-prime ring with involution $*$ which is of the second kind. If $t_1^2 \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

Proof. $t_1^2 \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, after linearizing we get, $t_1t_2 + t_2t_1 \in \mathfrak{J}_Z$ for all $t_1, t_2 \in \mathfrak{R}$. Since $*$ is of the second kind, there exist $0 \neq c \in \mathfrak{J}_Z \cap \mathfrak{J}_S$. Replacing t_2 by c, we have $t_1c \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, since $\mathfrak{R}$ is 2-torsion free. $[t_1c, r] = 0$ for all $r \in \mathfrak{R}$, implies $[t_1, r]c = 0$. Now by using Lemma 2.4, we get $[t_1, r] = 0$ for all $t_1, r \in \mathfrak{R}$, implies $\mathfrak{R}$ is commutative. □

Fact 2.4. Let $\mathfrak{R}$ be a 2-torsion free $*$-prime rings with involution $*$ which is of the second kind, if $\mathfrak{R}$ is normal, then $\mathfrak{R}$ is commutative.

Proof. Since $\mathfrak{R}$ is normal, i. e., $hk = kh$ where $h \in \mathfrak{J}_H$ *and* $k \in \mathfrak{J}_S$ respectively. Take any $t_1 \in \mathfrak{R}$, then $t_1 - t_1^* \in \mathfrak{J}_S$.

$$h(t_1 - t_1^*) = (t_1 - t_1^*)h, \quad \text{for all } t_1 \in \mathfrak{R} \text{ and } h \in \mathfrak{J}_H. \tag{2.1}$$

Take $s \in \mathfrak{J}_S \cap \mathfrak{J}_Z$, then $s(t_1 + t_1^*) \in \mathfrak{J}_S$ for all $t_1 \in \mathfrak{R}$, so by normality of $\mathfrak{R}$, we have $hs(t_1 + t_1^*) = s(t_1 + t_1^*)h$ for all $t_1 \in \mathfrak{R}$ and $h \in \mathfrak{J}_H$

$$s\{h(t_1 + t_1^*) - (t_1 + t_1^*)h\} = 0, \quad \text{for all } t_1 \in \mathfrak{R} \text{ and for all } h \in \mathfrak{J}_H. \tag{2.2}$$

So by Lemma 2.2, we have either $s = 0$ or $h(t_1 + t_1^*) = (t_1 + t_1^*)h$. First case is not possible, since $*$ is of the second kind and latter case together with (2.1), gives $ht_1 = t_1h$ for all $t_1 \in \mathfrak{R}$ and $h \in \mathfrak{J}_H$. Replacing t_1 by t_2 gives

$$ht_2 = t_2h, \quad \text{for all } t_2 \in \mathfrak{R} \text{ and } h \in \mathfrak{J}_H. \tag{2.3}$$

Replacing h by $t_1 + t_1^*$ in (2.3), we get

$$\{t_1 + t_1^*\}t_2 = t_2\{t_1 + t_1^*\} \quad \text{for all } t_1, t_2 \in \mathfrak{R}. \tag{2.4}$$

Now we take $s \in \mathfrak{J}_S \cap \mathfrak{J}_Z$, then $s(t_1 - t_1^*) \in \mathfrak{J}_H$ and using (2.3), we have $s\{(t_1 - t_1^*)t_2 - t_2(t_1 - t_1^*)\} = 0$ for all $t_1, t_2 \in \mathfrak{R}$. By Lemma 2.2, we have either $s = 0$ or $(t_1 - t_1^*)t_2 = t_2(t_1 - t_1^*)$ but first case is not possible, since $*$ is of the second kind and latter case implies

$$(t_1 - t_1^*)t_2 = t_2(t_1 - t_1^*) \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.5}$$

Using (2.4), together with (2.5), we get, $t_1t_2 = t_2t_1$ for all $t_1, t_2 \in \mathfrak{R}$. □

Fact 2.5. Let $\mathfrak{R}$ be a $*$-prime rings with involution $*$ which is of the second kind, then $*$ is centralizing iff $\mathfrak{R}$ is commutative.

Proof. Let

$$[t_1, t_1^*] \in \mathfrak{J}_Z \text{ for all } t_1 \in \mathfrak{R}. \tag{2.6}$$

Linearizing (2.6)

$$[t_1, t_2^*] + [t_2, t_1^*] \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.7}$$

Replacing t_2 by t_2^*, we get

$$[[t_1, t_2], t_1] + [[t_2^*, t_1^*], t_1] = 0 \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.8}$$

Displacing t_2 by t_2t_1 in (2.8), we get

$$\begin{aligned} &[[t_1, t_2], t_1]t_1 + t_1^*[[t_2^*, t_1^*], t_1] + \\ &[t_1^*, t_1][t_2^*, t_1^*] = 0 \text{ for all } t_1, t_2 \in \mathfrak{R}. \end{aligned} \tag{2.9}$$

