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A CONSTRUCTION OF A LATTICE BY SUBSTITUTION SUM OF A LATTICE AND BOOLEAN ALGEBRA

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Abstract Formal Concept Analysis (FCA) is an applied branch of lattice theory, widely used in computer science that derives implicit relationships between objects described through a set of attributes on the one hand and these attributes on the other. Any finite lattice can be generated by a formal concept, which can be obtained from the formal context of objects and attributes. In this paper, given a finite lattice L with $|(L)| = |(L)| = n$ where n is the number of atoms in the Boolean lattice B_n, we construct a formal context obtained by substitution sum $\begin{array}{c|c} L & L \\ \hline \varnothing & B_n \end{array}$ and study the structural properties of the lattice $BS(L)$ generated by the above substitution sum. Interestingly, the lattice L and the Boolean lattice B_n remain as separate entities in the lattice $BS(L)$. Further, we have proved that the lattice $BS(L)$ is complemented.

1 Introduction

Formal Concept Analysis (FCA) is an important mathematical application of computer science that is highly used in knowledge representation, knowledge acquisition, linguistics, and data visualization [20], [13], [5]. It helps in processing a wide class of data types by providing a framework in which various data analysis techniques can be formulated. The formal context in FCA consists of a binary relation between the set of objects and the set of attributes. Every formal context $\mathbb{K}$ is isomorphic to $\langle J(L), M(L), I\rangle$ and every formal context generates a unique concept lattice [2].

The substitution sum and substitution product were introduced by Luksch and Wille for the concept analytic evaluation of pair comparison tests [2] and further described in detail by Stephan [8], [9]. Wille and Ganter further compiled all these various types of formal contexts and have characterized the corresponding concept lattices, one of them being the substitution sum in which a context of any lattice is placed in an empty cell of another context where there is no object attribute relation [2]. Doubling constructions by replacing the interval in a lattice by its direct product were first introduced by Alan Day to prove the Whitman's structure theorem for free lattices in a simpler manner [4]. It is also noted that, by deriving the decomposition algorithm, W. Geyer has described all the contexts that corresponds to the doubling constructions using convex sets [6]. Furthermore, he has given an algebraic characterization to these generated lattices. In their paper, K Bertet et al. have characterized the tableau for the $\mathcal{CN}$ class and then described an algorithm to recognize whether a lattice belongs to the class $\mathcal{CN}$of all lattices [1]. Furthermore, they have characterized the tableau and provided the recognition algorithm for the class of bounded lattices and distributive lattices.

Convex sublattice $CS(L)$ of the lattice L was first introduced by K.M. Koh [3], [10]. A new partial order $\leq$ on $CS(L)$ was defined by S. Lavanya et al. and they proved that the lattice L and the corresponding convex sublattice $CS(L)$ of L is in the same equational class [11]. Furthermore, corresponding to a congruence relation Θ on lattice L, Ramananda H.S. et al. defined a congruence relation ψ_θ on $CS(L)$ and proved that $CS(L/\theta) \cong CS(L)/\psi_\theta$ [12]. Also, they interestingly proved that L/Θ and $CS(L)/\psi_\theta$ are in the same equational class. Additionally, Ramananda H.S. has defined bounds on the cardinality of $CS(L)$ in terms of join irreducible elements [14]. By using the concept of substitution sum, the authors in their paper [16] have proved that the formal context obtained by substitution sum $\begin{array}{c|c} L & L \\ \hline L & \text{X} \end{array}$ generated the convex sublattice $CS(L)$ of a lattice L. Further, in the same paper, the authors have proved that the formal context $\begin{array}{c|c} L & L \\ \hline \varnothing & L \end{array}$ generates the lattice $TS(L)$ obtained by one-step doubling construction of the lattice L and discussed the structural properties of this lattice.

