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Hyper near-vector spaces

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Abstract In this paper we take a first look at hyper near-vector spaces. We define a hyper near-vector space having similar properties to Johannes André's near-vector space. We define important concepts including independence, the notion of a basis, regularity, subhyperspaces and as a highlight prove that there is a Decomposition Theorem for these spaces.

1 Introduction

Near-vector spaces are a generalisation of vector spaces, as near-rings are a generalisation of rings. A number of authors have defined the notion of a near-vector space in different ways, including Beidleman [5], Karzel [11] and André [2]. We will focus on André's near-vector spaces in this paper. These have a geometric origin, see for example [3], [4], and more recently [10].

Hyperstructures are algebraic structures with a multi-values operator. Such structures have mathematical applications in areas such as fuzzy set theory, hypergraphs, and latices, and further applications in chemistry and physics (see [7] for a historical development of the theory of hyperstructures and their link to other fields). Hyper nearrings and hyper vector spaces have been defined and studied (see [8] and [13], for example). A natural question would therefore be to determine whether such a generalisation is possible for near-vector spaces.

In this paper we define and study hyper near-vector spaces that have similar properties to André's near-vector spaces. Important algebraic concepts including independence, the notion of a basis, regularity and subhyperspaces are defined. We give some interesting first examples of hyper near-vector spaces. Most notably, we prove that there is a Decomposition Theorem for these spaces into maximal regular subhyperspaces.

In Section 2 we give the required preliminary material for this paper, while in Section 3 we define our hyper near-vector space, give some examples and prove our main results. For any set S, we will write S^* for $S \setminus \{0\}$ throughout this paper.

2 Preliminary Material

We begin with the preliminary material on near-vector spaces, hypergroups and hyper vector spaces we will need in the paper.

Definition 2.1. [2, Definition 4.1] A *near-vector space* is a pair (V, A) which satisfies the following conditions.

(i) $(V, +)$ is a group and A is a set of endomorphisms of V.

(ii) A contains the endomorphisms 0, id and $-id$ (hereafter simply $0, 1, -1$).

(iii) A^* is a subgroup of the group Aut(V).

(iv) If $\alpha x = \beta x$ with $x \in V$ and $\alpha, \beta \in A$, then $\alpha = \beta$ or $x = 0$, i.e. *A acts fixed point free on V.*

(v) The *quasi-kernel* $Q(V)$ of V, generates V as a group. Here,

$$Q(V) = \{x \in V \,|\, \forall \alpha, \beta \in A, \, \exists \gamma \in A \text{ such that } \alpha x + \beta x = \gamma x\} .$$

If there is no room for confusion we will write Q for $Q(V)$.

André defined independence in [2] in terms of a dependence relation.

Definition 2.2. Let X be a set and let $\mathcal{P}(X)$ be the set of all subsets of X. A relation between X and $\mathcal{P}(X)$, denoted by $v \triangleleft M$, with $v \in X$ and $M \subseteq X$, is a *dependence relation* if the following conditions are satisfied (where $u, v, w \in X$ and $M, N \subseteq X$).
(D_1) $v \in M$ implies that $v \triangleleft M$.
(D_2) $w \triangleleft M$ and $v \triangleleft N$ for each $v \in M$, implies that $w \triangleleft N$.
(D_3) $v \triangleleft M$ and the falsehood of $v \triangleleft M \setminus \{u\}$ (denoted $v \not\triangleleft M \setminus \{u\}$), implies that $u \triangleleft (M \setminus \{u\}) \cup \{v\}$.

Now let (V, A) be a near-vector space and Q its quasi-kernel. André (in [2, p.302]) defines a relation between Q and 2^Q as follows.

(i) $v \triangleleft \emptyset$ if $v = 0$.

(ii) $v \triangleleft M$, $\emptyset \neq M \subseteq Q$, if and only if there exists $u_i \in M$ and $\lambda_i \in A$ $(i = 1, 2, \ldots, n)$ such that

$$v = \sum_{i=1}^{n} \lambda_i u_i. \tag{2.1}$$

This relation is a dependence relation.

Theorem 2.3. *[2, Definition 3.1] Let Q be the quasi-kernel of the near-vector space (V, A). Then the relation defined in (2.1) is a dependence relation between Q and $\mathcal{P}(Q)$.*

Definition 2.4. [2, p.302] A subset E of Q is *independent* if there is no $v \in E$ such that $v \triangleleft E \setminus \{v\}$.

The *dimension* of the near-vector space, dim(V), is uniquely determined by the cardinality of an independent generating set for $Q(V)$, called a *basis* of V (see [2] for further details).

Next we give all the hyperstructure definitions we will need. For further reference, we refer the reader to [7].

Definition 2.5. Let J be a nonempty set. A mapping $\circ : J \times J \to \mathcal{P}^*(J)$, where $\mathcal{P}^*(J)$ is the set of all nonempty subsets of J, is called a *hyperoperation* on J.

From the above definition, if A and B are two nonempty subsets of J and $x \in J$, then $A \circ B = \cup_{a \in A_{b \in B}} a \circ b$, $x \circ A = \{x\} \circ A$, $A \circ x = A \circ \{x\}$. From now on we will write $\{x\}$ and x interchangeably, when there is no room for confusion.

Definition 2.6. [7] A *quasicanonical hypergroup* is a pair $(H, +)$, where $+$ is a hyperoperation on H satisfying the following.

(i) $(H, +)$ is a *hypergroup,* i.e.

  a. $a + (b + c) = (a + b) + c$ for all $a, b, c \in H$ ($(H, +)$ is a *semihypergroup*)

  b. $a + H = H + a = H$ for all $a \in H$ ($(H, +)$ is a *quasihypergroup*)

(ii) H has a *scalar identity*, i.e. there exists $0 \in H$ such that, for all $x \in H$, $x + 0 = \{x\}$.

(iii) Every element has a *unique inverse*, i.e. for all $x \in H$, there exists a unique $-x \in H$ such that $0 \in x + (-x)$.

(iv) H is *reversible*, i.e. if $x \in y + z$, then $z \in (-y) + x$.

If H is commutative, (i.e. $a + b = b + a$ for all $a, b \in H$), then H is called a *canonical hypergroup*.

Definition 2.7. A non-empty subset K of a canonical hypergroup H is a *canonical subhypergroup* if K is also a quasi-canonical hypergroup.

We note that it is well-known that canonical subhypergroups are closed under intersection.

Definition 2.8. Let $(H, +)$ and $(K, \circ)$ be canonical hypergroups with scalar identities 0 and e respectively. Let $f : H \to K$.

- f is a *homomorphism* if for all $x, y \in H$, $f(x + y) \subseteq f(x) \circ f(y)$ and $f(0) = e$.
- f is a *good homomorphism* if for all $x, y \in H$, we have $f(x + y) = f(x) \circ f(y)$ and $f(0) = e$.
- f is an *isomorphism* if it is a homomorphism and its inverse f^{-1} is a homomorphism.
- f is an *endomorphism* if $(K, \circ) = (H, +)$ and f is a homomorphism.
- f is an *automorphism* if it is an isomorphism and an endomorphism.

As with any algebraic structure, the automorphisms of a canonical hypergroup form a group, which, for a hypergroup H we will denote $\text{Aut}(H)$.

A proof in [7] is presented that shows a homomorphism is an isomorphism if and only if it is bijective and good.

In 1990 [8], Dašić introduced the concept of hypernear-rings.

Definition 2.9. [8, Definition 1] A triple $(R, +, \cdot)$ is called a *hypernear-ring* if the following axioms hold.

- $(R, +)$ is a quasicanonical hypergroup.
- $(R, \cdot)$ is a semigroup having 0 as a left absorbing element, i.e. $x \cdot 0 = 0$ for all $x \in R$.
- The multiplication $\cdot$ is distributive with respect to the hyperoperation $+$ on the left side, i.e. $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$.

If in addition, $(R^*, \cdot)$ is a quasicanonical hypergroup, $(R, +, \cdot)$ is called a *hypernear-field*.

In 1990, Tallini [13] introduced the notion of a hyper vector space over a field, while Vougiouklis [15] introduced weak hyper vector spaces. Recently, Al Tahan and Davvaz [1] introduced a hyper vector space over a Krasner hyperfield. This is the definition of a hyper vector space. We will see that it is most fitting since we will show that every hyper vector space is a hyper near-vector space.

Definition 2.10. [12] A *Krasner hyperring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms for all $x, y, z \in R$.

(i) $(R, +)$ is a canonical hypergroup.

(ii) $(R, \cdot)$ is a semigroup having zero as bilaterally absorbing element, i.e. $x \cdot 0 = 0 \cdot x = 0$.

(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$

(iv) $(x + y) \cdot z = x \cdot y + x \cdot z$

A commutative Krasner hyperring $(R, +, \cdot)$ with identity 1 is a *Krasner hyperfield* if $(R^*, \cdot)$ is a group.

It is clear that a $(R, +, \cdot)$ is a Krasner hyperring if it is a hypernear-ring with commutative $+$ and $\cdot$.

Definition 2.11. [1, Definition 3.1] Let F be a Krasner hyperfield. A canonical hypergroup $(V, +)$ together with a map $\cdot : F \times V \to V$ is called a *hyper vector space over* F if for all $a, b \in F$ and $x, y \in V$ the following conditions hold.

(i) $a \cdot (x + y) = a \cdot x + a \cdot y$

(ii) $(a + b) \cdot x = a \cdot x + b \cdot x$

(iii) $a \cdot (b \cdot x) = (ab) \cdot x$

(iv) $a \cdot (-x) = (-a) \cdot x = -(a \cdot x)$

(v) $x = 1 \cdot x$

We will also need the definition of a weak hyper vector space.

Definition 2.12. [1, Definition 3.2] Let F be a Krasner hyperfield. A canonical hypergroup $(V, +)$ together with a map $\cdot : F \times V \to V$ is called a *weak hyper vector space over* F if for all $a, b \in F$ and $x, y \in V$ the following conditions hold.