Using (2.8) in (2.9), we receive

$$\begin{aligned} &[[t_1, t_2], t_1]t_1 - t_1^*[[t_2, t_1], t_1] + \\ &[t_1^*, t_1][t_2^*, t_1^*] = 0 \text{ for all } t_1, t_2 \in \mathfrak{R}. \end{aligned} \tag{2.10}$$

Taking t_2t_1 for t_2 in above equation, we attain

$$\begin{aligned} &[[t_1, t_2], t_1]t_1^2 - t_1^*[[t_2, t_1], t_1]t_1 \\ &+ [t_1^*, t_1]t_1^*[t_2^*, t_1^*] = 0 \text{ for all } t_1, t_2 \in \mathfrak{R}. \end{aligned} \tag{2.11}$$

Using (2.10) in (2.11), and replacing t_1 by t_1^* and t_2 by t_2^*, we have

$$[t_1, t_1^*]\{t_1[t_2, t_1] - [t_2, t_1]t_1^*\} = 0 \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.12}$$

Exchanging t_2 by t_2t_1 in (2.12), we capture

$$[t_1, t_1^*]\{t_1[t_2, t_1]t_1 - [t_2, t_1]t_1t_1^*\} = 0 \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.13}$$

Invoking (2.12) in (2.13), we obtain

$$[t_1, t_1^*][t_2, t_1]\{-t_1t_1^* + t_1^*t_1\} = 0 \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.14}$$

Last relation further implies

$$[t_1, t_1^*]^2\mathfrak{R}[t_2, t_1] = (0) \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.15}$$

Replacing t_1 by t_1^* and t_2 by t_2^* in (2.15), we find

$$[t_1, t_1^*]^2\mathfrak{R}[t_2, t_1] = (0) = [t_1, t_1^*]^2\mathfrak{R}\,[t_2, t_1]^*, \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.16}$$

By definition of $*$-prime ring, we get

$$[t_1, t_1^*]^2 = 0 \quad \text{or} \quad [t_1, t_2] = 0 \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.17}$$

Later case suggests that $\mathfrak{R}$ is commutative, by first case we have

$$[t_1, t_1^*]^2 = 0 \text{ for all } t_1 \in \mathfrak{R}. \tag{2.18}$$

Since $[t_1, t_1^*] \in \mathfrak{J}_Z \cap \mathfrak{J}_H$ and by Lemma 2.2, we get

$$[t_1, t_1^*] = 0 \text{ for all } t_1 \in \mathfrak{R}. \tag{2.19}$$

Using Fact 2.4, $\mathfrak{R}$ is commutative. □

Fact 2.6. Let $\mathfrak{R}$ be a 2-torsion free $*$-prime ring with involution $*$ of the second kind, with $char(\mathfrak{R}) \neq 2$. Then $t_1 \circ t_1^* \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$ iff $\mathfrak{R}$ is commutative.

Proof. By the given condition

$$t_1 \circ t_1^* \in \mathfrak{J}_Z \text{ for all } t_1 \in \mathfrak{R}. \tag{2.20}$$

Linearizing above

$$t_1 \circ t_2^* + t_2 \circ t_1^* \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.21}$$

Last relation further implies

$$[t_1 \circ t_2^*, r] + [t_2 \circ t_1^*, r] = 0 \text{ for all } t_1, t_2, r \in \mathfrak{R}. \tag{2.22}$$

Replacing t_2 by t_2^* in (2.22), we found

$$[t_1 \circ t_2, r] + [t_2^* \circ t_1^*, r] = 0 \text{ for all } t_1, t_2, r \in \mathfrak{R}. \tag{2.23}$$

Taking t_1 in place of t_2 in (2.23), we grasp

$$[t_1^2, r] + [(t_1^*)^2, r] = 0 \text{ for all } t_1, r \in \mathfrak{R}. \tag{2.24}$$

Assuming $t_2 \in \mathfrak{J}_Z \setminus \{0\}$ and $t_1 = t_1^2$ in (2.22), we have

$$[t_1^2, r]t_2 + [(t_1^*)^2, r]t_2^* = 0 \text{ for all } t_1, r \in \mathfrak{R}. \tag{2.25}$$

Making use of (2.24) in (2.25), we obtain

$$[t_1^2, r]\{t_2 - t_2^*\} = 0 \quad \text{for all } t_1, t_2, r \in \mathfrak{R}. \tag{2.26}$$

$\{t_2 - t_2^*\} \in \mathfrak{J}_S \cap \mathfrak{J}_Z$, by using Lemma 2.4, we have either $[t_1^2, r] = 0$ or $\{t_2 - t_2^*\} = 0$, latter case is not possible since $*$ is of the second kind, first case implies

$$[t_1^2, r] = 0 \quad \text{for all } t_1, r \in \mathfrak{R}. \tag{2.27}$$

So, $t_1^2 \in \mathfrak{Z}(\mathfrak{R})$ for all $t_1 \in \mathfrak{R}$. Using Lemma 2.3, $\mathfrak{R}$ is commutative. □

Fact 2.7. Let $\mathfrak{R}$ be a 2-torsion free $*$-prime ring and $\psi \neq 0$ is centralizing derivation on $\mathfrak{R}$ commutes with $*$, then $\mathfrak{R}$ is commutative.

