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Structural Properties and Laplacian Spectrum of Equal-Square Graph of Finite Groups

Pankaj Rana*, Shilpa Aggarwal, Amit Sehgal and Pooja Bhatia

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Abstract Equal-square graph of finite group G is a simple finite undirected graph with vertex set G, in which two distinct vertices a, b are adjacent if and only if $a^2 = b^2$. In this research paper we have studied various structural properties such as connectivity, vertex degree, girth, clique number, independent number, chromatic number and matching number of Equal-square graph of finite groups. We have also calculated Laplacian polynomial and domination number of Equal-square graph of various finite groups.

1 Introduction

Studying graphs associated with groups is growing area of research due to their importance in various fields. Various types of graph are studied in association with finite groups towards study of their properties such as degree, girth, clique, chromatic number, matching, independent number, etc. In [1, 2, 3] various properties such as degree, connectedness, girth, clique, independent number, chromatic number, etc are studied for complement graph of the square power graph, cubic power graph of finite abelian group and line graphs associated to the unit graphs of rings whereas square power graph of $\mathbb{Z}_n$ is studied in [4]. Degree of a vertex in k^{th}- power graph is calculated in [5]. Various properties of Co-Prime Order graph of a finite groups are studied in [6]. Vertex degree in the power graph of a finite abelian group is given in [7]. S. N. Singh calculated the laplacian polynomial of power graph for $\mathbb{Z}_{p^m}^n$ in [8]. Perfect Italian domination number of graphs is studied in [9] whereas idempotent graph of rings is studied in [10].
S. U. Rehman, et al. introduced and gave representation of equal-square graph for finite groups in [11]. From [11] we have 'Theorem 1. A group G has odd order iff ES(G) is empty.' , 'Theorem 3. For a cyclic group G of order $2n$, $ES(G) = 2K_2$.' and 'Theorem 4. $ES(D_n) = K_{n+2} + (\frac{n-2}{2})K_2$, if n is even and $K_{n+1} + (n-1)K_1$ if n is odd.'. We have further studied the structural properties such as degree, girth, self-centred, clique, chromatic number, matching number, independent number, weakly perfectness of equal-square graph for finite groups. We finally calculated domination number and laplacian polynomial of Equal-square graph of various finite groups.
Equal-square graph of finite group is finite simple undirected graph in which any pair of vertices r, s are adjacent if and only if $r^2 = s^2$. It is represented as $ES(G)$. A graph is said to be connected if and only if we have path between every distinct vertices pair. If we have edge between every distinct vertices pair then graph is said to be complete and denoted as K_n where n is number of vertices in that graph. Let u be any vertex of graph then number of vertices with which u have edge in that graph is known as degree of vertex u and denoted as $deg(u)$. The length of shortest cycle in graph $ES(G)$ is known as girth of $ES(G)$ and denoted as $gr(ES(G))$. A clique of $ES(G)$ is the complete subgraph of $ES(G)$ and number of vertices in maximal complete subgraph is known as clique number of $ES(G)$, denoted as $\omega(ES(G))$. A set of vertices such that no pair of vertices in set have edge between them is known as independent set

and number of vertices in maximal independent set is known as independent number denoted as $\beta(ES(G))$. We have denoted vertex set of graph $ES(G)$ by $V(ES(G))$ and its order by $|V(ES(G)|$. Vertex coloring of equal-square graph means the mapping f : $V(ES(G)) \to M$. Elements of M are known as colors. If $|M| = k$, then f is known by k-coloring. If the colors assigned are different to vertices having edge, then coloring is known as proper-coloring and graph with proper k-coloring is known as $k-$colorable. Chromatic number[12, 13] is the value of least k such that $ES(G)$ is $k-$colorable. If clique number and chromatic number of any graph are equal then graph is called weakly perfect. Independent edge set or matching of graph $ES(G)$ is a set of edges E of graph $ES(G)$, in which no edges pair have common vertex. A vertex is said to be matched if that vertex is incident to any edge in matching. Matching containing largest possible edges is known as Maximum matching and number of edges in maximum matching is called matching number, denoted by $\mu(ES(G))$. If all the vertices of the graph are saturated by matching then it matching is called perfect matching. For finite natural number n, dihedral group of order $2n$ is given as $D_n = \{r^i, s^j | o(r) = n, o(s) = 2, srs^{-1} = r^{-1}\}$ which is also known as group of symmetries of a regular n-gon. For simplicity we have used $D_n = \{r_0, r_{\frac{360}{n}}, r_{\frac{2\times360}{n}}, r_{\frac{3\times360}{n}}, \cdots, r_{\frac{(n-1)\times360}{n}}, s_1, s_2, \cdots, s_n\}$ with r_0 as identity element in which $r_{\frac{(i-1)\times360}{n}}$ for $1 \le i \le n$ are rotation elements and s_j for $1 \le j \le n$ are reflection elements.