(i) $a \cdot (x + y) \subseteq a \cdot x + a \cdot y$

(ii) $(a + b) \cdot x \subseteq a \cdot x + b \cdot x$

(iii) $a \cdot (b \cdot x) = (ab) \cdot x$

(iv) $a \cdot (-x) = (-a) \cdot x = -(a \cdot x)$

(v) $x = 1 \cdot x$

Note that, given a weak hyper vector space V over a Krasner hyper field F, one may construct for each $a \in F$ a map $a : V \to V$ such that $a(v) = a \cdot v$, which by the first property of a weak hyper vector space is a homomorphism of V. Now, if $a \neq 0$, then it has an inverse a^{-1}, since

$$a(a^{-1}(v)) = a(a^{-1} \cdot v) = a \cdot (a^{-1} \cdot v) = (aa^{-1}) \cdot v = 1 \cdot v = v$$

for all $v \in V$. It follows that each nonzero a is an isomorphism, and is therefore a good homomorphism. Hence $a \cdot (x + y) = a(x + y) = a(x) + a(y) = a \cdot x + a \cdot y$. Furthermore, $0 \cdot (x + y) = 0 = 0 + 0 = 0 \cdot x + 0 \cdot y$. It follows that, for all $a \in F$ and all $x, y \in V$, $a \cdot (x + y) = a \cdot x + a \cdot y$.

3 Hyper near-vector spaces

In this section we define our hyper near-vector space, give some examples and prove our main results. In order to do this we begin by replacing the additive group of vectors in Theorem 2.1 with a canonical hypergroup and define the scalar multiplication as a group of endomorphisms as before. This is similar to how Theorem 2.11 generalizes a vector space. The notion of the quasi-kernel is generalized in a suitable way so that its elements maintain the structural properties of André's near-vector space.

3.1 Definition and preliminary results

We now give our hyper analogue for André's near-vector space.

Definition 3.1. A *hyper near-vector space* is a pair (V, A) which satisfies the following conditions.

(i) $(V, +)$ is a canonical hypergroup and A is a set of endomorphisms of V.

(ii) A contains the endomorphisms 0, 1 and -1.

(iii) $A^* = A \setminus \{0\}$ is a subgroup of the group $\mathrm{Aut}(V)$.

(iv) If $\alpha x = \beta x$ with $x \in V$ and $\alpha, \beta \in A$, then $\alpha = \beta$ or $x = 0$, i.e. *A acts fixed point free on V.*

(v) $V = \langle Q(V) \rangle$, i.e. V is generated additively by the *quasi-kernel,* $Q(V)$, where

$$Q(V) = \{x \in V \,|\, \forall \alpha, \beta \in A,\ \alpha x + \beta x \subseteq Ax\}.$$

Because A^* is a set of isomorphisms, and $0 \in A$ is itself good, it follows each endomorphism in A is good. We view A as the set of scalars of V.

In order to compare hyper near-vector spaces, we need the following definition.

Definition 3.2. Let (V, A_1) and (W, A_2) be hyper near-vector spaces over A. Then maps $\phi : V \to W$ and $\eta : A_1 \to A_2$ form a *homomorphism* if η is a semigroup isomorphism, $\phi(0) = 0$ and for any $x, y \in V$ and $\alpha \in A_1$ we have $\phi(x + y) \subseteq \phi(x) + \phi(y)$ and $\phi(\alpha x) = \eta(\alpha)\phi(x)$, and a *good homomorphism* if in addition $\phi(x) + \phi(y) \subseteq \phi(x + y)$.

We say that two hyper near-vector spaces (V, A) and (W, A) are *isomorphic* (written $(V, A) \cong (W, A)$) if there is a bijective good homomorphism $\phi : V \to W$.

If, in the definition above, we have that $A_1 = A_2$, often η is implicitly taken to be the identity map on A, unless expressly otherwise stated.

It is known that every vector space is a near-vector space with the quasi-kernel the entire space. We now prove the analogous result for hyper vector spaces.

Lemma 3.3. *Every hyper vector space is a hyper near-vector space.*

Proof. Let V be a hyper vector space over F.

(i) By definition $(V, +)$ is a canonical hypergroup and F is a set of endomorphisms of V.

(ii) F contains the endomorphisms 0, 1 and -1 by definition.

(iii) F^* is a subgroup of the group Aut(V) since for any $\alpha, \beta^{-1} \in F^*, \alpha\beta^{-1} \in F^*$ and it is not difficult to check that every $\alpha \in F^*$ is a bijection of $(V, +)$.

(iv) Suppose that $\alpha x = \beta x$ with $x \neq 0$. Then $0 \in \alpha x - \beta x = (\alpha - \beta)x$ so that $0 \in \alpha - \beta$. Thus $-\beta = -\alpha$, so by the uniqueness of inverses, $\alpha = \beta$. Thus F acts fixed point free on V, as explained under Theorem 2.12.

(v) $V = \langle Q(V) \rangle$, where $Q(V) = V$ by Definition 2.11 (2).

□

Below we give a first example of a hyper near-vector space.

Example 3.4. Let $V = \{0, a, b, c\}$ be a set with the hyperoperation $\oplus$ defined as follows:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	$\{0, a\}$	c	$\{b, c\}$
b	b	c	$\{0, b\}$	$\{a, c\}$
c	c	$\{b, c\}$	$\{a, c\}$	V

Then $(V, \oplus)$ is a canonical hypergroup (See Example 12 in [6]). If we now take $A = \{0, 1\}$, then since $-1 = 1$, we have that (V, A) is a hyper near-vector space. A quick check shows that $Q(V) = \{0, a, b\}$. We note that (V, A) is also a weak hyper vector space.

We now prove some useful properties of the quasi-kernel.

Lemma 3.5. *Let (V, A) be a hyper near-vector space. The quasi-kernel Q has the following properties.*

(a) $0 \in Q$.

(b) For $u \in Q^*$, if $\alpha u + \beta u = A'u \subseteq Au$, then A' is uniquely determined by α and β.

(c) If $u \in Q$ and $\lambda \in A$, then $\lambda u \in Q$, i.e. $Au \subseteq Q$.

(d) If $u \in Q$ and $\lambda_i \in A$, $i = 1, 2, \ldots, n$, then $\sum_{i=1}^{n} \lambda_i u = A'u \subseteq Q$ for some $A' \subseteq A$.

Proof.

(a) Let $\alpha, \beta \in A$. Then $\alpha 0 + \beta 0 = 0 + 0 = \{0\} = A0$. Thus $0 \in Q$.

(b) Suppose that for all $\alpha, \beta \in A$ we have that $\alpha u + \beta u = A'u$ and $\alpha u + \beta u = A''u$, where $A', A'' \subseteq A$. If $\alpha \in A'$, then $\alpha u \in A'u$ and $\alpha u \in A''u$. Thus $\alpha u = \alpha' u$ for some $\alpha' \in A''$. Since $u \neq 0$, by the fixed point free property, we have that $\alpha = \alpha'$. Thus $A' \subseteq A''$. Similarly, it can be shown that $A'' \subseteq A'$, so that $A' = A''$.

(c) Suppose $u \in Q$ and $\lambda \in A$. There are two cases to consider:
Case 1: $\lambda = 0$
Then $\lambda u = 0u = 0 \in Q$ by (a).
Case 2: $\lambda \neq 0$
Let α, β be elements of A. Then

$$\begin{aligned} \alpha(\lambda u) + \beta(\lambda u) &= (\alpha\lambda)u + (\beta\lambda)u \\ &= \lambda' u \text{ for some } \lambda' \in A \text{ since } u \in Q. \end{aligned}$$

Since $\lambda \neq 0$, $\lambda' u = (\lambda' \lambda^{-1})\lambda u$. Thus $\lambda u \in Q$, so $Au \subseteq Q$.

(d) We prove the result using induction on n. From (c), if $u \in Q, \lambda u \in Q$ for $\lambda \in A$. Now suppose that $\sum_{i=1}^{k} \lambda_i u \subseteq Au$, say $\sum_{i=1}^{k} \lambda_i u = A'u$. Then

$$\begin{aligned}\sum_{i=1}^{k} \lambda_i u &= A'u + \lambda_{k+1} u \\ &= \bigcup_{\lambda \in A'} (\lambda u + \lambda_{k+1} u) \subseteq Au.\end{aligned}$$

□

Lemma 3.6. *Let (V, A) and (W, A) be hyper near-vector spaces over A and $\phi : V \to W$ be a good homomorphism. Then $\phi(Q(V)) \subseteq Q(W)$.*

Proof. Let $u \in Q(V)$ and $\alpha, \beta \in A$. Then $\alpha u + \beta u \subseteq Au$, so that $\alpha\phi(u) + \beta\phi(u) = \phi(\alpha u + \beta u) \subseteq \phi(Au) = A\phi(u)$. It follows $\phi(u) \in Q(W)$. □

3.2 An addition on A

In [2], a special addition on the group of scalars is introduced. We do the same below.

Definition 3.7. Let (V, A) be a hyper near-vector space. For $u \in Q^*$, we define an operation $+_u$ on A as follows. For all $\alpha, \beta \in A$,

$$\alpha +_u \beta = A',$$

where $\alpha u + \beta u = A'u$.

Example 3.8. Returning to Example 3.4 we have that for all $\alpha, \beta \in A$,

$$\alpha +_a \beta = \alpha +_b \beta.$$

We now prove that the addition on A results in it having the structure of a canonical hypergroup.

Lemma 3.9. *Let (V, A) be a hyper near-vector space. Then $(A, +_u)$ is a canonical hypergroup for each $u \in Q^*$.*

Proof. By the uniqueness of A' in Theorem 3.5, we have that $+_u$ is well-defined. Let $\alpha, \beta, \gamma \in A$. We verify each of the axioms for a canonical hypergroup.