Proof. By the given condition

$$[t_1, \psi(t_1)] \in \mathfrak{J}_Z \quad \text{for all } t_1 \in \mathfrak{R}. \tag{2.28}$$

Replacing t_1 by $t_1 + t_2$, we get

$$[t_1, \psi(t_2)] + [t_2, \psi(t_1)] \in \mathfrak{J}_Z \quad \text{for all } t_1, t_2 \in \mathfrak{R}. \tag{2.29}$$

Substituting t_1^2 in place of t_2 in above equation, we obtain

$$[t_1, \psi(t_1^2)] + [t_1^2, \psi(t_1)] \in \mathfrak{J}_Z \quad \text{for all } t_1 \in \mathfrak{R}. \tag{2.30}$$

And from definition of derivation, we have

$$[t_1, \psi(t_1^2)] = [t_1^2, \psi(t_1)] \quad \text{for all } t_1 \in \mathfrak{R}. \tag{2.31}$$

Invoking (2.30) in (2.31) and using $char(\mathfrak{R}) \neq 2$, we obtain

$$[t_1^2, \psi(t_1)] \in \mathfrak{J}_Z \quad \text{for all } t_1 \in \mathfrak{R}. \tag{2.32}$$

$$[[t_1^2, \psi(t_1)], r] = 0 \quad \text{for all } t_1, r \in \mathfrak{R}. \tag{2.33}$$

Using (2.28) and $char(\mathfrak{R}) \neq 2$, we obtain

$$[t_1, \psi(t_1)][t_1, r] = 0 \quad \text{for all } t_1, r \in \mathfrak{R}. \tag{2.34}$$

Replacing r by ru; where $u \in \mathfrak{R}$ and using (2.34), we get

$$[t_1, \psi(t_1)] \;\; \mathfrak{R} \, [t_1, u] = (0) \quad \text{for all } t_1, u \in \mathfrak{R}. \tag{2.35}$$

Since u is an arbitrary elements of $\mathfrak{R}$, then we take $\psi(t_1)$ in place of u

$$[t_1, \psi(t_1)] \;\; \mathfrak{R} \, [t_1, \psi(t_1)] = (0) \quad \text{for all } t_1 \in \mathfrak{R}. \tag{2.36}$$

Every $*$-prime ring is semiprime, hence we have $[t_1, \psi(t_1)] = 0$, for all $t_1 \in \mathfrak{R}$. On Linearizing we found

$$[t_1, \psi(t_2)] + [t_2, \psi(t_1)] = 0 \quad \text{for all } t_1, t_2 \in \mathfrak{R}. \tag{2.37}$$

Further implies

$$[t_1, \psi(t_2)] = [\psi(t_1), t_2] \quad \text{for all } t_1, t_2 \in \mathfrak{R}. \tag{2.38}$$

Now define $\psi_c(t_1) = [t_1, c]$ for all $t_1 \in \mathfrak{R}$ is called inner derivation. (2.38), implies

$$\psi_{\psi(t_2)} = \psi_{t_2} \circ \psi, \;\; where \;\; \psi \;\; \text{is derivation and} \;\; \psi_{t_2} \;\; \text{is inner derivation.} \tag{2.39}$$

Posner's first theorem for $*$-prime rings states that iterate of derivation is a derivation if atleast one of them is 0, see [3, Theorem 3.1], so we have either $\psi_{t_2} = 0$ or $\psi = 0$, latter case is not possible by our assumption, hence the first case implies $t_2 \in \mathfrak{J}_Z$ for all $t_2 \in \mathfrak{R}$. Hence $\mathfrak{R}$ is commutative. □

Theorem 2.8. *Let $\mathfrak{R}$ be a noncommutative $*$-prime rings with involution $*$ which is of the second kind, with $char(\mathfrak{R}) \neq 2$, if ψ_1, ψ_2 are derivations of $\mathfrak{R}$ such that $\psi_1 * = *\psi_1$, or $(\psi_2 * = *\psi_2)$ satisfying $\psi_1(t_1)t_1^* - t_1^*\psi_2(t_1) \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\psi_1 = \psi_2 = 0$.*

Proof. Let on contrary there exist nonzero derivation ψ_1 and ψ_2 satisfying

$$\psi_1(t_1)t_1^* - t_1^*\psi_2(t_1) \in \mathfrak{J}_Z \;\; for \;\; all \;\; t_1 \in \mathfrak{R}. \tag{2.40}$$

Linearizing above, we achieve

$$\begin{aligned} &\psi_1(t_1)t_2^* + \psi_1(t_2)t_1^* - t_1^*\psi_2(t_2) \\ &- t_2^*\psi_2(t_1) \in \mathfrak{J}_Z \text{ for all } \;\; t_1, t_2 \in \mathfrak{R}. \end{aligned} \tag{2.41}$$