The association of a graph with algebraic structures such as rings, modules, or lattices has garnered significant attention in research. A graph known as the identity-filter graph is constructed by Shahabaddin Ebrahimi Atani for any lattice L, offering a characterization of lattices based on properties derived from these graphs [18]. Similarly, Vikas Kulal et al. have constructed the annihilator ideal graph for any lattice L, where the vertices represent non-trivial annihilator ideals of L. Two vertices, I and J, are connected if the annihilator of either I or J contains a non-zero element of the other, establishing a comprehensive understanding of lattice properties through graph theory [19]. Expanding upon this framework, the authors have constructed a graph of a lattice L with respect to an ideal I of L and obtained some results based on the adjacency matrices [15]. The authors have also obtained the determinant of the formal context of atomic amalgams of two Boolean algebras. These results are then leveraged to analyze the properties of adjacency matrices for complete graphs [17].

In this paper, notations and definitions of Lattice theory and Formal Concept Analysis used in the paper are given in Section 2. In Section 3, the structural properties of the lattice BS(L) are introduced and the characterization of the meet and join irreducible elements of BS(L) is given. The properties of the lattice BS(L) are discussed in Section 4.

2 Notations and Definitions

A partially ordered relation $P = (X, \leq)$ or poset is an ordering which is reflexive, antisymmetric, and transitive. A poset $(L, \leq)$ is a lattice if every pair of elements has a join $\vee$ and a meet $\wedge$. A distributive lattice in which every element x has a unique complement x' is called a Boolean Lattice.

We say that $a \prec b$ in L if an element a in L is covered by an element b. An element x that has a unique upper cover is called a *meet-irreducible element of* L. In particular if x is covered by x^+, then we denote it by $x \prec x^+$ and the set consisting of all meet irreducible elements of a lattice L is denoted by (L). Similarly, an element y that has a unique lower cover is called a *join-irreducible element in* L. In particular, if y^- is covered by y, then we denote it by $y^- \prec y$ and the set consisting of all join irreducible elements of a lattice L is denoted by (L). Every element that covers the minimum element 0 in a lattice is called the atom and is denoted as $A(L)$. Dually, every element in L that is covered by the maximum element 1 is called the coatom and is denoted as $CoA(L)$.(For further reading, refer[7]).

We now introduce the reader to some definitions of formal concept analysis used in this paper. For a better understanding of the definitions and examples, the reader may refer to Formal Concept Analysis by B.Ganter and R. Wille[2]. The formal context $\mathbb{K} := \langle G, M, I \rangle$ is a binary relation I between two sets G, the object set and M, the attribute set. gIm or $(g, m) \in I$ denotes that the object $g \in G$ has an attribute $m \in M$. Let $\mathbb{K}_1 := \langle G_1, M_1, I_1 \rangle$ and $\mathbb{K}_2 := \langle G_2, M_2, I_2 \rangle$ be the formal contexts such that $(g_1, m_1) \notin I_1$ in $\mathbb{K}_1$. Suppose that $G_2 \neq \emptyset \neq M_2$ and $G_1 \setminus g_1 \cap G_2 = \emptyset = (M_1 \setminus m) \cap M_2$. The substitution sum of $\mathbb{K}_1$ with $\mathbb{K}_2$ on (g_1, m_1) is defined to be the context $\mathbb{K}_1(g_1, m_1)\mathbb{K}_2 := \langle G_0, M_0, I_0 \rangle$ with $G_0 = (G_1 \setminus g_1) \cup G_2), M_0 = (M_1 \setminus m_1) \cup M_2, I_0 = (h_1, n_1) \in I_1, h_1 \neq g_1, n_1 \neq m_1 \cup G_2 \times g_1^{I_1} \cup m_1^{I_1} \times M_2 \cup I_2$ (For further reading, see page 150, [2]).