(i) (a) Let $u \in Q^*$. Then

$$\begin{aligned}(\alpha +_u (\beta +_u \gamma))u &= \alpha u + (\beta +_u \gamma)u \\ &= \alpha u + (\beta u + \gamma u) \\ &= (\alpha u + \beta u) + \gamma u \text{ (since V is a semihypergroup)} \\ &= ((\alpha +_u \beta) +_u \gamma)u.\end{aligned}$$

Since $u \neq 0$, by the fixed point free property, we have that

$$\alpha +_u (\beta +_u \gamma) = (\alpha +_u \beta) +_u \gamma.$$

(b)

$$\begin{aligned}(\alpha +_u A)u &= \alpha u + Au \\ &= \bigcup_{\beta\in A} (\alpha u + \beta u).\end{aligned}$$

We want to show that $\alpha +_u A = A$. Since for all $\beta \in A$, we have that $\alpha u + \beta u \subseteq Au$, we have that $(\alpha +_u A)u = \bigcup_{\beta\in A}(\alpha u + \beta u) \subseteq Au$. Thus by the fixed point free property, $\alpha +_u A \subseteq A$. Now let $\lambda \in A$. Then

$$\begin{aligned}\lambda u \in \alpha u - \alpha u + \lambda u &= \alpha u + (-\alpha +_u \lambda)u \\ &= (\alpha +_u A')u \text{ where } A' = -\alpha +_u \lambda.\end{aligned}$$

Now we have that $\lambda u \in (\alpha +_u A')u$ where $\alpha +_u A' \subseteq \alpha +_u A$. Thus $\lambda u \in (\alpha +_u A)u$ and by using the fixed point free property we have that $\lambda \in \alpha +_u A$.

(ii)

$$\begin{aligned}(\alpha +_u \beta)u &= \alpha u + \beta u \\ &= \beta u + \alpha u \text{ (since } (V,+) \text{ is commutative)} \\ &= (\beta +_u \alpha)u.\end{aligned}$$

Hence, by the fixed point free property, $\alpha +_u \beta = \beta +_u \alpha$.

(iii) We claim 0 is the scalar identity of $(A, +_u)$.

$$\begin{aligned}(\alpha +_u 0)u &= \alpha u + 0u \\ &= \alpha u + 0 \\ &= \{\alpha u\} \\ &= \{\alpha\}\, u\end{aligned}$$

Hence, by the fixed point free property, $\alpha +_u 0 = \{\alpha\}$.

(iv) We claim $-\alpha := (-1) \circ \alpha$ is the unique inverse of α in $(A, +_u)$.

$$\begin{aligned}(\alpha +_u (-\alpha))u &= \alpha u + (-1)(\alpha u) \\ &= \alpha u - \alpha u\end{aligned}$$

Now, since $-\alpha u$ is the unique inverse of αu in $(V, +)$, we have that $0u = 0 \in \alpha u - \alpha u = (\alpha +_u (-\alpha))u$. It follows that $0 \in \alpha +_u (-\alpha)$ by the fixed point free property. For uniqueness, suppose $0 \in \alpha +_u \lambda$ for some $\lambda \in A$. Then $0 = 0u \in (\alpha +_u \lambda)u = \alpha u + \lambda u$, hence, from the uniqueness of the inverse of αu in $(V, +)$, it follows that $\lambda u = -\alpha u$. By the fixed point free property, it follows that $\lambda = -\alpha$.

(v) Suppose $\alpha \in \beta +_u \gamma$. Then $\alpha u \in (\beta +_u \gamma)u = \beta u + \gamma u$. Since $(V, +)$ is reversible, it follows $\gamma u \in -\beta u + \alpha u = (-\beta +_u \alpha)u$, and so $\gamma \in -\beta +_u \alpha$ by the fixed point free property.

□

In fact, we can show more, i.e. we have a hyper-nearfield.

Lemma 3.10. *Let $u \in Q^*$. Then $(A, +_u, \circ)$ is a hyper-nearfield.*

Proof. Since $(A, +_u)$ is a canonical hypergroup, and $(A^*, \circ)$ is a group by definition, it remains to be shown that the left distributive law holds and that $0 \in A$ is bilaterally absorbing. Let $\alpha, \beta, \gamma \in A$, then

$$\begin{aligned}\alpha \circ (\beta +_u \gamma)u &= \alpha(\beta u + \gamma u)\\ &= (\alpha \circ \beta)u + (\alpha \circ \gamma)u\\ &= (\alpha \circ \beta +_u \alpha \circ \gamma)u.\end{aligned}$$

By the fixed point free property it follows that $\alpha \circ (\beta +_u \gamma) = \alpha \circ \beta +_u \alpha \circ \gamma$. Furthermore $(0 \circ \alpha)u = 0(\alpha u) = 0 = 0u$ and $(\alpha \circ 0)u = \alpha(0u) = \alpha 0 = 0 = 0u$, hence by the fixed point free property, $0\alpha = \alpha 0 = 0$ for all $\alpha \in A$. □

Next we show that for any nonzero element of the quasi-kernel, the hyper near-field from the previous lemma is isomorphic to all of those where the addition is defined in terms of scalar multiples of it.

Lemma 3.11. *For every $u \in Q^*$ and $\lambda \in A^*$, $(A, +_u, \circ) \cong (A, +_{\lambda u}, \circ)$.*

Proof. Let $u \in Q^*$ and $\lambda \in A^*$. Define $\theta : (A, +_{\lambda u}, \circ) \to (A, +_u, \circ)$ so that $\theta(\alpha) = \lambda^{-1}\alpha\lambda$. Let $\alpha, \beta \in A$, then

$$\begin{aligned}\theta(\alpha +_{\lambda u} \beta)u &= \lambda^{-1}(\alpha +_{\lambda u} \beta)\lambda u\\ &= \lambda^{-1}(\alpha\lambda u + \beta\lambda u)\\ &= \lambda^{-1}\alpha\lambda u + \lambda^{-1}\beta\lambda u\\ &= \theta(\alpha)u + \theta(\beta)u\\ &= (\theta(\alpha) +_u \theta(\beta))u.\end{aligned}$$

Therefore $\theta(\alpha +_{\lambda u} \beta) = \theta(\alpha) +_u \theta(\beta)$ by the fixed point free property. Furthermore,

$$\begin{aligned}\theta(\alpha\beta) &= \lambda^{-1}\alpha\beta\lambda\\ &= \lambda^{-1}\alpha(\lambda\lambda^{-1})\beta\lambda\\ &= \theta(\alpha)\theta(\beta).\end{aligned}$$

Hence θ is a homomorphism. But $\lambda(\theta(\alpha))\lambda^{-1} = \lambda\lambda^{-1}\alpha\lambda\lambda^{-1} = \alpha$, and $\theta(\lambda\alpha\lambda^{-1}) = \lambda^{-1}(\lambda\alpha\lambda^{-1})\lambda = \alpha$, so that θ is invertible, with $\theta^{-1} : \alpha \mapsto \lambda\alpha\lambda^{-1}$. Hence θ is an isomorphism. □

3.3 Independence and a basis for $Q(V)$

[2] defined independence in terms of a dependence relation. We follow the same route to defining independence.

Definition 3.12. Let (V, A) be a hyper near-vector space. We define a relation between Q and $\mathcal{P}(Q)$ as follows:
$v \lhd M \subseteq Q$ if there exists $n \in \mathbb{N}$, $u_i \in M$ for $i \in \{1, \ldots, n\}$, and $\lambda_1, \ldots, \lambda_n \in A$ such that

$$v \in \sum_{i=1}^{n} \lambda_i u_i.$$

Theorem 3.13. *The relation defined in Definition 3.12 is a dependence relation.*

Proof. (D1) Let $v \in M$. Then since $v \in \{v\} = v + 0v$, we have that $v \lhd M$.

(D2) Suppose that $w \lhd M$ and $v \lhd N$ for all $v \in M$, where M and N are subsets of Q. Then $w \in \sum_{i=1}^{n_i} \lambda_i v_i$ for some $v_i \in M$ and $\lambda_i \in A$, $i \in \{1, \ldots, n\}$, and so, for each $i \in \{1, \ldots, n\}$, $v_i \in \sum_{j=1}^{n} \eta_{ji} u_{ji}$, where $u_{ij} \in N$ and $\eta_{ij} \in A$ for all $j \in \{1, \ldots, n_i\}$. Now $\lambda_i v_i \in \lambda_i \sum_{j=1}^{n_i} \eta_{ji} u_{ji}$ and thus $\sum_{i=1}^{n_i} \lambda_i v_i \subseteq \sum_{i=1}^{n} \lambda_i (\sum_{j=1}^{n_i} \eta_{ji} u_{ji})$.
Thus $w \in \sum_{i=1}^{n} \lambda_i (\sum_{j=1}^{n_i} \eta_{ji} u_{ji}) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \lambda_i \eta_{ji} u_{ji}$, so that $w \lhd N$.

(D3) Let $v \lhd M$ and $v \ntriangleleft M \setminus \{u\}$. Then $v \in \sum_{i=1}^{n} \lambda_i u_i$ where $u_i \in M$ for $i \in \{1, \ldots, n\}$. Since $v \ntriangleleft M \setminus \{u\}$, we must have that u is equal to one of the u_i. To see this, suppose it is not the case, then $\{u_1, \ldots, u_n\} \subseteq M \setminus \{u\}$ and $v \in \sum_{i=1}^{n} \lambda_i u_i$, so $v \lhd M \setminus \{u\}$, a contradiction. Suppose then, without loss of generality, that $u = u_1$. Then $v \in \lambda_1 u + \sum_{i=2}^{n} \lambda_i u_i$. So there exists $x \in \sum_{i=2}^{n} \lambda_i u_i$ such that $v \in \lambda_1 u + x = x + \lambda_1 u$. Thus $\lambda_1 u \in (-x) + v$ by the reversibility property. This implies that $u \in \lambda_1^{-1}(-x + v) \subseteq \lambda^{-1}(-\sum_{i=2}^{n} \lambda_i u_i + v) = \sum_{i=2}^{n} -\lambda_1^{-1} \lambda_i u_i + \lambda_1^{-1} v$. Thus $\{u_2, \ldots, u_n\} \subseteq M \setminus \{u\}$, so that $\{u_2, \ldots, u_n\} \cup \{v\} \subseteq (M \setminus \{u\}) \cup \{v\}$.