Replacing t_2 by t_2^* in (2.41), we have

$$\begin{aligned} &\psi_1(t_1)t_2 + \psi_1(t_2^*)t_1^* - t_1^*\psi_2(t_2^*) \\ &- t_2\psi_2(t_1) \in \mathfrak{J}_Z \text{ for all } \;\; t_1, t_2 \in \mathfrak{R}. \end{aligned} \tag{2.42}$$

Replacing t_2 by t_2h, where $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, we receive

$$\begin{aligned} &\psi_1(t_1)t_2h + \psi_1(t_2^*)t_1^*h + t_2^*t_1^*\psi_1(h) - t_1^*\psi_2(t_2^*)h \\ &- t_1^*t_2^*\psi_2(h) - t_2\psi_2(t_1)h \in \mathfrak{J}_Z \text{ for all } \;\; t_1, t_2 \in \mathfrak{R}. \end{aligned} \tag{2.43}$$

Invoking (2.42) in (2.43) and using Lemma 2.2, we have

$$t_2^*t_1^*\psi_1(h) - t_1^*t_2^*\psi_2(h) \in \mathfrak{J}_Z \text{ for all } \;\; t_1, t_2 \in \mathfrak{R}. \tag{2.44}$$

Substituting t_2 by h, where $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, we get

$$h\, t_1^*\psi_1(h) - t_1^*\; h\; \psi_2(h) \in \mathfrak{J}_Z \text{ for all } \;\; t_1, t_2 \in \mathfrak{R}. \tag{2.45}$$

Last relation further implies

$$h\{t_1^*\psi_1(h) - t_1^*\psi_2(h)\} \in \mathfrak{J}_Z \text{ for all } \;\; t_1, t_2 \in \mathfrak{R}. \tag{2.46}$$

Replacing t_1 by t_1^* in (2.46) and using Lemma 2.2, we obtain

$$t_1\{\psi_1(h) - \psi_2(h)\} \in \mathfrak{J}_Z \quad \text{for all } \;\; t_1 \in \mathfrak{R}. \tag{2.47}$$

Last relation further implies

$$[t_1, r]\{\psi_1(h) - \psi_2(h)\} = 0 \quad \text{for all} \quad t_1, r \in \mathfrak{R}. \tag{2.48}$$

Replacing r by ru, where $u \in \mathfrak{R}$ and using (2.48), we have

$$[t_1, r]\ u\ \{\psi_1(h) - \psi_2(h)\} = 0 \quad \text{for all} \quad t_1, r, u \in \mathfrak{R}. \tag{2.49}$$

Last relation further implies

$$\begin{aligned}&[t_1, r]\ \mathfrak{R}\ \{\psi_1(h) - \psi_2(h)\} = (0)\\ &= [t_1, r]^*\ \mathfrak{R}\ \{\psi_1(h) - \psi_2(h)\} \quad \text{for all} \quad t_1, r, u \in \mathfrak{R}.\end{aligned} \tag{2.50}$$

By definition of $*$-prime ring we have either $[t_1, r] = 0$ or $\{\psi_1(h) - \psi_2(h)\} = 0$, first case implies commutative of $\mathfrak{R}$ which is not possible by our assumption. Latter case implies $\psi_1(h) = \psi_2(h)$ and by (2.44), we have

$$\{t_2^* t_1^* - t_1^* t_2^*\}\psi_2(h) \in \mathfrak{J}_Z \text{ for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.51}$$

Last relation further implies

$$[t_2^*, t_1^*]\ \ \psi_2(h) \in \mathfrak{J}_Z \text{ for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.52}$$

Last relation further implies

$$[[t_2, t_1], r]\ \ \psi_2(h) = 0 \text{ for all} \quad t_1, t_2, r \in \mathfrak{R}. \tag{2.53}$$

Since $*$ commutes with ψ_2, then $\psi_2(h) \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, so by Lemma 2.2, we have either $\psi_2(h) = 0$ or $[t_2, t_1], r] = 0$. First case is not possible because $*$ is of second kind, latter case implies

$$[t_2, t_1] \in \mathfrak{J}_Z \quad \text{for all} \quad t_1, t_2, r \in \mathfrak{R}. \tag{2.54}$$

In particular substituting t_2 by t_1^*, we achieve

$$[t_1^*, t_1] \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.55}$$

By Fact 2.5, $\mathfrak{R}$ is commutative, which is a contradiction to our assumption. So, we have either $\psi_1 = 0$ or $\psi_2 = 0$.
Let on contrary assume $\psi_1 = 0$ and $\psi_2 \neq 0$ then by equation (2.40), we have

$$t_1^* \psi_2(t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.56}$$

Linearizing above, we have

$$t_1^* \psi_2(t_2) + t_2^* \psi_2(t_1) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.57}$$