3 Formal context of the lattice $BS(L)$

In this section, we shall investigate the lattice $BS(L)$ obtained by substitution sum $\begin{array}{c|c} L & L \\ \hline \varnothing & B_n \end{array}$ and introduce some results that provide the structural properties of the lattice $BS(L)$.
Throughout this paper, a lattice L under consideration is a finite lattice, with the maximum

element of L denoted by 1 and minimum element of L denoted by 0 such that $|(L)| = |(L)| = n$ where n is the number of atoms in the Boolean lattice B_n.

Lemma 3.1. *Let L be a lattice with $|(L)| = |(L)| = n$ and B_n be the Boolean lattice with n atoms. Then $\mid BS(L) \mid = \mid L \mid + \mid B_n \mid -2$.*

Proof. By the construction of $BS(L)$, clearly,$\mid BS(L) \mid \geq \mid L \mid + \mid B_n \mid -2$.
Let $x \in BS(L)$, $x \neq 0, x \neq 1$. Let m_i and m'_i be the meet irreducible elements of L and B_n respectively, for $1 \leq i \leq n$. There are two possibilities.

Case (1) If $x = \wedge m_i$ where $m'_i \in B_n$, $1 \leq i \leq k$, $k < n$. Then clearly, $x \in B_n$.

Case (2) Let $x = \wedge m_j \wedge m'_j$, $1 \leq j \leq t$, $t \leq n$, $m_j \in (L)$, $m'_j \in (B_n)$. Let $y = \wedge m_j$. Then $y \in L$. Hence the proof. □

Remark 3.2. The above lemma shows that every element of BS(L) comes from either L or B_n. The substitution sum will not create new elements in BS(L). This will be used in the proof characterization of irreducible elements in BS(L).

Lemma 3.3. *For any $x \in L$, $y \in B_n$, $x \neq 0, 1$, $y \neq 0, 1$. Then either $x \parallel y$ or $x < y$ in $BS(L)$.*

Proof. Suppose that $y = \wedge S$, where $S = \{m_1, m_2, ..., m_k : k < n\}$, $k < n$ and $y < x$, for some $x \in L$.
Since $y < x$, $x = \wedge S'$, where $S' \subset S$. Since B_n is a Boolean lattice, $\wedge S' = z' \in B_n$, contradicting that $x \in L$. Therefore,for any $x \in L$, $y \in B_n$, $x \neq 0, 1$, $y \neq 0, 1$. Then either $x \parallel y$ or $x < y$ in $BS(L)$. □

Remark 3.4. From Lemma 3.3, we observe that there exists no $y \in B_n$ such that $y < x$.

Lemma 3.5. *If $x \prec y$ in L or B_n, then $x \prec y$ in $BS(L)$.*

Proof. Let $x = m_1 \wedge m_2 \wedge ... \wedge m_k$, $k < n$. Since $x \prec y$ in L, without loss of generality, we assume $y = m_2 \wedge m_3 \wedge ... \wedge m_k$, $k < n$.
Then in $BS(L)$, $x = m_1 \wedge m_2 \wedge ... \wedge m_k \wedge m'_1 \wedge m'_2 \wedge ... \wedge m'_k$ and $y = m_2 \wedge m_3 \wedge ... \wedge m_k \wedge m'_2 \wedge m'_3 \wedge ... \wedge m'_k$, $k < n$.
Now, if $x < c < y$ in $BS(L)$, by lemma 3.3, $c \in L$, contradiction to the fact that $x \prec y$ in L.
Now, if $x \in B_n$, then $x = m'_1 \wedge m'_2 \wedge ... \wedge m'_k$ and $y = m'_2 \wedge m'_3 \wedge ... \wedge m'_k$, $k < n$.
Note that, x and y have the same representation in $BS(L)$. Clearly $x < y$ in $BS(L)$. □

Lemma 3.6. *If $x \in L$, then there exists $y \in B_n$ such that $x \prec y$ in $BS(L)$.*

Proof. Let $x \in L$. Then $x = m_1 \wedge m_2 \wedge ... \wedge m_k \wedge m'_1 \wedge m'_2 \wedge ... \wedge m'_k$ and $y = m'_1 \wedge m'_2 \wedge ... \wedge m'_k$ in $BS(L)$ for $k \leq n$. Note that y is an element of B_n and $x < y$. Clearly $x \underset{BS(L)}{\prec} y$. □

The following theorems give a characterization of the irreducible elements of $BS(L)$.