□

We can now use André's notion of independence, (see [2]) i.e. we say that a subset M of Q is *independent* if and only if for all $x \in M$ we have that $x \ntriangleleft M \setminus \{x\}$.

The next result will be useful.

Lemma 3.14. *A subset M of Q is independent if and only if for all $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in M$ with $u_i \neq u_j$ when $i \neq j$ and $\lambda_i \in A$ for $i \in \{1, \ldots, n\}$ if*

$$0 \in \sum_{i=1}^{n} \lambda_i u_i,$$

then $\lambda_i = 0$ for $i \in \{1, \ldots, n\}$.

Proof. Suppose that $M \subseteq Q$ is independent and that $0 \in \sum_{i=1}^{n} \lambda_i u_i$ where $u_i \in M$ and $\lambda_i \in A$ for $i \in \{1, \ldots, n\}$. Assume, without loss of generality, that $\lambda_1 \neq 0$. Then $0 \in \lambda_1 u_1 + \sum_{i=2}^{n} \lambda_i u_i$. Thus $-\lambda_1 u_1 \in \sum_{i=2}^{n} \lambda_i u_i$, otherwise $0 \notin \sum_{i=1}^{n} \lambda_i u_i$. Then we have that $u_1 \in (-\lambda_1)^{-1} \sum_{i=2}^{n} \lambda_i u_i$. Thus $u_1 \in \sum_{i=2}^{n} (-\lambda_1)^{-1} \lambda_i u_i$, so that $u_1 \lhd M \setminus \{u_1\}$, a contradiction. Conversely, suppose that $M \subseteq Q$ such that for any $u_1, \ldots, u_j \in M$ with $u_i \neq u_j$ when $i \neq j$ we have that $0 \in \sum_{i=1}^{n} \lambda_i u_i$ implies that $\lambda_i = 0$ for $i \in \{1, \ldots, n\}$. Let $x \in M$ and suppose that $x \lhd M \setminus \{x\}$, then there exist $u_1, \ldots, u_n \in M \setminus \{x\}$ and $\lambda_i \in A$ such that $x \in \sum_{i=1}^{n} \lambda_i u_i$. Then $0 \in \sum_{i=1}^{n} \lambda_i u_i - x = \sum_{i=1}^{n} \lambda_i u_i + (-1)x$, a contradiction since $-1 \neq 0$. □

We define a *basis* for a hyper near-vector space in the standard way, as in [2], from the above dependence relation, i.e. it is an independent generating set for the quasi-kernel. As for near-vector spaces, we show that every vector in the hyper near-vector space has a unique representation in terms of the basis elements.

Lemma 3.15. *Let (V, A) be a hyper near-vector space, and let $B = \{u_i \mid i \in I\}$ be a basis of Q. Then each $u \in V$ is an element of a unique linear combination of elements of B, i.e. there exists $\lambda_i \in F$, with $\lambda_i \neq 0$ for at most a finite number of $i \in I$, which are uniquely determined by u and B, such that*

$$u \in \sum_{i \in I} \lambda_i u_i.$$

Proof. Since $\langle Q(V) \rangle = V$, we know that there exists $x_1, \ldots, x_n \in Q(V)$ such that $u \in \sum_{j=1}^{n} x_j$. Since B is a basis for $Q(V)$, it follows for all $j \in \{1, \ldots, n\}$, $x_j \lhd B$, so that $x_j \in \sum_{i \in I} \lambda_{ij} u_i$, where $\lambda_{ij} \in A$ for all $i \in I$. It follows

$$
\begin{aligned}
u &\in \sum_{j=1}^{n} x_j \\
u &\in \sum_{j=1}^{n} \sum_{i \in I} \lambda_{ij} u_i \\
u &\in \sum_{i \in I} (\lambda_{i1} u_i + \ldots + \lambda_{in} u_i) \\
u &\in \sum_{i \in I} (\lambda_{i1} +_{u_i} \ldots +_{u_i} \lambda_{in}) u_i
\end{aligned}
$$

It follows there exists $\eta_i \in \lambda_{i1} +_{u_i} \ldots +_{u_i} \lambda_{in}$ such that $u \in \sum_{i \in I} \eta_i u_i$.

For uniqueness, suppose that $u \in \sum_{i \in I} \lambda_i u_i = \sum_{i \in I} \lambda'_i u_i$ for the index set I. Then $0 \in u + (-u) \subseteq \sum_{i \in I} \lambda_i u_i - \sum_{i \in I} \lambda'_i u_i = \sum_{i \in I} A_i u_i$ where $A_i = \lambda_i +_{u_i} (-\lambda'_i) \subseteq A$.

Thus there exists $\eta_i \in A_i$ such that $0 \in \sum_{i \in I} \eta_i u_i$. This implies that $\eta_i = 0$ for all $i \in I$. It follows for each $i \in I$, $0 \in \lambda_i +_{u_i} (-\lambda'_i)$, i.e. $-\lambda'_i$ is the unique inverse of λ_i. Thus for each $i \in I$, $\lambda_i = \lambda'_i$. □

The unique linear combination above is referred to as the *decomposition* of u in terms of a basis B of Q. A basis B of Q is referred to as a basis of V, as B generates V. The following result is an analogue of Lemma 3.2 in [2].

Lemma 3.16. *Let (V, A) be a hyper near-vector space with basis $B = \{b_i \mid i \in I\}$, and let $\lambda_i \in A^*$ for all $i \in I$. Then $B' = \{\lambda_i b_i \mid i \in I\}$ is a basis of V.*

Proof. Suppose there exists $\eta_i \in A$ such that $0 \in \sum_{i \in I} \eta_i(\lambda_i b_i) = \sum_{i \in I} (\eta_i \lambda_i) b_i$. Then, since B is independent, $\eta_i \lambda_i = 0$ for all $i \in I$, and thus $\eta_i = \eta_i \lambda_i \lambda_i^{-1} = 0 \lambda_i^{-1} = 0$ for all $i \in I$. Hence B' is independent. Furthermore, if $x \in Q$ has decomposition $x \in \sum_{i \in I} \alpha_i b_i$, then $x \in \sum_{i \in I} \alpha_i (\lambda_i^{-1} \lambda_i) b_i = \sum_{i \in I} (\alpha_i \lambda_i^{-1}) \lambda_i b_i$, so that $x \lhd B'$, hence B' generates Q (and therefore V). It follows B' is a basis for Q. □

André showed that every near-vector space (V, A) with basis $B = \{b_i \mid i \in I\}$ is isomorphic to the set of families $(x_i)_{i \in I}$ where x_i is in A for each $i \in I$ and $x_i = 0$ for some cofinite subset of I. We prove the analogue in the next result.

Theorem 3.17. *Let (V, A) be a hyper–near-vector space with basis $B = \{b_i \mid i \in I\}$. Let*

$$A^{(I)} = \{(\lambda_i)_{i \in I} \mid 0 \neq \lambda_i \in A \text{ for at most finitely many } i \in I\}.$$

For $(\alpha_i)_{i \in I}, (\beta_i)_{i \in I} \in A^{(I)}$, define $(\alpha_i)_{i \in I} + (\beta_i)_{i \in I} = \{(\gamma_i)_{i \in I} \mid \gamma_i \in \alpha_i +_{b_i} \beta_i\}$ and $\lambda(\alpha_i)_{i \in I} = (\lambda \alpha_i)_{i \in I}$. Then $V \cong A^{(I)}$.

Proof. Define $\phi : V \to A^{(I)}$ so that, if $v \in \sum_{i \in I} \lambda_i b_i$, with at most finitely many λ_i's nonzero, then $\phi(v) = (\lambda_i)_{i \in I}$. Take $v, w \in V$ with decompositions $v \in \sum_{i \in I} \lambda_i b_i$ and $w \in \sum_{i \in I} \eta_i b_i$. Let $u \in v + w$. Then $u \in \sum_{i \in I} \lambda_i b_i + \sum_{i \in I} \eta_i b_i = \sum_{i \in I} (\lambda_i +_{b_i} \eta_i) b_i$, so that $u \in \sum_{i \in I} \gamma_i b_i$, for some $\gamma_i \in \lambda_i +_{b_i} \eta_i$. It follows $\phi(u) = (\gamma_i)_{i \in I} \in (\lambda_i)_{i \in I} + (\eta_i)_{i \in I} = \phi(v) + \phi(w)$, so that $\phi(v + w) \subseteq \phi(v) + \phi(w)$. Conversely, suppose $(\gamma_i)_{i \in I} \in (\lambda_i)_{i \in I} + (\eta_i)_{i \in I} = \phi(v) + \phi(w)$. Then

$\gamma_i \in \lambda_i +_{b_i} \eta_i$, so that $\sum_{i\in I} \gamma_i b_i \subseteq \sum_{i\in I}(\lambda_i +_{b_i} \eta_i) b_i = v+w$. Let $u \in \sum_{i\in I} \gamma_i b_i$. Then $u \in v+w$ and $\phi(u) = (\gamma_i)_{i\in I}$. It follows $(\gamma_i)_{i\in I} = \phi(u) \in \phi(v+w)$, so that $\phi(v)+\phi(w) = \phi(v+w)$.

Next, note that $\lambda w \in \lambda \sum_{i\in I} \eta_i b_i = \sum_{i\in I} \lambda\eta_i b_i$, so that $\phi(\lambda w) = (\lambda\eta_i)_{i\in I} = \lambda(\eta_i)_{i\in I} = \lambda\phi(w)$.