Replacing t_2 by $t_2 h$, where $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$ in (2.57) and using (2.57), we have

$$t_1^* t_2 \psi_2(h) \in \mathfrak{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.58}$$

Last relation further implies

$$[t_1^* t_2, r]\psi_2(h) = 0 \quad \text{for all} \quad t_1, t_2, r \in \mathfrak{R}. \tag{2.59}$$

Since $*$ commutes with ψ_2, then $\psi_2(h) \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, so by Lemma 2.2, we have either $\psi_2(h) = 0$ or $[t_1^* t_2, r] = 0$. First case is not possible because $*$ is of second kind, latter case implies

$$t_1^* t_2 \in \mathfrak{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathfrak{R}. \tag{2.60}$$

Inparticular taking $t_2 = t_1$, in the last equation, we get

$$t_1^* t_1 \in \mathfrak{J}_Z \quad \text{for all} \quad t_1 \in \mathfrak{R}. \tag{2.61}$$

Replacing t_1 by t_1^*, in the last relation, we get

$$t_1 t_1^* \in \mathfrak{J}_Z \text{ for all } t_1 \in \mathfrak{R}. \tag{2.62}$$

Using (2.61) and (2.62), we get

$$[t_1, t_1^*] \in \mathfrak{J}_Z \text{ for all } t_1 \in \mathfrak{R}. \tag{2.63}$$

By Fact 2.5, we have $\mathfrak{R}$ is commutative, which is a contradiction. Hence $\psi_1 = 0$ implies $\psi_2 = 0$. □

Theorem 2.9. *Let $\mathfrak{R}$ be a noncommutative $*$-prime rings with involution $*$ which is of the second kind with char$(\mathfrak{R}) \neq 2$, if ψ_1, ψ_2 are derivations of $\mathfrak{R}$ such that $\psi_1 * = *\psi_1$, or $(\psi_2 * = *\psi_2)$ satisfying $\psi_1(t_1^*)t_1 - t_1^*\psi_2(t_1) \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\psi_1 = \psi_2 = 0$.*

Proof. Let on contrary there exist nonzero derivation ψ_1 and ψ_2 satisfying

$$\psi_1(t_1^*)t_1 - t_1^*\psi_2(t_1) \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.64}$$

Linearizing above, we achieve

$$\begin{aligned}&\psi_1(t_1^*)t_2 + \psi_1(t_2^*)t_1 - t_1^*\psi_2(t_2)\\ &- t_2^*\psi_2(t_1) \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}.\end{aligned} \tag{2.65}$$

Replacing t_2 by $t_2 h$, where $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, we receive

$$\begin{aligned}&\psi_1(t_1^*)t_2 h - t_1^*\psi_2(t_2)h - t_1^* t_2 \psi_2(h) + \psi_1(t_2^*)t_1 h +\\ &t_2^* t_1 \psi_1(h) + t_2^*\psi_2(t_1)h \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}.\end{aligned} \tag{2.66}$$

Using (2.65) in (2.66), we obtain

$$t_2^* t_1 \psi_1(h) - t_1^* t_2 \psi_2(h) \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.67}$$

Taking $t_2 = t_1$ in (2.67), we gain

$$t_1^* t_1 \{\psi_1(h) - \psi_2(h)\} \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.68}$$

Last relation further implies,

$$[t_1^* t_1, r]\{\psi_1(h) - \psi_2(h)\} = 0 \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.69}$$

Replacing r by ru, where $u \in \mathfrak{R}$ and using (2.69), we have

$$\begin{aligned}&[t_1^* t_1, r]\,\mathfrak{R}\,\{\psi_1(h) - \psi_2(h)\} = 0\\ &= [t_1^* t_1, r]^*\,\mathfrak{R}\,\{\psi_1(h) - \psi_2(h)\} \text{ for all } t_1, t_2 \in \mathfrak{R}.\end{aligned} \tag{2.70}$$

By the definition of $*$-prime rings we have, either $[t_1^* t_1, r] = 0$ or $\{\psi_1(h) - \psi_2(h)\} = 0$, later case together with (2.67), gives us

$$\{t_2^* t_1 - t_1^* t_2\}\,\psi_1(h) \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.71}$$

Last relation further implies,

$$[t_2^* t_1 - t_1^* t_2, r]\,\psi_1(h) = 0 \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.72}$$

Since $*$ commutes with ψ_1, then $\psi_1(h) \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, so by Lemma 2.2, we have either $\psi_1(h) = 0$ or $[t_2^* t_1 - t_1^* t_2, r] = 0$. First case is not possible because $*$ is of second kind, later case implies

$$t_2^* t_1 - t_1^* t_2 \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.73}$$

Taking $t_1 s$ in place of t_1, where $0 \neq s \in \mathfrak{J}_Z \cap \mathfrak{J}_S$, so by using Lemma 2.2, we obtain

$$t_2^* t_1 + t_1^* t_2 \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.74}$$