Theorem 3.7. *The atoms of BS(L) satisfy the following: $A(BS(L)) = A(L) \cup A(B_n)$.*

Proof. Follows from Lemma 3.1, 3.2 and 3.3. □

Theorem 3.8. *The meet irreducible elements of BS(L) satisfy the relation*
$(BS(L)) = CoA(L) \cup CoA(B_n)$.

Proof. Let $x \in CoA(L)$. Without loss of generality, $x = m_1$ in L. Then $x = m_1 \wedge m'_1$ in $BS(L)$. Then $x \prec m'_1$ uniquely in $BS(L)$. Therefore $x \in (BS(L))$. Let $x \in CoA(B_n)$. By lemma 3.3, Clearly, $x \in (BS(L))$. Therefore, $CoA(L) \cup CoA(B_n) \subseteq (BS(L))$.
Let $x \in (BS(L))$. If $x \in B_n$, then $x \in (B_n)$ if and only if $x \in CoA(B_n)$. If $x \in L$, and suppose that $x \notin CoA(L)$. Then $x \prec z$, $z \neq 1$ in L. By Lemma 3.5, $x \prec z$ in $BS(L)$. Also, by Lemma 3.6, there exists a unique $y \in B_n$ such that $x \underset{BS(L)}{\prec} y$. This shows that x has one more covering in $BS(L)$. □

Theorem 3.9. *The join irreducible elements of $BS(L)$ satisfy the following :*
$(BS(L)) = (L) \bigcup (B_n)$

Proof. Suppose that $a \in L$ be an atom in L. Then $a \in (L)$ and $0 \prec a$ in L. By Lemma 3.7 atoms of L remain as atoms in $BS(L)$, hence join irreducible in $BS(L)$.
Suppose that $a \in (L)$ and a is not an atom. Let a^- be the unique lower cover of a in $\in L$. Then $a^- \underset{BS(L)}{\prec} a$ by Lemma 3.6. Further, there exists no $b_i \in B_n$ such that $b_i \underset{BS(L)}{\prec} a$ by Lemma 3.3. Therefore, $a^- \prec a$ is unique cover in $BS(L)$.
The join irreducible elements of B_n are atoms in B_n. The atoms of B_n remain as atoms of $BS(L)$. □

Example 3.1. A lattice L is depicted in Figure 2, the formal context of L is given in Figure 1. The Boolean lattice B_4 is depicted in Figure 4, the formal context of B_4 is given in Figure 3. The context table and corresponding lattice $BS(L)$ obtained from substitution sum of L and B_4 is given in Figure 5 and Figure 6, respectively.