Finally, to show ϕ is surjective, for any $(\alpha_i)_{i\in I} \in A^{(I)}$, let $u \in \sum_{i\in I} \alpha_i b_i$, then $\phi(u) = (\alpha_i)_{i\in I}$. For injectivity, suppose $\phi(v) = \phi(w)$. Then $(0)_{i\in I} \in \phi(v) - \phi(w) = \phi(v-w)$. Let $x \in V \setminus \{0\}$, then $x = \sum_{i\in I} \lambda_i b_i$ such that $\lambda_j \neq 0$ for some $j \in I$. It follows $\phi(x) = (\lambda_i)_{i\in I} \neq (0)_{i\in I}$, since $\lambda_j \neq 0$. Therefore $\ker\phi = \{0\}$, so that $0 \in v-w$, hence $v = w$. □

Corollary 3.18. *Let (V, A) be a hyper–near-vector space with basis $B = \{b_i \mid i \in I\}$. Suppose $x, y \in V$ such that x and y have the same decomposition in terms of B, i.e. $x, y \in \sum_{i\in I} \lambda_i b_i$ for some $\lambda_i \in A$ for each $i \in I$. Then $x = y$.*

Proof. Take $\phi : V \to A^{(I)}$ from the previous theorem. Then $\phi(x) = \phi(y) = (\lambda_i)_{i\in I}$. Since ϕ is injective, this implies $x = y$. □

The above result reveals more: suppose $U = \{u_1, \dots, u_n\}$ is independent, and consider the sum $\sum_{i=1}^{n} \lambda_i u_i$. Since U is independent, it is contained in a basis B of Q, and hence $\sum_{i=1}^{n} \lambda_i u_i$ is the decomposition of some unique element by the corollary above, i.e. $\sum_{i=1}^{n} \lambda_i u_i = \{x\}$. It therefore is clear that any *independent sum* (a linear combination of independent elements of $Q(V)$) contains exactly one element.

3.4 Compatibility and regularity

Regularity and compatibility are central to the study of near-vector spaces. We define these below and develop the theory as André does in [2].

Definition 3.19. Let (V, A) be a hyper near-vector space. The elements u, v of Q^* are called *compatible* (u cp v) if there exists a $\lambda \in A^*$ such that $+_u = +_{\lambda v}$.

We note that for a near-vector space (V, A), two vectors $u, v \in Q^*$ are said to be compatible if there exists a $\lambda \in A^*$ such that $u + \lambda v \in Q$ and it is shown that this is equivalent to there existing a $\lambda \in A^*$ such that $+_u = +_{\lambda v}$. This is not the case for hyper near-vector spaces. Referring back to Example 3.4, a cp b, but $a \oplus b = c \notin Q^*$, so we do not have the second statement. We will motivate our choice of definition a bit later in the paper.

Next we show that compatibility induces an equivalence relation on Q^*, a fact that becomes central to the proof of the Decomposition Theorem, as we will see.

Lemma 3.20. *The compatibility relation cp is an equivalence relation on Q^*.*

Proof. (i) Reflexivity

It is clear that for all $u \in Q^*$, we have that $+_u = +_{1u}$.

(ii) Symmetry

Suppose that $+_u = +_{\lambda v}$ for $\lambda \in A^*$. Now let $\alpha, \beta \in A$, then

$$
\begin{aligned}
(\alpha +_{\lambda^{-1}u} \beta)\lambda^{-1}u &= \alpha\lambda^{-1}u + \beta\lambda^{-1}u \\
&= (\alpha\lambda^{-1} +_u \beta\lambda^{-1})u \\
&= (\alpha\lambda^{-1} +_{\lambda v} \beta\lambda^{-1})u.
\end{aligned}
$$

Thus, since $u \neq 0$, we have that $(\alpha +_{\lambda^{-1}u} \beta)\lambda^{-1} = \alpha\lambda^{-1} +_{\lambda v} \beta\lambda^{-1}$. Next we have,

$$\begin{aligned}(\alpha\lambda^{-1} +_{\lambda v} \beta\lambda^{-1})\lambda v &= \alpha\lambda^{-1}\lambda v + \beta\lambda^{-1}\lambda v \\ &= \alpha v + \beta v \\ &= (\alpha +_v \beta)v.\end{aligned}$$

Thus by the fixed point free property, $(\alpha\lambda^{-1} +_{\lambda v} \beta\lambda^{-1})\lambda = \alpha +_v \beta$. Hence we finally have that $\alpha +_{\lambda^{-1}u} \beta = [(\alpha +_{\lambda^{-1}u} \beta)\lambda^{-1}]\lambda = (\alpha\lambda^{-1} +_{\lambda v} \beta\lambda^{-1})\lambda = \alpha +_v \beta$.

(iii) Transitivity
Suppose that $+_v = +_{\lambda u}$ and $+_u = +_{\lambda' w}$ for $\lambda, \lambda' \in A^*$. Then since $+_v = +_{\lambda u}$, we have that $+_{\lambda^{-1}v} = +_u = +_{\lambda' w}$, so that $+_v = +_{\lambda\lambda' w}$.

□

We give a second example of a hyper near-vector space.

Example 3.21. Let $X = \{0, 1\}$ with the hyperoperation $+_X$ defined as follows:

$+_X$	0	1
0	0	1
1	1	X

It is not difficult to verify that $(X, +_X)$ is a canonical hypergroup.
Take $V = X \times \mathbb{Z}_2$, with $\oplus$ defined for all $(a, b), (a', b') \in V$, by

$$(a, b) \oplus (a', b') = \{(x, y) | x \in a +_X a' \text{ and } y \in b +_{\mathbb{Z}_2} b'\}.$$

We then have the following table for $(V, \oplus)$:

$\oplus$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$\{(0,0),(1,0)\}$	$\{(0,1),(1,1)\}$
$(1,1)$	$(1,1)$	$(1,0)$	$\{(0,1),(1,1)\}$	$\{(1,0),(0,0)\}$

$(V, \oplus)$ is a canonical hypergroup and if we take $A = \{0, 1\}$, then since $-1 = 1$, we have that (V, A) is a hyper near-vector space. A quick check shows that $Q(V) = \{(0, 0), (0, 1), (1, 0)\}$. In addition, $+_{(0,1)} \neq +_{(1,0)}$, since $+_{(0,1)} = +_{\mathbb{Z}_2}$, while $+_{(1,0)} = +_X$. Thus $(0, 1)$ is not compatible with $(1, 0)$. We note that (V, A) is not a weak hyper vector space, and therefore also not a hyper vector space.

Lemma 3.22. *Let (V, A), (W, A) be hyper near-vector spaces and $\phi : V \to W$ a good homomorphism. Let $u, v \in Q(V)^*$. Then the following properties hold.*

(i) If $W = V$, then $\phi(u)$ cp u if and only if $\phi(u) \neq 0$.

(ii) $\phi(u)$ cp $\phi(v)$ if and only if $\phi(u) \neq 0 \neq \phi(v)$ and u cp v.

Proof. By Theorem 3.6 we know $\phi(u), \phi(v) \in Q(W)$.

(i) Suppose $V = W$, and suppose u cp $\phi(u)$. Then $\phi(u) \neq 0$, since cp is an equivalence relation on $Q(V)^*$. Conversely, suppose $\phi(u) \neq 0$ and let $\alpha, \beta \in A$. Then the following holds.

$$\begin{aligned} \alpha u + \beta u &= (\alpha +_u \beta)u \\ \phi(\alpha u + \beta u) &= \phi((\alpha +_u \beta)u) \\ \alpha\phi(u) + \beta\phi(u) &= (\alpha +_u \beta)\phi(u) \\ (\alpha +_{\phi(u)} \beta)\phi(u) &= (\alpha +_u \beta)\phi(u) \end{aligned}$$

Since $\phi(u) \neq 0$, it follows by the fixed-point-free property that $\alpha +_{\phi(u)} \beta = \alpha +_u \beta$. Hence $+_{\phi(u)} = +_u$, so that $\phi(u)$ cp u.

(ii) Suppose $\phi(u)$ cp $\phi(v)$. Then $\phi(u) \neq 0 \neq \phi(v)$, since cp is an equivalence relation on $Q(W)^*$. Let $\lambda \in A^*$ such that $+_{\phi(u)} = +_{\lambda\phi(v)}$. Then by the same argument as above we have $+_u = +_{\phi(u)}$ and $+_{\lambda v} = +_{\phi(\lambda v)} = +_{\lambda\phi(v)}$, hence $+_u = +_{\lambda v}$ so that u cp v.

Conversely, if u cp v and $\phi(u) \neq 0 \neq \phi(v)$, let $\lambda \in A^*$ such that $+_u = +_{\lambda v}$. Then once again we have $+_{\phi(u)} = +_u$ and $+_{\lambda\phi(v)} = +_{\phi(\lambda v)} = +_{\lambda v}$, hence $+_{\phi(u)} = +_{\lambda\phi(v)}$. It follows that $\phi(u)$ cp $\phi(v)$.

□

The next result shows that each vector in the quasi-kernel is compatible with each basis vector in its decomposition.

Lemma 3.23. *Let (V, A) be a hyper near-vector space and let $u_1, \ldots, u_n$ be independent elements in Q. Let $u \in \sum_{i=1}^{n} \lambda_i u_i$ such that $u \in Q$ for some $\lambda_1, \ldots, \lambda_n \in A^*$. Then u cp u_i for all $i \in \{1, \ldots, n\}$.*

Proof. Let $\alpha, \beta \in A$. Since $u \in Q$ we know that there exists $A' \subseteq A$ such that $\alpha u + \beta u = A'u$. We know that, since $\lambda_1, \ldots, \lambda_n$ are nonzero and $u_1, \ldots, u_n$ are independent, u is nonzero, so $A' = \alpha +_u \beta$ is uniquely defined by α and β. Now, let $\gamma \in A'$, then:

$$\begin{aligned} &\gamma u \in \alpha u + \beta u \\ &0 \in \alpha u + \beta u - \gamma u \\ &0 \in \alpha \sum_{i=1}^{n} \lambda_i u_i + \beta \sum_{i=1}^{n} \lambda_i u_i - \gamma \sum_{i=1}^{n} \lambda_i u_i \\ &0 \in \sum_{i=1}^{n} (\alpha\lambda_i u_i + \beta\lambda_i u_i - \gamma\lambda_i u_i) \\ &0 \in \sum_{i=1}^{n} (\alpha +_{\lambda_i u_i} \beta +_{\lambda_i u_i} (-\gamma))\lambda_i u_i \end{aligned}$$

It follows that there exists $\eta_1, \ldots, \eta_n \in A$ such that $\eta_i \in (\alpha +_{\lambda_i u_i} \beta +_{\lambda_i u_i} (-\gamma))$ and $0 \in \sum_{i=1}^{n} \eta_i \lambda_i u_i$. But then $\eta_i \lambda_i = 0$ for each $i \in \{1, \ldots, n\}$ by Theorem 3.14, and hence $\eta_i = 0$ for each $i \in \{1, \ldots, n\}$. Now, since $0 \in (\alpha +_{\lambda_i u_i} \beta +_{\lambda_i u_i} (-\gamma))$, it follows that $\gamma \in \alpha +_{\lambda_i u_i} \beta$ for each i. Hence $A' \subseteq \alpha +_{\lambda_i u_i} \beta$.