Combining (2.73) and (2.74) and using char$(\mathfrak{R}) \neq 2$, we have

$$t_2^* t_1 \in \mathfrak{J}_Z \;\text{ for all }\; t_1, t_2 \in \mathfrak{R}. \tag{2.75}$$

Replacing t_2^* by t_1, we achieve

$$t_1^2 \in \mathfrak{J}_Z \;\text{ for all }\; t_1 \in \mathfrak{R}. \tag{2.76}$$

By Lemma 2.3, $\mathfrak{R}$ is commutative, which is not true by our assumption. First case implies

$$t_1 t_1^* \in \mathfrak{J}_Z \;\text{ for all }\; t_1 \in \mathfrak{R}. \tag{2.77}$$

Replacing t_1 by t_1^*, we get

$$t_1^* t_1 \in \mathfrak{J}_Z \;\text{ for all }\; t_1 \in \mathfrak{R}. \tag{2.78}$$

Taking together (2.77) and (2.78), we obtain

$$[t_1,\; t_1^*] \in \mathfrak{J}_Z \;\text{ for all }\; t_1 \in \mathfrak{R}. \tag{2.79}$$

By Fact 2.5, $\mathfrak{R}$ is commutative, which is not true by our assumption. So we have either $\psi_1 = 0$ or $\psi_2 = 0$.
Let on contrary assume $\psi_1 = 0$ and $\psi_2 \neq 0$ then by equation (2.64), we have

$$t_1^* \psi_2(t_1) \in \mathfrak{J}_Z \;\text{ for all }\; t_1 \in \mathfrak{R}. \tag{2.80}$$

Linearizing above, we have

$$t_1^* \psi_2(t_2) + t_2^* \psi_2(t_1) \in \mathfrak{J}_Z \;\text{ for all }\; t_1, t_2 \in \mathfrak{R}. \tag{2.81}$$

Replacing t_2 by $t_2 h$, where $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$ in (2.81) and using (2.81), we have

$$t_1^* t_2 \psi_2(h) \in \mathfrak{J}_Z \;\text{ for all }\; t_1, t_2 \in \mathfrak{R}. \tag{2.82}$$

Last relation further implies

$$[t_1^* t_2, r]\psi_2(h) = 0 \;\text{ for all }\; t_1, t_2, r \in \mathfrak{R}. \tag{2.83}$$

Since $*$ commutes with ψ_2, then $\psi_2(h) \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, so by Lemma 2.2, we have either $\psi_2(h) = 0$ or $[t_1^* t_2, r] = 0$. First case is not possible because $*$ is of second kind, latter case implies

$$t_1^* t_2 \in \mathfrak{J}_Z \;\text{ for all }\; t_1, t_2 \in \mathfrak{R}. \tag{2.84}$$

Inparticular taking $t_2 = t_1$, in the last equation, we get

$$t_1^* t_1 \in \mathfrak{J}_Z \;\text{ for all }\; t_1 \in \mathfrak{R}. \tag{2.85}$$

Replacing t_1 by t_1^*, in the last relation, we get

$$t_1 t_1^* \in \mathfrak{J}_Z \;\text{ for all }\; t_1 \in \mathfrak{R}. \tag{2.86}$$

Using (2.85) and (2.86), we get

$$[t_1, t_1^*] \in \mathfrak{J}_Z \;\text{ for all }\; t_1 \in \mathfrak{R}. \tag{2.87}$$

By Fact 2.5, we have $\mathfrak{R}$ is commutative, which is a contradiction. Hence $\psi_1 = 0$ implies $\psi_2 = 0$. □

Corollary 2.10. *Let $\mathfrak{R}$ be a $*$-prime rings with involution $*$ which is of the second kind with char$(\mathfrak{R}) \neq 2$, if ψ_1, ψ_2 are derivations on $\mathfrak{R}$ such that $\psi_1 * = *\psi_1$, or ($\psi_2 * = *\psi_2$) satisfying $\psi_1(t_1^*)t_1 - t_1^*\psi_2(t_1) \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$, then $\mathfrak{R}$ is commutative*

Theorem 2.11. *Let $\mathfrak{R}$ be a $*$-prime rings with involution $*$ which is of the second kind with $char(\mathfrak{R}) \neq 2$, if ψ_1, ψ_2 are derivations on $\mathfrak{R}$ such that $\psi_1 * = *\psi_1$, or ($\psi_2 * = *\psi_2$), then following assertions are equivalent.*
(1) $\psi_1(t_1) \circ \psi_2(t_1^*) - t_1 \circ t_1^* \in \mathfrak{J}_Z$ *for all* $t_1 \in \mathfrak{R}$.
(2) $\psi_1(t_1) \circ \psi_2(t_1^*) + t_1 \circ t_1^* \in \mathfrak{J}_Z$ *for all* $t_1 \in \mathfrak{R}$.
(3) $[\psi_1(t_1), \psi_2(t_1^*)] - t_1 \circ t_1^* \in \mathfrak{J}_Z$ *for all* $t_1 \in \mathfrak{R}$.
(4) $[\psi_1(t_1), \psi_2(t_1^*)] + t_1 \circ t_1^* \in \mathfrak{J}_Z$ *for all* $t_1 \in \mathfrak{R}$.
(5) $\mathfrak{R}$ *is commutative.*