2 Structural Properties of $ES(G)$

Theorem 2.1. *Let $ES(D_n)$ be equal-square graph of dihedral group D_n with identity element r_0 and x be any vertex in $ES(D_n)$ then*

$$deg(x) = \begin{cases} n+1, & \text{if } n \text{ is even number and } x^2 = r_0, \\ 1, & \text{if } n \text{ is even number and } x^2 \neq r_0, \\ n, & \text{if } n \text{ is odd number and } x^2 = r_0, \\ 0, & \text{if } n \text{ is odd number and } x^2 \neq r_0. \end{cases}$$

Proof. Let $D_n = \{r_0, r_{\frac{360}{n}}, r_{\frac{2\times360}{n}}, \cdots, r_{\frac{(n-1)\times360}{n}}, s_1, s_2, \cdots, s_n\}$ be dihedral group of order $2n$ with n rotation elements $r_{\frac{(i-1)\times360}{n}}$ for $1 \le i \le n$ and n reflection elements s_j for $1 \le n$. $ES(D_n)$ be equal-square graph of dihedral group D_n and x be any vertex in $ES(D_n)$.

Case 1. When n is even number and $x^2 = r_0$

In this case we have $n+2$ elements $r_0, r_{180}, s_1, s_2, \cdots, s_n$ in D_n of order 2 with $r_0^2 = r_{180}^2 = s_1^2 = s_2^2 = \cdots = s_n^2 = r_0$. We have no other element $x \in D_n$ for which we have $x^2 = r_0$. Thus we have $n+2$ vertices in this component of $ES(D_n)$ which are adjacent with each other and form the K_{n+2} component of $ES(D_n)$. So we have $deg(x) = n+1$ if n is even number and $x^2 = r_0$.

Case 2. When n is even number and $x^2 \neq r_0$

In this case we have $n-2$ elements $r_{\frac{360}{n}}, r_{\frac{2\times360}{n}}, \cdots, r_{\frac{(\frac{n}{2}-1)\times360}{n}}, r_{\frac{(\frac{n}{2}+1)\times360}{n}}, \cdots, r_{\frac{(n-1)\times360}{n}}$. We have $\frac{n-2}{2}$ pairs (x, y) of elements $(r_{\frac{360}{n}}, r_{\frac{(\frac{n}{2}+1)\times360}{n}}), (r_{\frac{2\times360}{n}}, r_{\frac{(\frac{n}{2}+2)\times360}{n}}), \cdots,$ $(r_{\frac{(\frac{n}{2}-1)\times360}{n}}, r_{\frac{(n-1)\times360}{n}})$ with $x^2 = y^2 \neq r_0$ and one pair elements square is not equal to any other pair elements square. Hence we have $deg(x) = 1$ if n is even number and $x^2 \neq r_0$.

Case 3. When n is odd number and $x^2 = r_0$

In this case we have $n+1$ elements $r_0, s_1, s_2, \cdots, s_n$ in D_n of order 2 with $r_0^2 = s_1^2 = s_2^2 = \cdots = s_n^2 = r_0$. We have no other element $x \in D_n$ for which we have $x^2 = r_0$. Thus we have $n+1$ vertices in this component of $ES(D_n)$ which are adjacent with each other and form the K_{n+1} component of $ES(D_n)$. So we have $deg(x) = n$ if n is odd number and $x^2 = r_0$.