Conversely, suppose without loss of generality $\alpha +_{\lambda_1 u_1} \beta \not\subseteq A'$. We know that $u \in \sum_{i=1}^{n} \lambda_i u_i$, so $\lambda_1 u_1 \in \sum_{i=2}^{n} \lambda_i u_i - u$. If $\{u_2, \ldots, u_n, u\}$ is independent, it follows that $\alpha +_{\lambda_1 u_1} \beta \subseteq \alpha +_u \beta = A'$, contradicting the assumption. Let $\eta, \eta_2 \ldots, \eta_n \in A$ such that $0 \in \eta u + \sum_{i=2}^{n} \eta_i u_i$. Then

$$0 \in \eta u + \sum_{i=2}^{n} \eta_i u_i$$

$$0 \in \eta \sum_{i=1}^{n} \lambda_i u_i + \sum_{i=2}^{n} \eta_i u_i$$

$$0 \in \eta \lambda_1 u_1 + \sum_{i=2}^{n} (\eta \lambda_i u_i + \eta_i u_i)$$

$$0 \in \eta \lambda_1 u_1 + \sum_{i=2}^{n} (\eta \lambda_i +_{u_i} \eta_i) u_i$$

It follows there exist $\xi_2, \ldots \xi_n \in A$ such that $\xi_i \in \eta\lambda_i +_{u_i} \eta_i$ for all $i \in \{2, \ldots n\}$ and $0 \in \eta\lambda_1 u_1 + \sum_{i=2}^{n} \xi_i u_i$. But since $u_1 \ldots, u_n$ are independent, it follows $\eta\lambda_1 = \xi_i = 0$ for all $i \in \{2, \ldots, n\}$, so that $\eta = 0$ and $0 \in \eta\lambda_i +_{u_i} \eta_i = 0 +_{u_i} \eta_i = \{\eta_i\}$, hence $\eta_i = 0$ for all $i \in \{2, \ldots, n\}$. It follows $u, u_2, \ldots, u_n$ are independent, so $\alpha +_{\lambda_1 u_1} \beta \subseteq \alpha +_u \beta = A'$, a contradiction. Hence $\alpha +_u \beta = \alpha +_{\lambda_i u_i} \beta$ for all $\alpha, \beta \in A$, and so $+_u = +_{\lambda_i u_i}$ for all $i \in \{1, \ldots, n\}$, hence u cp u_i for all $i \in \{1 \ldots, n\}$. □

We now define regularity.

Definition 3.24. Let (V, A) be a hyper near-vector space. V is said to be *regular* if every pair of nonzero elements of the quasi-kernel are compatible.

Example 3.25. Referring back to Examples 3.4 and 3.21, Theorem 3.4 is regular and Theorem 3.21 is non-regular.

As with near-vector spaces, we can prove that regularity is determined by the regularity of the basis elements.

Theorem 3.26. *A near vector space V is regular if and only if there exists a basis which consists of mutually pairwise compatible vectors.*

Proof. Suppose V is regular. Then, by definition, any two vectors of Q^* are compatible. Therefore, every basis of Q consists of mutually pairwise compatible vectors.
Conversely, suppose there exists a basis $B = \{u_i | i \in I\}$ of mutually pairwise compatible vectors. Let $u, v \in Q^*$, then $u \in \sum_{i=1}^{n} \lambda_i u_i$ with $u_i \in B$ for some $\lambda_1, \ldots, \lambda_n \in A$ and $v \in \sum_{i=1}^{n} \eta_i u_i$ with $u_i \in B$ for some $\eta_1, \ldots, \eta_n \in A$. Since the u_i for $i \in \{1, \ldots, n\}$ are independent, we can apply Lemma 3.23. Thus u is compatible to each u_i for $i \in \{1, \ldots, n\}$. Similarly, v is compatible to each u_i for $i \in \{1, \ldots, n\}$. Thus by the transitivity of the compatibility relation, we have that u is compatible to v. □

The following result is an analogue of Theorem 4.2 in [2].

Theorem 3.27. *Let (V, A) be a near-vector space and $u \in Q^*$. Let F be the nearfield defined by $(F, +, \cdot) = (A, +_u, \circ)$. Then V is regular if and only if $V \cong F^{(I)}$, as defined in Theorem 3.17, with $(\alpha_i)_{i \in I} + (\beta_i)_{i \in I} = \{(\gamma_i)_{i \in I} \,|\, \gamma_i \in \alpha_i +_u \beta_i\}$ and $\lambda(\alpha_i)_{i \in I} = (\lambda\alpha_i)_{i \in I}$.*

Proof. Suppose V is regular, then there exists a basis $B = \{b_i \,|\, i \in I\}$ of V such that $u \in B$. Since V is regular, u cp b_i for all $i \in I$, therefore there exists $\lambda_i \in A^*$ such that $+_u = +_{\lambda_i b_i}$ for all $i \in I$. Therefore by Theorem 3.17, $V \cong A^{(I)}$ with $(\alpha_i)_{i\in I} + (\beta_i)_{i\in I} = \{(\gamma_i)_{i\in I} \,|\, \gamma_i \in \alpha_i +_{\lambda_i b_i} \beta_i\}$ for all $(\alpha_i)_{i\in I} + (\beta_i)_{i\in I} \in A^{(I)}$. But since $+_u = +_{\lambda_i b_i}$, it follows $(\alpha_i)_{i\in I} + (\beta_i)_{i\in I} = \{(\gamma_i)_{i\in I} \,|\, \gamma_i \in \alpha_i +_u \beta_i\}$ for all $(\alpha_i)_{i\in I} + (\beta_i)_{i\in I} \in A^{(I)}$, so that $A^{(I)} = F^{(I)}$, and so $V \cong F^{(I)}$.

Conversely, suppose $V \cong F^{(I)}$. Let $\phi : V \mapsto F^{(I)}$ be an isomorphism and let $b_j = \phi^{-1}((\delta_{ij})_{i\in I})$ for all $j \in I$, where δ_{ij} is the Kronecker delta symbol. Then $b_j \in Q(V)$ and $+_{b_j} = +_u$ for all $j \in I$:

$$\begin{aligned}
\alpha b_j + \beta b_j &= \alpha\phi^{-1}((\delta_{ij})_{i\in I}) + \beta\phi^{-1}((\delta_{ij})_{i\in I}) \\
&= \phi^{-1}(\alpha(\delta_{ij})_{i\in I}) + \phi^{-1}(\beta(\delta_{ij})_{i\in I}) \\
&= \phi^{-1}((\alpha\delta_{ij})_{i\in I} + (\beta\delta_{ij})_{i\in I}) \\
&= \phi^{-1}(\alpha\delta_{ij} +_u \beta\delta_{ij})_{i\in I}) \\
&= \phi^{-1}(((\alpha +_u \beta)\delta_{ij})_{i\in I}) \text{ (Since } \delta_{ij} \in \{0,1\}\text{, it satisfies the right distributive law.)} \\
&= \phi^{-1}((\alpha +_u \beta)(\delta_{ij})_{i\in I}) \\
&= (\alpha +_u \beta)\phi^{-1}((\delta_{ij})_{i\in I}) \\
&= (\alpha +_u \beta) b_j.
\end{aligned}$$

Moreover, $B = \{b_i \,|\, i \in I\}$ is a basis for V. To see this, if $0 \in \sum_{j\in I} \lambda_j b_j$, then $(0)_{j\in I} = \phi(0) \in \phi(\sum_{j\in I} \lambda_j b_j) = \sum_{j\in I} \lambda_j \phi(b_j) = \sum_{j\in I} \lambda_j (\delta_{ij})_{i\in I}) = \{(\lambda_i)_{i\in I}\}$, hence $\lambda_j = 0$ for all $j \in I$ and B is independent. Furthermore, if $x \in Q$ and $\phi(x) = (\eta_i)_{i\in I}$, then $\phi(x) \in \sum_{j\in I} \eta_j (\delta_{ij})_{i\in I} = \phi(\sum_{i\in I} \eta_i b_i)$, and so $x \in \sum_{i\in I} \eta_i b_i$, since ϕ is injective. It follows $x \triangleleft B$ and thus B generates Q (and therefore V). Hence B is a basis consisting of mutually pairwise compatible vectors, so that V is regular by Theorem 3.26. □

The above result motivates our choice of definition for compatibility. Referring back to Example 3.4, we have a basis of mutually compatible vectors, namely $B = \{a, b\}$. Should we have chosen the alternative definition, these two would not be compatible and so the hyper near-vector space would not be regular. This would not correspond to the above result, as $V \cong X^2$ where X is defined as in Theorem 3.21.

3.5 Subhyperspaces of V and the Decomposition Theorem

Next we define the notion of a subhyperspace, the final missing requirement to prove an analogue of the Decomposition Theorem.

Definition 3.28. If (V, A) is a hyper near-vector space and $\emptyset \neq V' \subseteq V$ is such that V' is the canonical subhypergroup of $(V, +)$ generated additively by $AX = \{ax \,|\, x \in X, a \in A\}$, where X is an independent subset of $Q(V)$, then we say that (V', A) is a *subhyperspace* of (V, A), or simply V' is a *subhyperspace* of V if A is clear from the context.

If $(V, +)$ is generated additively by AX, we will write $V = \langle AX \rangle$.

Lemma 3.29. *Let (V, A) be a hyper near-vector space and V' be a subhyperspace of V. Then $Q(V') = V' \cap Q(V)$.*

Proof. Suppose $v \in V' \cap Q(V)$, then $v \in V'$ and $v \in Q(V)$, so that for all $\alpha, \beta \in Q(V)$, $\alpha v + \beta v \subseteq Av$. It follows $v \in Q(V')$.