Proof. Clearly; (5) $\implies$ $(1-4)$.
If $\psi_1 = 0$ or $\psi_2 = 0$ then then above relation reduces to $t_1 \circ t_1^* \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$. Then $\mathfrak{R}$ is commutative by Fact 2.6.
Now we assuming $\psi_1 \neq 0$ and $\psi_2 \neq 0$.
(1) $\implies$ (5) Given that

$$\psi_1(t_1) \circ \psi_2(t_1^*) - t_1 \circ t_1^* \in \mathfrak{J}_Z \text{ for all } t_1 \in \mathfrak{R}. \tag{2.88}$$

Linearizing above, we receive

$$\begin{aligned} &\psi_1(t_1) \circ \psi_2(t_2^*) + \psi_1(t_2) \circ \psi_2(t_1^*) - t_1 \circ t_2^* \\ &- t_2 \circ t_1^* \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \end{aligned} \tag{2.89}$$

Replacing t_2 by $t_2 h$; where $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, so by using (2.89), we get

$$\begin{aligned} &\{\psi_1(t_1) \circ t_2^*\}\psi_2(h) + \{t_2 \circ \psi_2(t_1^*)\}\psi_1(h) \\ &\in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \end{aligned} \tag{2.90}$$

Putting h in place of t_2 where; $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, so by using Lemma 2.2, we get

$$\psi_1(t_1)\psi_2(h) + \psi_2(t_1^*)\psi_1(h) \in \mathfrak{J}_Z \text{ for all } t_1 \in \mathfrak{R}. \tag{2.91}$$

Putting s in place of t_2 in (2.90), where, $0 \neq s \in \mathfrak{J}_Z \cap \mathfrak{J}_S$, so by using Lemma 2.2, we get

$$-\psi_1(t_1)\psi_2(h) + \psi_2(t_1^*)\psi_1(h) \in \mathfrak{J}_Z \text{ for all } t_1 \in \mathfrak{R}. \tag{2.92}$$

Combining (2.91) and (2.92) and using char($\mathfrak{R}$) $\neq 2$, we obtain

$$\psi_2(t_1)\psi_1(h) \in \mathfrak{J}_Z \text{ for all } t_1 \in \mathfrak{R}. \tag{2.93}$$

Last relation further implies,

$$\{[\psi_2(t_1), r]\}\ \psi_1(h) = 0 \text{ for all } t_1, r \in \mathfrak{R}. \tag{2.94}$$

Since $*$ commutes with ψ_1, then $\psi_1(h) \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, so by Lemma 2.2, we have either $\psi_1(h) = 0$ or $[\psi_2(t_1), r] = 0$. First case is not possible because $*$ is of second kind, later case implies

$$[\psi_2(t_1), r] = 0 \text{ for all } t_1, r \in \mathfrak{R}. \tag{2.95}$$

In particular taking $r = t_1$, we have

$$[\psi_2(t_1), t_1] = 0 \text{ for all } t_1 \in \mathfrak{R}. \tag{2.96}$$

By Fact 2.7, $\mathfrak{R}$ is commutative.
(2) $\implies$ (5) Given that

$$\psi_1(t_1) \circ \psi_2(t_1^*) + t_1 \circ t_1^* \in \mathfrak{J}_Z \text{ for all } t_1 \in \mathfrak{R}. \tag{2.97}$$

Linearizing above, we receive

$$\begin{aligned} &\psi_1(t_1) \circ \psi_2(t_2^*) + \psi_1(t_2) \circ \psi_2(t_1^*) + t_1 \circ t_2^* \\ &+ t_2 \circ t_1^* \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \end{aligned} \tag{2.98}$$

Replacing t_2 by t_2h; where $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$ and using (2.98), we get

$$\{\psi_1(t_1) \circ t_2^*\}\psi_2(h) + \{t_2 \circ \psi_2(t_1^*)\}\psi_1(h) \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.99}$$

Above equation is same as (2.90), so by the same argument $\mathfrak{R}$ is commutative.

(3) $\Longrightarrow$ (5) Given that

$$[\psi_1(t_1),\ \psi_2(t_1^*)] - t_1 \circ t_1^* \in \mathfrak{J}_Z \text{ for all } t_1 \in \mathfrak{R}. \tag{2.100}$$

Taking $t_1 = t_1 + t_2$ in above, we obtain

$$[\psi_1(t_1),\ \psi_2(t_2^*)] + [\psi_1(t_2),\ \psi_2(t_1^*)] - t_1 \circ t_2^* - t_2 \circ t_1^* \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.101}$$

Replacing t_2 by t_2h in (2.101) where $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, so by using last equation, we gain

$$[\psi_1(t_1),\ t_2^*]\ \psi_2(h) + [t_2, \psi_2(t_1^*)]\psi_1(h) \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.102}$$