Case 4. When n is odd number and $x^2 \neq r_0$

In this case we have $n-1$ elements $r_{\frac{360}{n}}, r_{\frac{2\times360}{n}}, \cdots, r_{\frac{(n-1)\times360}{n}}$. Among these $n-1$ elements we have no pair of elements (x, y) for which $x^2 = y^2$ and so forming the $(n-1)$ K_1 components of $ES(D_n)$ Hence we have $deg(x) = 1$ if n is odd number and $x^2 \neq r_0$.

Hence the required result. □

Theorem 2.2. *Let $ES(D_n)$ be equal-square graph of dihedral group D_n then*
(i) $ES(D_n)$ is connected iff $n \in \{1, 2\}$.
(ii) $ES(D_n)$ is complete iff $n \in \{1, 2\}$.
(iii) $ES(D_n)$ is self-centered iff $n \in \{1, 2\}$.

Proof. **(i)** Let $ES(D_n)$ is connected. Using [11, Theorem 4] we have $ES(D_n) = K_{n+2} + (\frac{n-2}{2})K_2$ for even n and $K_{n+1} + (n-1)K_1$ for odd values of n. Thus $ES(D_n)$ is disconnected graph for even values of $n \geq 4$ and odd values of $n \geq 3$. Hence $ES(D_n)$ is connected if $n \in \{1, 2\}$.
Conversely, Using from [11, Theorem 4] we have for $n = 1$, $ES(D_1) = K_2$ and for $n = 2$, $ES(D_2) = K_4$. So if $n \in \{1, 2\}$ then $ES(D_n)$ is connected.
Hence the required result.
(ii) Let $ES(D_n)$ is complete graph then it is should be connected. But from Theorem 2.2(i) we have $ES(D_n)$ is disconnected for $n \geq 3$. So $n \ngeq 3$. Now for $n \in \{1, 2\}$, by using [11, Theorem 4] we have $ES(D_1) = K_2$ and $ES(D_2) = K_4$. Hence $n \in \{1, 2\}$.
Conversely, For $n \in \{1, 2\}$ we have $ES(D_1) = K_2$ and $ES(D_2) = K_4$. Thus $ES(D_n)$ is complete. Hence the required result. □

Theorem 2.3. *Let $ES(D_n)$ be equal-square graph of dihedral group D_n then girth*
$$gr(ES(D_n)) = \begin{cases} 3 \text{ if } n \neq 1, \\ \infty \text{ if } n = 1. \end{cases}$$

Proof. Let $ES(D_n)$ be equal-square graph of dihedral group D_n.
Case 1. When $n \neq 1$
By using [11, Theorem 4] we have K_{n+2} as a component of $ES(D_n)$ for even values of n and K_{n+1} i case of odd values of n. Thus for $n \neq 1$. we have cycle of length 3 in $ES(D_n)$. Hence $gr(D_n) = 3$ if $n \neq 1$.
Case 2. When $n = 1$
Using [11, Theorem 4], we have $ES(D_1) = K_2$. Thus we have no cycle in $ES(D_n)$. Hence $gr(ES(D_n)) = \infty$ if $n = 1$. □

Theorem 2.4. *Let $ES(D_n)$ be equal-square graph of dihedral group D_n then clique number*
$$\omega(ES(D_n)) = \begin{cases} n+2 \text{ if } n \text{ is even number,} \\ n+1 \text{ if } n \text{ is odd number.} \end{cases}$$

Proof. Let $ES(D_n)$ be equal-square graph of dihedral group D_n.
Case 1. When n is even number
Using [11] we have K_{n+2} and K_2 two kinds of components in $ES(D_n)$. Thus we have K_{n+2} maximal complete sub-graph in $ES(D_n)$. Hence $\omega(ES(D_n)) = n + 2$ if n is even number.
Case 2. When n is odd number
Using [11] we have K_{n+1} and K_1 two kinds of components in $ES(D_n)$. Thus we have K_{n+1} maximal complete sub-graph in $ES(D_n)$. Hence $\omega(ES(D_n)) = n + 1$ if n is odd number. □