	m_1	m_2	m_3	m_4
a		X		X
b	X		X	
c		X		
d	X			

Figure 1. Formal Context of L

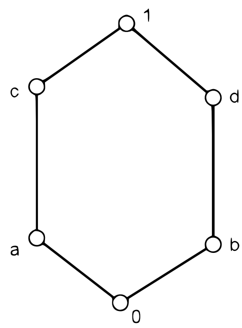

Figure 2. Lattice L

	m_1'	m_2'	m_3'	m_4'
a		X	X	X
b	X		X	X
c	X	X		X
d	X	X	X	

Figure 3. Formal Context of B_4

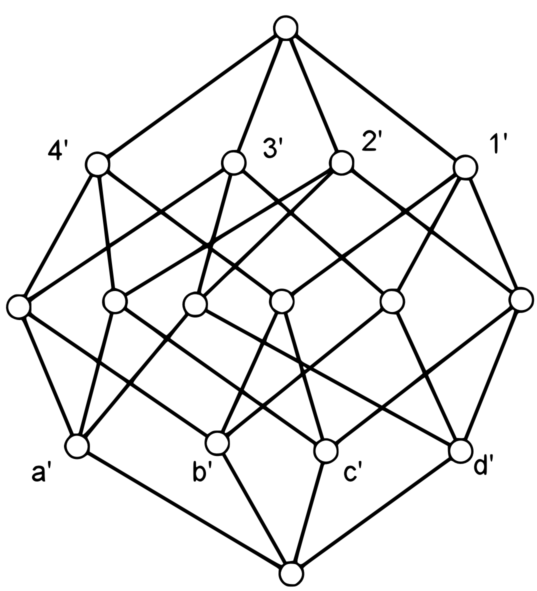

Figure 4. Lattice B_4

4 Properties of BS(L)

Theorem 4.1. *For any lattice with more than 2 elements, the corresponding $BS(L)$ is always a complemented lattice.*

Proof. Let $a \in BS(L)$. We prove that a has a complement.

Case 1. Suppose $a \in B_n$. Since B_n is a complemented lattice, there exists a unique a' such that $a \wedge a' = 0$ and $a \vee a' = 1$ in B_n. The same a' serve as complement of a in $BS(L)$.

	m_1	m_2	m_3	m_4	m_1'	m_2'	m_3'	m_4'
a		X		X		X		X
b	X		X		X		X	
c		X				X		
d	X				X			
a'						X	X	X
b'					X		X	X
c'					X	X		X
d'					X	X	X	

Figure 5. Context of $BS(L)$

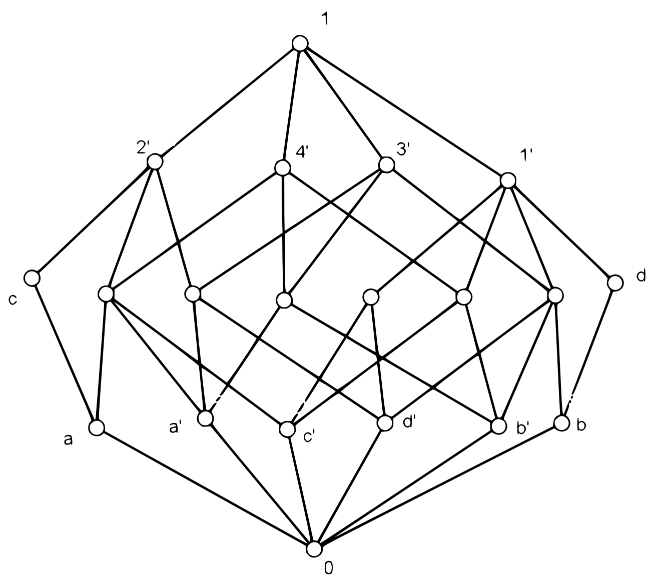

Figure 6. Lattice $BS(L)$

Case 2. Suppose $a \in L$, $a \neq 0$ and $a \neq 1$. Let $b \in B_n$ be such that $a \prec b$ in $BS(L)$. Let b' be the complement of $b \in B_n$. We prove that b' is a complement of a in $BS(L)$. Since $a < b$ in $BS(L)$, we have $0 = a \wedge b' < b \wedge b' = 0$. Therefore, $a \wedge b' = 0$.
Now, we prove that $a \vee b' = 1$. Observe that $a \prec b$ is the single bridge between L and B_n, we have $a \vee b'$=$b \vee b' = 1$.
Hence, BS(L) is a complemented Lattice. □

5 Conclusion

This construction differs from doubling construction in that L might not constitute a convex sublattice of B_n; nevertheless, a one-to-one mapping exists from L to B_n. A general characterization of the lattice generated by the substitution sum $\frac{L_1 \mid L_2}{L_3 \mid L_4}$ is suggested for further study.

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