Replacing t_2 by t_2s in (2.102), where $0 \neq s \in \mathfrak{J}_Z \cap \mathfrak{J}_S$, we obtain

$$-[\psi_1(t_1),\ t_2^*]\ \psi_2(h) + [t_2, \psi_2(t_1^*)]\psi_1(h) \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.103}$$

By using (2.102) and (2.103), we achieve

$$[\psi_1(t_1),\ t_2^*]\ \psi_2(h) \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.104}$$

Last relation further implies,

$$[[\psi_1(t_1),\ t_2^*], r]\ \psi_2(h) = 0 \text{ for all } t_1, t_2, r \in \mathfrak{R}. \tag{2.105}$$

Since $*$ commutes with ψ_2, then $\psi_2(h) \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, so by Lemma 2.2, we have either $\psi_2(h) = 0$ or $[[\psi_1(t_1),\ t_2^*], r] = 0$. First case is not possible because $*$ is of the second kind, latter case implies

$$[\psi_1(t_1),\ t_2^*] \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.106}$$

In particular taking $t_2 = t_1^*$, we have

$$[\psi_1(t_1),\ t_1] \in \mathfrak{J}_Z \text{ for all } t_1 \in \mathfrak{R}. \tag{2.107}$$

By Fact 2.7, $\mathfrak{R}$ is commutative.

(4) $\Longrightarrow$ (5) Given that

$$[\psi_1(t_1),\ \psi_2(t_1^*)] + t_1 \circ t_1^* \in \mathfrak{J}_Z \text{ for all } t_1 \in \mathfrak{R}. \tag{2.108}$$

Taking $t_1 = t_1 + t_2$ in above, we obtain

$$[\psi_1(t_1),\ \psi_2(t_2^*)] + [\psi_1(t_2),\ \psi_2(t_1^*)] + t_1 \circ t_2^* + t_2 \circ t_1^* \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.109}$$

Replacing t_2 by t_2h; where $0 \neq h \in \mathfrak{J}_Z \cap \mathfrak{J}_H$, so by using (2.109), we get

$$\{\psi_1(t_1) \circ t_2^*\}\psi_2(h) + \{t_2 \circ \psi_2(t_1^*)\}\psi_1(h) \in \mathfrak{J}_Z \text{ for all } t_1, t_2 \in \mathfrak{R}. \tag{2.110}$$

Above equation is same as (2.102), so by the same argument $\mathfrak{R}$ is commutative. □

Corollary 2.12. *Let $\mathfrak{R}$ be a $*$-prime rings with involution $*$ which is of the second kind, with char$(\mathfrak{R}) \neq 2$, if ψ_1, ψ_2 are derivations on $\mathfrak{R}$ such that $\psi_1 * = *\psi_1$, or ($\psi_2 * = *\psi_2$), then following assertions are equivalent.*

(1) $\psi_1(t_1) \circ \psi_2(t_2) - t_1 \circ t_2 \in \mathfrak{J}_Z$ *for all* $t_1, t_2 \in \mathfrak{R}$.
(2) $\psi_1(t_1) \circ \psi_2(t_1) + t_1 \circ t_2 \in \mathfrak{J}_Z$ *for all* $t_1, t_2 \in \mathfrak{R}$.
(3) $[\psi_1(t_1), \psi_2(t_2)] - t_1 \circ t_2 \in \mathfrak{J}_Z$ *for all* $t_1, t_2 \in \mathfrak{R}$.
(4) $[\psi_1(t_1), \psi_2(t_2)] + t_1 \circ (t_2) \in \mathfrak{J}_Z$ *for all* $t_1, t_2 \in \mathfrak{R}$.
(5) $\mathfrak{R}$ *is commutative.*

As it is well-known that the zero-divisor is impossible in the center of a prime ring, but in $*$-prime rings centre is not free from zero divisor. The follwing example explain that the above fact.

Example 2.13. Consider $\mathfrak{R} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\}$, define $*$ in such away, $*\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right) = \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}$. It is easy to verify that $\mathfrak{R}$ is $*$ -prime ring with involution $*$. For any non zero a, $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in \mathfrak{J}_Z$, and for any nonzero b, $\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \in \mathfrak{R}$ and $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This shows the fact.

The following examples shows second kind is necessary in Theorem 2.11.

Example 2.14. Consider $\mathfrak{R} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{Z} \right\}$, define $*$ in such away, $*\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. It is easy to verify that $\mathfrak{R}$ is $*$ -prime ring with involution $*$ of the first kind. Moreover, we define ψ_1 and ψ_2 as $\psi_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$ and $\psi_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}$, here ψ_1 and ψ_2 satisfy the condition $\psi_1(t_1) \circ \psi_2(t_1^*) - t_1 \circ t_1^* \in \mathfrak{J}_Z$ for all $t_1 \in \mathfrak{R}$. However, $\mathfrak{R}$ is noncommutative.

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