Theorem 2.5. *Let $ES(D_n)$ be equal-square graph of dihedral group D_n then independent number,*
$$\beta(ES(D_n)) = \begin{cases} \frac{n}{2} \text{ if } n \text{ is even number,} \\ n \text{ if } n \text{ is odd number.} \end{cases}$$

Proof. Let $ES(D_n)$ be equal-square graph of dihedral group D_n.
Case 1. When n is even number
Using [11] we have $\frac{n-2}{2}$ components of K_2 type and one component of K_{n+2} type. Thus $\frac{n-2}{2}$ vertices from K_2 type components(one vertex from each K_2 component) and one vertex from K_{n+2} type component forms the maximal independent set. Hence $\beta(ES(D_n)) = \frac{n-2}{2} + 1 = \frac{n}{2}$.
Case 2. When n is odd number
Using [11] we have $n - 1$ components of K_1 type and one component of K_{n+1} type. Thus $n - 1$

vertices from K_1 type components and one vertex from K_{n+1} type component forms the maximal independent set. Hence $\beta(ES(D_n)) = n - 1 + 1 = n$. □

Theorem 2.6. *Let $ES(D_n)$ be equal-square graph of dihedral group D_n then chromatic number,*
$$\chi(ES(D_n)) = \begin{cases} n+2 \text{ if } n \text{ is even number,} \\ n+1 \text{ if } n \text{ is odd number.} \end{cases}$$

Proof. Let $ES(D_n)$ be equal-square graph of dihedral group D_n.
Case 1. When n is even number
Using [11] we have $ES(D_n) = K_{n+2} + (\frac{n-2}{2})K_2$. So we need 2 colors for proper coloring of $\frac{n-2}{2}$ components of K_2 type component and $n+2$ colors for proper coloring of one K_{n+2} type component. Thus using $n+2$ colors we can do proper coloring of K_{n+2} component and using any 2 colors from already used $n+2$ colors we can do proper coloring of $\frac{n-2}{2}$ components of K_2 type. Thus we need atleast $n+2$ colors for proper coloring of $ES(D_n)$. Hence $\chi(ES(D_n)) = n+2$ when n is even number.
Case 2. When n is odd number
Using [11] we have $ES(D_n) = K_{n+1} + (n-1)K_1$. So we need 1 color for proper coloring of $n-1$ components of K_1 type component and $n+1$ colors for proper coloring of one K_{n+1} type component. Thus using $n+1$ colors we can do proper coloring of K_{n+1} component and using any 1 color from already used $n+1$ colors we can do proper coloring of $n-1$ components of K_1 type. Thus we need atleast $n+1$ colors for proper coloring of $ES(D_n)$. Hence $\chi(ES(D_n)) = n+1$ when n is odd number. □

Theorem 2.7. *Let $ES(D_n)$ be equal-square graph of dihedral group D_n then matching number,*
$$\mu(ES(D_n)) = \begin{cases} n \text{ if } n \text{ is even number,} \\ \frac{n+1}{2} \text{ if } n \text{ is odd number.} \end{cases}$$

Proof. Let $ES(D_n)$ be equal-square graph of dihedral group D_n. Using [11, Theorem 4] we have $ES(D_n) = K_{n+2} + (\frac{n-2}{2})K_2$ for even n and $K_{n+1} + (n-1)K_1$ for odd values of n.
Case 1. When n is even number
In this case we have one K_{n+2} component and $(\frac{n-2}{2})-$ K_2 components. Thus we have $(\frac{n+2}{2})$ edges from K_{n+2} component and $(\frac{n-2}{2})$ edges from $\frac{n-2}{2}-$ K_2 components forming together the maximal set of edges from $ES(D_n)$ such that no pair of edges have any common vertex. Thus $\mu(ES(D_n)) = \frac{n+2}{2} + \frac{n-2}{2} = n$.
Case 2. When n is odd number
In this case we have one K_{n+1} component and $(n-1)-$ K_1 components. Thus $(\frac{n+1}{2})$ edges from K_{n+1} component forms the maximal set of edges from $ES(D_n)$ such that no pair of edges have any common vertex. Thus $\mu(ES(D_n)) = \frac{n+1}{2}$. □