Conversely, suppose $v \in Q(V')$. Then $v \in V'$ and for all $\alpha, \beta \in A$, $\alpha v + \beta v \subseteq Av$. It follows $v \in Q(V)$ and so $v \in Q(V) \cap V'$. □

Corollary 3.30. *Let (V,A) be a hyper near-vector space, and suppose U and W are subhyperspaces of V. Then $U \subseteq W$ if and only if $Q(U) \subseteq Q(W)$.*

Proof. Suppose $U \subseteq W$, then $Q(V) \cap U \subseteq Q(V) \cap W$, hence $Q(U) \subseteq Q(W)$. Conversely, if $Q(U) \subseteq Q(W)$, let $X \subseteq Q(V)$ such that $U = \langle AX \rangle$. Then $X \subseteq Q(V) \cap U = Q(U) \subseteq Q(W)$, so that X is an independent subset of $Q(W)$. It follows there exists a basis X' for $Q(W)$ such that $X \subseteq X'$. Therefor $AX \subseteq AX'$, and hence $U = \langle AX \rangle \subseteq \langle AX' \rangle = W$. □

In the next proposition we prove when the union of two subhyperspaces will be a subhyperspace.

Proposition 3.31. *Let (V,A) be a hyper near-vector space and W_1, W_2 subhyperspaces of V. Then $W_1 \cup W_2$ is a subhyperspace of V if and only if $W_1 \subseteq W_2$ or $W_1 \subseteq W_2$.*

Proof. Suppose without loss of generality that $W_1 \subseteq W_2$, where $W_2 = \langle AX \rangle$ with X an independent subset of $Q(V)$. Then $W_1 \cup W_2 = W_2$ so we are done.
Conversely, suppose that $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$. Then there exist $x, y \in V$ such that $x \in W_1, y \in W_2, x \notin W_2, y \notin W_1$. Since $W_1 \cup W_2$ is assumed to be a subhyperspace, having $x + y \subseteq W_1 \cup W_2$ implies that for all $z \in x + y$, we have $z \in W_1 \cup W_2$. Without loss of generality, suppose that $z \in W_1$. Then $z \in x + y$ implies that $y \in z - x \subseteq W_1$, a contradiction.
□

We end off with the analogue of the Decomposition Theorem for hyper-near vector spaces. André proves in [2] that every near-vector space is isomorphic to the direct sum of its maximal regular subspaces. However, this result does not generalize to hyper near-vector spaces; in fact, the direct sum of hyper near-vector spaces is not defined in the category theoretical sense. Instead, we show that any finite-dimensional hyper near-vector space can be expressed as the direct product of its maximal regular subhyperspaces. This result does not generalize to arbitrary hyper near-vector spaces as arbitrary direct products are not defined even for near-vector spaces. First, we show that finite direct products are defined for hypersubspaces of hyper near-vector spaces.

Theorem 3.32. *Let (V,A) be a near-vector space, $I = \{1, \ldots, n\}$ and suppose $\{V_i \mid i \in I\}$ is a set of subhyperspaces of V. Define*

$$\prod_{i=1}^{n} V_i = \{(v_i)_{i \in I} \mid \forall i \in I\,[v_i \in V_i]\},$$

with addition defined as $(v_i)_{i\in I} + (w_i)_{i \in I} = \{(u_i)_{i\in I} \mid \forall i \in I[u_i \in v_i + w_i]\}$ and scalar multiplication defined componentwise. Then $(\prod_{i=1}^{n} V_i, A)$ is a hyper near-vector space, and it is a direct product of $\{V_i \mid i \in I\}$, with projection maps $\pi_j : \prod_{i=1}^{n} V_i \to V_j$ defined by $\pi_j((v_i)_{i\in I}) = v_j$ for all $j \in I$.

Proof. It is routine to show $\prod_{i=1}^{n} V_i$ is a hyper near-vector space with neutral element $(0)_{i\in I}$, and that π_i is a good homomorphism for each $i \in I$. To show that $\prod_{i=1}^{n} V_i$ is indeed the direct product of $\{V_1, \ldots, V_n\}$, let W be a hyper near-vector space, and let $f_i : W \to V_i$ be homomorphisms. Define $f : W \to \prod_{i=1}^{n} V_i$ such that $f(w) = (f_i(w))_{i\in I}$. Suppose for some $x + y \in W$ that $(u_i)_{i \in I} \in f(x+y)$. Then there exists some $w \in x + y$ such that $f(w) = (u_i)_{i\in I}$. It follows that $u_i = f_i(w) \in f_i(x+y) \subseteq f_i(x) + f_i(y)$, so that $(u_i)_{i\in I} \in f(x) + f(y)$. Furthermore, $f(\alpha w) = (f_i(\alpha w))_{i\in I} = (\alpha f_i(w))_{i \in I} = \alpha(f_i(w))_{i\in I} = \alpha f(w)$ for all $\alpha \in A$ and $w \in W$. Hence f is a homomorphism. Furthermore, $(\pi_j \circ f)(w) = \pi_j((f_i(w))_{i\in I}) = f_j(w)$, so that $\pi_j \circ f = f_j$ for all $j \in I$. Finally, to show uniqueness, suppose $g : W \to \prod_{i=1}^{n} V_i$ such that$\pi_j \circ g = f_j$ for all $j \in I$. Then, for $w \in W$, $g(w) = (f_i(w))_{i\in I} = f(w)$, so that $g = f$. □

Theorem 3.33. *Let (V, A) be a finite-dimensional hyper–near-vector space. Then V is isomorphic to the direct product of maximal regular subhyperspaces, with each $u \in Q^*$ being in exactly one of these maximal regular subhyperspaces.*

Proof. Let $\{Q_i \,|\, i \in I\}$ be the partition of Q^* into its compatible elements, and define $B_i = B \cap Q_i$, where B is a basis of V. Define $V_i = \langle AB_i \rangle$. By definition V_i is a subhyperspace of V with basis B_i. Since $B_i \subseteq Q_i$, it follows that B_i consists of mutally pairwise compatible vectors, so that V_i is a regular subhyperspace for all $i \in I$. Furthermore, if $V_i \subset W \subseteq V$, where W is a regular subhyperspace of V, then W has a basis of mutually pairwise compatible vectors (by Theorem 3.26) properly containing B_i and properly contained in B, a contradiction, since B_i contains all vectors of B that lie in the partition Q_i. Hence the V_i subhyperspaces are maximal.

Let $u \in Q^*$. Then, since Q^* is partitioned by Q_i's, $i \in I$, it follows that $u \in Q_j$ for exactly one $j \in I$. We wish to show that $u \in V_j$. Let $u \in \sum_{i=1}^{n} \lambda_i b_i$ for some $b_1, \ldots, b_n \in B$ and $\lambda_1, \ldots, \lambda_n \in A^*$. Then by Lemma 3.23, u cp b_i for each $i \in \{1, \ldots, n\}$. It follows that $b_i \in Q_j$ for each $i \in \{1, \ldots, n\}$, so $b_i \in B \cap Q_j = B_j$ for all $i \in \{1, \ldots, n\}$. It follows $u \in \sum_{i=1}^{n} \lambda_i b_i \subseteq \langle AB_j \rangle = V_j$.

Now, suppose $u \in V_k$ for some $k \in I$ such that $j \neq k$. Then, because the unique expression (by Lemma 3.15) for u in terms of the basis B is $u \in \sum_{i=1}^{n} \lambda_i b_i$, $b_1, \ldots, b_n \in B_k$, we have that $b_1, \ldots, b_n \in Q_k$ — a contradiction, since $b_1, \ldots, b_n \in Q_j$ and $Q_j \cap Q_k = \emptyset$. Hence u lies in exactly one V_i, $i \in I$.

Define now $f : \prod_{i=1}^{n} V_i \to V$ such that $f((u_i)_{i \in I}) \in \sum_{i \in I} u_i$. Since V is finite-dimensional, I is finite, so that the sum $\sum_{i \in I} u_i$ is defined. To show that f is well-defined, note that, for each $i \in I$, u_i is the unique element such that $u_i \in \sum_{b_{ij} \in B_i} \lambda_{ij} b_{ij}$ for some distinct $b_{ij} \in B_i$ and $\lambda_{ij} \in A$ (see paragraph below Theorem 3.18). It follows that $\sum_{i \in I} u_i = \sum_{i \in I} \sum_{b_{ij} \in B_i} \lambda_{ij} b_{ij}$. But $B_i \cap B_k = \emptyset$ for all $i, k \in I$ where $i \neq k$, so $\sum_{i \in I} \sum_{b_{ij} \in B_i} \lambda_{ij} b_{ij}$ is a linear combination of distinct basis elements and so contains only one element.

Now suppose $x \in f((u_i)_{i \in I} + (v_i)_{i \in I})$. It follows there exist $x_i \in u_i + v_i$ for all $i \in I$ such that $x = f((x_i)_{i \in I}) \in \sum_{i \in I} x_i \subseteq \sum_{i \in I} (u_i + v_i) = \sum_{i \in I} u_i + \sum_{i \in I} v_i = f((u_i)_{i \in I}) + f((v_i)_{i \in I})$. Conversely, if $x \in f((u_i)_{i \in I}) + f((v_i)_{i \in I})$, then $x \in \sum_{i \in I} u_i + \sum_{i \in I} v_i = \sum_{i \in I} (u_i + v_i)$, therefore for all $i \in I$ there exists $x_i \in u_i + v_i$ such that $x \in \sum_{i \in I} x_i$. But then $x = f((x_i)_{i \in I}) \in f((u_i)_{i \in I} + (v_i)_{i \in I})$. Finally, $f(\alpha (u_i)_{i \in I}) = f((\alpha u_i)_{i \in I}) \in \sum_{i \in I} \alpha u_i = \alpha \sum_{i \in I} u_i$. But $\alpha f((u_i)_{i \in I}) \in \alpha \sum_{i \in I} u_i$, so $f(\alpha (u_i)_{i \in I}) = \alpha f((u_i)_{i \in I})$, and so f is a good homomorphism.