Corollary 2.8. *Let $ES(D_n)$ be equal-square graph of dihedral group D_n then $ES(D_n)$ have perfect matching if and only if n is even number.*

Proof. Let $ES(D_n)$ be equal-square graph of dihedral group D_n and $ES(D_n)$ have perfect matching then we have
matching number $\mu(ES(D_n)) = \frac{\text{number of vertices in } ES(D_n)}{2} = \frac{2n}{2} = n$. But from Theorem 2.7 we have $\mu(ES(D_n)) = n$ when n is even number and $\mu(ES(D_n)) = \frac{n+1}{2}$ when n is odd number. Thus $ES(D_n)$ have perfect matching if n is even number.
Conversely, when n is even number using Theorem 2.7 we have $ES(D_n)$ have perfect matching. Hence the required result. □

Theorem 2.9. *Let D_n be dihedral group then square power graph of D_n, $\Gamma_{sq}(G)$ is weakly perfect.*

Proof. From Theorem 2.4 and Theorem 2.6 we have $\omega(\Gamma_{sq}(G)) = \chi(\Gamma_{sq}(G))$. Hence $\Gamma_{sq}(D_n)$ is weakly perfect. □

Theorem 2.10. *Let G be finite group of odd order and $ES(G)$ be equal-square graph of G then*
(i) $ES(G)$ is disconnected graph.
(ii) $deg(x) = 0\ \forall x \in V(ES(G))$.
(iii) Girth, $gr(ES(G)) = \infty$.
(iv) Clique number, $\omega(ES(G)) = 1$.
(v) Independent number, $\beta(ES(G)) = |G|$.
(vi) Chromatic number, $\chi(ES(G)) = 1$.
(vii) $ES(G)$ is weakly-perfect.
(viii) Matching number, $\mu(ES(G)) = 0$.

Proof. Let G be finite group of odd order n and $ES(G)$ be equal-square graph of G. Then by using [11, Theorem 1] we have $ES(G) = nK_1$.
(i) As $ES(G) = nK_1$. Thus $ES(G)$ is disconneted.
(ii) Let $x \in V(ES(G))$. As we have no pair of vertices adjacent in $ES(G)$, so $deg(x) = 0$ $\forall x \in V(ES(G))$.
(iii) In $ES(G)$ we have no cycle. Hence $gr(ES(G)) = \infty$.
(iv) As we have no pair of adjacent vertices in $ES(G)$, so maximal complete subgraph of $ES(G)$ is K_1. Hence $\beta(ES(G)) = |G|$.
(v) As we have no pair of adjacent vertices in $ES(G)$ so maximal independent set is vertex set of $ES(G)$ itself. Hence $\beta(ES(G)) = |V(ES(G))| = |G|$.
(vi) We need atleast one color for proper coloring of K_1 component in $ES(G)$ and so atleast one color to do proper coloring of $ES(G)$. So $\chi(ES(G)) = 1$.
(vii) From Theorem 2.10(iv) and (vi), we have $\omega(\Gamma_{sq}(G)) = \chi(\Gamma_{sq}(G))$. Hence $\Gamma_{sq}(G)$ is weakly perfect.
(viii) As $ES(G) = nK_1$ so we have no edge in graph. Thus Matching number, $\mu(ES(G)) = 0$. □

Theorem 2.11. *Let G be cyclic group of even order $2n$ and $ES(G)$ be equal-square graph of G then*
(i) $ES(G)$ is disconnected graph.
(ii) $deg(x) = 1\ \forall x \in G$.
(iii) Girth, $gr(ES(G)) = \infty$.
(iv) Clique number, $\omega(ES(G)) = 2$.
(v) Independent number, $\beta(ES(G)) = \frac{|G|}{2}$.
(vi) Chromatic number, $\chi(ES(G)) = 2$.
(vii) $ES(G)$ is weakly-perfect.
(viii) Matching number, $\mu(ES(G)) = n$.

Proof. Let G be cyclic group of even order $2n$ and $ES(G)$ be equal-square graph of G. Then from [11, Theorem 3] we have $ES(G) = nK_2$.
(i) As we have $ES(G) = nK_2$. So $ES(G)$ is disconnected.
(ii) Let $x \in V(ES(G))$. In $ES(G)$ we have only K_2 type components. Thus $deg(x) = 1\ \forall x \in G$.
(iii) We have no cycle in $ES(G)$ so $gr(ES(G)) = \infty$.
(iv) As we have only K_2 type components in $ES(G)$. So we have K_2 as maximal complete subgraph in $ES(G)$. Thus $\omega(ES(G)) = 2$.
(v) We have n components of K_2 type in $ES(G)$. n vertices, one from each K_2 component forms the maximal independent set. Thus $\beta(ES(G)) = n = \frac{|G|}{2}$.
(vi) For proper coloring of K_2, we need atleast 2 colors. With two colors we can do proper coloring of $n - K_2$ components in $ES(G)$. Hence $\chi(ES(G)) = 2$.
(vii) From Theorem 2.11(iv) and (vi) we have we $\omega(\Gamma_{sq}(G)) = \chi(\Gamma_{sq}(G))$. Hence $\Gamma_{sq}(G)$ is weakly perfect.
(viii) As $ES(G) = nK_2$ so we have maximal independent set of edges having n edges. Hence Matching number, $\mu(ES(G)) = n$. □

Theorem 2.12. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 = Z_2^m$ and $ES(G)$ be equal-square graph of G then*
(i) $ES(G)$ is connected graph.

(ii) $deg(x) = 2^m - 1 \forall x \in G$.
(iii) Girth, $gr(ES(G)) = 3$.
(iv) Clique number, $\omega(ES(G)) = 2^m$.
(v) Independent number $= 1$.
(vi) Chromatic number, $\chi(ES(G)) = 2^m$.
(vii) $ES(G)$ *is weakly-perfect.*
(viii) Matching number, $\mu(ES(G)) = 2^{m-1}$.

Proof. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 = Z_2^m$ and $ES(G)$ be equal-square graph of G then from [11, Example 4] we have $ES(G) = K_{2^m}$. Hence we have the required results. □

3 Laplacian Spectrum and Domination Number

Let $ES(G)$ be equal-square graph of group G. Laplacian matrix, $L(ES(G))$ is the difference of vertex degree diagonal matrix and adjacency matrix of $ES(G)$. Characteristic polynomial of $L(ES(G))$ is known as Laplacian polynomial denoted as $\ominus(ES(G), x)$. Laplacian polynomial of complete graph with n vertices K_n is $\ominus(K_n, x) = x(x-n)^{n-1}$. If graph ES is disjoint union of $ES_1, ES_2, \cdots, ES_k$ then $\ominus(ES, x) = \prod_{i=1}^{k} \ominus(ES_i, x)$ [14]. Minimum dominating set S is the minimal subset of vertices such that every vertex of graph either lie in S or adjacent with atleast one vertex in S. Number of vertices in minimum dominating set for G is known as domination number, denoted as $\gamma(G)$. Domination number is studied in [15] for idempotent divisor graphs associated with commutative rings.

Theorem 3.1. *Let* $ES(D_n)$ *be equal-square graph of dihedral group* D_n *of order* $2n$*. Then*
$$\ominus(ES(G), x) = \begin{cases} x^{\frac{n}{2}}(x-2)^{\frac{n-2}{2}}(x-n-2)^{n+1} & \text{when } n \text{ is even number}, \\ x^n(x-n-1)^n & \text{when } n \text{ is odd number}. \end{cases}$$