Furthermore, f is bijective: if $u \in V$, suppose it has decomposition $\sum_{i \in I} \sum_{b_{ij} \in B_i} \lambda_{ij} b_{ij}$. Define u_i to be the unique element with decomposition $\sum_{b_{ij} \in B_i} \lambda_{ij} b_{ij}$. Then $u_i \in V_i$, and $f((u_i)_{i \in I}) \in \sum_{i \in I} u_i \subseteq \sum_{i \in I} \sum_{b_{ij} \in B_i} \lambda_{ij} b_{ij}$. But $u \in \sum_{i \in I} \sum_{b_{ij} \in B_i} \lambda_{ij} b_{ij}$, so $u = f((u_i)_{i \in I})$ and f is surjective. Furthermore, if $f((u_i)_{i \in I}) = f((v_i)_{i \in I})$, then $\sum_{i \in I} u_i = \sum_{i \in I} v_i$, so that $0 \in \sum_{i \in I} u_i - \sum_{i \in I} v_i = \sum_{i \in I} (u_i - v_i)$. It follows there exists $w_i \in u_i - v_i$ such that $0 \in \sum_{i \in I} w_i$. Let w_i have decomposition $\sum_{b_{ij} \in B_i} \lambda_{ij} b_{ij}$. Then $0 \in \sum_{i \in I} w_i \subseteq \sum_{i \in I} \sum_{b_{ij} \in B_i} \lambda_{ij} b_{ij}$. It follows $\lambda_{ij} = 0$ for all $i \in I$ and $b_{ij} \in B_i$. But then $w_i \in \sum_{b_{ij} \in B_i} = \{0\}$, so $w_i = 0$ for all $i \in I$. Hence $0 \in u_i - v_i$, and so $u_i = v_i$ for all $i \in I$. It follows $(u_i)_{i \in I} = (v_i)_{i \in I}$ and hence f is injective.

It follows f is bijective and a good homomorphism, so that f is an isomorphism. Hence $V \cong \prod_{i=1}^{n} V_i$.

□

The above decomposition is unique up to the order of the subhyperspaces, as will be shown in the next result.

Theorem 3.34. *Let V be a hyper near-vector space, and suppose*

$$\prod_{i=1}^{n} V_i \cong V \cong \prod_{j=1}^{m} V'_j$$

where V_i and V'_j are maximal regular subspaces for all $i \in \{1, \ldots, n\}$ and $j \in \{1 \ldots, m\}$. Then $m = n$ and there exists $\sigma \in S_n$ such that $V_i = V'_{\sigma(i)}$.

Proof. Let $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, m\}$, and let $Q_i = Q(V_i)^*$ for some $i \in I$. Then Q_i is a maximal set of compatible vectors of $Q(V)$. If not, there exists $u \in Q(V)^* \setminus Q_i$ such that u cp v for all $v \in Q_i$. But then $u \notin Q(V_i) = V_i \cap Q(V)$, so it follows $u \notin V_i$. But then $V_i = \langle Q_i \rangle \subsetneq \langle Q_i \cup \{u\} \rangle$, contradicting its maximality. It follows $Q_i \in Q^*/\text{cp}$. To show $\{Q_i \,|\, i \in I\} = Q^*/\text{cp}$, suppose $u \in Q^*$ such that $u \notin Q_i$ for any $i \in \{1, \ldots, n\}$. Since $V \cong \prod_{i=1}^{n} V_i$, it follows that there exists an isomorphism $\phi : V \to \prod_{i=1}^{n} V_i$. Let $\phi(u) = (u_1, \ldots, u_n)$, then $\pi_i(\phi(u)) = u_i$ for all $i \in I$.

Consider the sum $\sum_{i \in I} v_i$ for some $v_i \in V_i$ for each $i \in \{i \ldots, n\}$. Suppose v_i has decomposition $\sum_{j=1}^{m_i} \lambda_{ij} b_{ij}$ where $B_i = \{b_{i1}, \ldots, b_{im_i}\}$ is some independent subset of $Q(V_i)^* = Q_i$. Then $B = \{b_{ij} \,|\, i \in I, 1 \leq j \leq m_i\}$ is independent. If not, then there is some minimal dependent subset of B, say $B' = \{b_k \,|\, k \in K\}$, such that B' is dependent. It follows that there exist $k' \in K$ such that $b_{k'} \lhd B' \setminus \{b_{k'}\}$, i.e. there exist some $K' \subseteq K \setminus \{k'\}$ and $\lambda_k \in A^*$ for each $k \in K'$ such that $b_{k'} \in \sum_{k \in K'} \lambda_k b_k$. Since B' is a minimally dependent set, $\{b_k \,|\, k \in K'\}$ is independent, so that $b_{k'}$ cp b_k for each $k \in K'$ by Theorem 3.23. It follows there is some $i \in I$ such that $\{b_k \,|\, k \in K'\} \cup \{b_{k'}\} \subseteq B_i$ — a contradiction, since B_i is independent and therefore has no dependent subsets. Hence B is independent, and so $\sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij} b_{ij}$ contains a unique element, say v. But $\sum_{i=1}^{n} v_i \subseteq \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij} b_{ij} = \{v\}$, so $\sum_{i=1}^{n} v_i = \{v\}$.

Define then the map $f : V \to V$ such that, if $\pi_i(\phi(v)) = v_i$, then $f(v)$ is the unique element in $\sum_{i=1}^{n} v_i$.

We show f is a good homomorphism. Let $x, y, z \in V$ such that $x \in y + z$ and $\{f(x)\} = \sum_{i=1}^{n} x_i$, $\{f(y)\} = \sum_{i=1}^{n} y_i$ and $\{f(z)\} = \sum_{i=1}^{n} z_i$, where $x_i, y_i, z_i \in V_i$ for each $i \in \{1, \ldots, n\}$. Since $x \in y + z$, it follows $x_i = \pi_i(\phi(x)) \in \pi_i(\phi(y + z)) = \pi_i(\phi(y)) + \pi_i(\phi(z)) = y_i + z_i$. It follows $\{f(x)\} = \sum_{i=1}^{n} x_i \subseteq \sum_{i=1}^{n} (y_i + z_i) = \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i = f(y) + f(z)$. It follows $f(x) \in f(y) + f(z)$, so that $f(y + z) \subseteq f(y) + f(z)$.

Conversely, if $x \in f(y) + f(z)$, with $\{f(y)\} = \sum_{i=1}^{n} y_i$ and $\{f(z)\} = \sum_{i=1}^{n} z_i$, where $y_i, z_i \in V_i$ for each $i \in \{1, \ldots, n\}$. Then $x \in \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i = \sum_{i=1}^{n} (y_i + z_i)$. It follows there exists $x_i \in y_i + z_i$ such that $x \in \sum_{i=1}^{n} x_i$. Since $y_i, z_i \in V_i$, it follows $x_i \in V_i$. Moreover, $y_i + z_i = \pi_i(\phi(y)) + \pi_i(\phi(z)) = \pi_i(\phi(y + z))$, since π_i and ϕ are good homomorphisms. It follows $x_i \in \pi_i(\phi(y + z))$, so that $\{x\} = \sum_{i=1}^{n} x_i \subseteq \sum_{i=1}^{n} \pi_i(\phi(y + z)) = \bigcup \{\sum_{i=1}^{n} \pi_i(\phi(x')) \,|\, x' \in y + z\} = \{f(x') \,|\, x' \in y + z\} = f(y + z)$. Hence $x \in f(y + z)$, and so f is a good homomorphism.

Consider now $f(u)$. Since f is a good homomorphism, $f(u) \in Q(V)$. Suppose $f(u) \neq 0$. Then u cp $f(u)$ by Theorem 3.22. Let u_i have decomposition $\sum_{j=1}^{m_i} \lambda_{ij} b_{ij}$ where each $b_{ij} \in V_i$. Then $f(u) \in \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij} b_{ij}$. Since $f(u) \neq 0$, it follows $\lambda_{ij} \neq 0$ for at least one $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m_i\}$. Therefore $f(u)$ cp b_{ij} by Theorem 3.23. But then u cp $f(u)$ cp $b_{ij} \in Q(V_i)^* = Q_i$, so that $u \in Q_i$ — a contradiction. Suppose then that $f(u) = 0$. Then $u_i = 0$ for each $i \in \{1, \ldots, n\}$, so that $\phi(u) = (0, \ldots, 0)$. It follows $u = 0$, since ϕ is an isomorphism — a contradiction. Hence there is no element $u \in Q(V)^*$ such that $u \notin Q_i$ for each $i \in \{1, \ldots, n\}$. It follows that $\{Q_i \,|1 \leq i \leq n\} = Q^*/\text{cp}$.

By a symmetric argument, $\{Q'_j \,|\, j \in J\} = Q^*/\text{cp}$, so that $\{Q_i \,|\, i \in I\} = \{Q'_j \,|\, j \in J\}$. It follows $n = m$ and for each $i \in I$ there is some $j \in I$ such that $Q_i = Q'_j$, so that $V_i = \langle Q_i \rangle =$

$\langle Q'_j \rangle = V'_j$.

□

Example 3.35. Returning to Example 3.4 we have that $V \cong V$ (because V is regular) while for Example 3.21, we have that $V \cong \mathbb{Z}^2 \times X$.

4 Conclusion

In this paper we took a first look at hyper near-vector spaces and their properties. We defined a hyper near-vector space that is similar to André's near-vector space for a particular hyper generalisation of a vector space defined as in [1]. As a highlight, we prove that there is a Decomposition Theorem for finite dimensional spaces of this type into maximal regular subhyperspaces.

An interesting avenue for future work would be to study the geometric structure of these spaces. André's near vector spaces have a geometric origin in the so-called nearaffine spaces. It remains to be investigated whether this geometry can be generalized to the hyper near-vector spaces. Furthermore, since there is more than one way to define the notion of a hyper vector space, one further avenue for future exploration, would be to look at what the corresponding near-vector spaces should be.

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