Proof. Let $ES(D_n)$ be equal-square graph of dihedral group D_n of order $2n$.
Case 1. When n is even number
From [11, Theorem 4] we have that $ES(D_n) = K_{n+2} + (\frac{n-2}{2})K_2$. So we have $\ominus(ES(D_n), x) = \ominus(K_{n+2}, x) \times \ominus(K_2, x)^{\frac{n-2}{2}} = x(x-n-2)^{n+1} \times [x(x-2)]^{\frac{n-2}{2}} = x^{\frac{n}{2}}(x-2)^{\frac{n-2}{2}}(x-n-2)^{n+1}$.
Case 2. When n is odd number
From [11, Theorem 4] we have that $ES(D_n) = K_{n+1} + (n-1)K_1$. So we have $\ominus(ES(D_n), x) = \ominus(K_{n+1}, x) \times \ominus(K_1, x)^{n-1} = x(x-n-1)^n \times x^{n-1} = x^n(x-n-1)^n$. □

Theorem 3.2. *Let* $ES(G)$ *be equal-square graph of finite odd order* n *abelian group* G*. Then* $\ominus(ES(G), x) = x^n$.

Proof. Let $ES(G)$ be equal-square graph of finite odd order n abelian group G. Using [11, Theorem 1] we have $ES(G) = nK_1$. So $\ominus(ES(G), x) = \ominus(K_1)^n = x^n$. □

Theorem 3.3. *Let* $ES(G)$ *be equal-square graph of cyclic group* G *of even order* $2n$*. Then* $\ominus(ES(G), x) = \ominus(K_2)^n = x^n(x-2)^n$.

Proof. Let $ES(G)$ be equal-square graph of cyclic group G of even order $2n$. Then by using [11, Theorem 3] we have $ES(G) = nK_2$. Thus $\ominus(ES(G), x) = \ominus(K_2, x)^n = [x(x-2)]^n = x^n(x-2)^n$. □

Theorem 3.4. *Let* $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 = Z_2^m$ *and* $ES(G)$ *be equal-square graph of* G *then* $\ominus(ES(G), x) = x(x-2^m)^{2^m-1}$.

Proof. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 = Z_2^m$ then $ES(G) = K_{2^m}$. Thus $\ominus(ES(G), x) = \ominus(K_{2^m}, x) = x(x-2^m)^{2^m-1}$. □

Theorem 3.5. *Let G be group and $ES(G)$ be equal square graph of G then domination number,*
(i)When $G = D_n$, $\gamma(G) = \begin{cases} \frac{n}{2} \text{ if } n \text{ is even,} \\ n \text{ if } n \text{ is odd.} \end{cases}$
(ii) When $|G|$ is odd, $\gamma(G) = |G|$.
(iii) When G is cyclic group of order $2n$, $\gamma(G) = n$.

Proof. (i) When $G = D_n$ and n is even number then we have one component of K_{n+2} type and $\frac{n-2}{2}$ components of K_2 type. So one vertex from K_{n+2} and one from each K_2 type components forms minimum dominating set of G. Thus $\gamma(G) = 1 + \frac{n-2}{2} = \frac{n}{2}$.
When n is odd, then we have one K_{n+1} and $(n-1)$ K_1 components. Thus $\gamma(G) = 1+n-1 = n$.
(ii) When $|G|$ is odd, we have $ES(G) = |G|K_1$. Thus $\gamma(G) = |G|$.
(iii) When G is cyclic group of order $2n$, then we have $ES(G) = nK_2$. Thus one vertex from every K_2 component combines to form the minimum dominating set. Hence $\gamma(G) = n$. □

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Author information

Pankaj Rana*, Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak-124021, (Haryana), India.
E-mail: pankajrana2034@gmail.com, ORCID: https://orcid.org/0000-0002-5319-932X

Shilpa Aggarwal, Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak-124021, (Haryana), India.
E-mail: shipsaggarwal07@gmail.com, ORCID: https://orcid.org/0009-0001-7913-6351

Amit Sehgal, Department of Mathematics, Pt. NRS Govt. College, Rohtak-124001, (Haryana), India.
E-mail: amit_sehgal_iit@yahoo.com, ORCID: https://orcid.org/0000-0001-7820-8037

Pooja Bhatia, Department of Mathematics, Baba Mastnath University, Asthal Bohar, Rohtak-124021, (Haryana), India.
E-mail: poojabudhiraja@bmu.ac.in