Palestine Journal of Mathematics

Vol 13(Special Issue III)(2024) , 162–167

On generalization of exact sequences in modules

Rajani S. and Kedukodi B.S. and Harikrishnan P. and Kuncham S.P.*

Communicated by Madeleine Al-Tahan

MSC 2010 Classifications: Primary18G99, 13C99.

Keywords and phrases:Exact sequence, U-exact sequence.

Abstract The notion of U-exact sequence in modules over rings was introduced in [20] as a generalization of $\{0\}$-exact sequence. In this paper, we prove further results on U-exact and V-coexact sequences where V is induced by U. As shown in the commutative diagram fig. 1, wherein if row-1 is U-exact and row-2 is $U^{'}$-exact, then we prove that the sequence $(0) \to ker\ f^{'} \to ker\ f \to ker\ f^{''}$ is $(ker\ f^{''} \cap U)$-exact, and the sequence $(0) \to coker\ f^{'} \to coker\ f \to coker\ f^{''}$ is $(\frac{U^{'}+Im\ f^{''}}{Im\ f^{''}})$-exact. We provide explicit examples of the existence of U-exact and $U^{'}$-coexactness.

1 Introduction

Let R be a ring and let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of R-modules and R-homomorphisms where $f(A) = g^{-1}(\{0\})$. In [20], the authors introduced U-exact sequence (or quasi-exact sequence), as an answer to a natural question that what if one substitutes an arbitrary submodule U of C in place of $\{0\}$ submodule in the above. The authors [20] proved a classical five lemma, in terms of U-exact sequences, and in [14], the authors studied U-split sequences and found relationships between U-split sequences and projective modules. Further in [15], the authors obtained some relationship between the quasi-exact sequence and superfluous (or essential) submodule. In case of module over nearrings, the essential ideals [24] and superfluous ideals [22] were studied. The notions studied in this paper lead to the generalization of some important aspects in homological algebra [21].

In this paper, we prove some extension results on U-exact and V-coexact sequences, those are different from the results proved by the authors in [14, 20]. In particular, we prove that if $K^{'} \to K \to K^{''} \to (0)$ is U-exact at K, then the corresponding $U^{'}$-coexactness of

$$(0) \to Hom(K^{''}, N) \to Hom(K, N) \to Hom(K^{'}, N)$$

can be obtained, where

$$U^{'} = \{h \in Hom(K^{''}, N) : h(U) = 0_N\}.$$

Further, as shown in the commutative diagram of fig. 1, wherein row-1 is U-exact and row-2 is $U^{'}$-exact, we prove that the sequence

$$(0) \to ker\ f^{'} \to ker\ f \to ker\ f^{''}$$

is $(ker\ f^{''} \cap U)$-exact, and the sequence

$$(0) \to coker\ f^{'} \to coker\ f \to coker\ f^{''}$$

is $(\frac{U^{'}+Im\ f^{''}}{Im\ f^{''}})$-exact.

We provide detailed proofs whenever the sequence is either U-exact or V-coexact plays an important role, however we skip the verification if the proof is similar to that of $\{0\}$-exact sequences. We assume $\{0\}$-exactness in the sequences whenever it is not specified. We denote set of all homomorphisms from R-module M to an R-module N by $\mathcal{H}(M, N)$.

We refer to [13, 16, 17] for standard definitions and notions on modules over rings (or commutative rings).

2 U-exact and U'-coexact sequences

We start this section with the necessary definitions from [20].

Definition 2.1. [20] A sequence of R-modules and R-homomorphisms

$$\cdots \to K_{i-1} \xrightarrow{f_i} K_i \xrightarrow{f_{i+1}} K_{i+1} \to \cdots$$

is U_{i+1}-exact (where U_{i+1} is a submodule of K_{i+1}) at K_i if
$Im\ f_i = f_{i+1}^{-1}(U_{i+1})$.

Definition 2.2. [20] A sequence $(0) \to P \xrightarrow{f} Q \xrightarrow{g} S \to (0)$ which is $\{0\}$-exact at P, U-exact at Q and $\{0\}$-exact at S is called U-exact, where U is a submodule of S.

Definition 2.3. A sequence $(0) \to P \xrightarrow{f} Q \xrightarrow{g} S \to (0)$ is V-coexact (V is a submodule of P) if f is one-one, g is onto and $ker\ g = f(V)$.

Theorem 2.4. *The sequence*

$$K^{'} \xrightarrow{u} K \xrightarrow{v} K^{''} \to (0) \tag{1}$$

of R-modules and R-homomorphisms is a U-exact sequence at K if and only if

$$(0) \to \mathcal{H}(K^{''}, N) \xrightarrow{\bar{v}} \mathcal{H}(K, N) \xrightarrow{\bar{u}} \mathcal{H}(K^{'}, N) \tag{2}$$

is $U^{'}$-coexact at $\mathcal{H}(K, N)$ where $U^{'} = \{h \in \mathcal{H}(K^{''}, N) : h(U) = 0_N\}$
for every R-module N.

Proof. Suppose $K^{'} \xrightarrow{u} K \xrightarrow{v} K^{''} \to (0)$ is U-exact at K and $\{0\}$-exact an $K^{''}$. Then $Im\ u = v^{-1}(U)$ and $v(K) = K^{''}$.
To prove $\bar{v}$ is a monomorphism, let $f^{''} \in ker\ \bar{v}$. Then $\bar{v}(f^{''}) = f_0$ where
$f_0 : K \to N$ is a trivial homomorphism, implying $f^{''} \circ v = f_0$. Since $v(K) = K^{''}$, it follows that $f^{''}(K^{''}) = 0_N$. Therefore, $\bar{v}$ is a monomorphism.
Next to prove $\bar{v}(U^{'}) = ker\ \bar{u}$, let $f \in ker\ \bar{u}$. Then $\bar{u}(f) = {f_0}^{'}$ where ${f_0}^{'}$ is a zero mapping in $\mathcal{H}(K^{'}, N)$, implying $f \circ u = {f_0}^{'}$.
Now $(f \circ u)(K^{'}) = {f_0}^{'}(K^{'}) = 0_N$. Since the sequence-1 is U-exact, it follows that $0_N = f(u(K^{'})) = f(v^{-1}(U))$, which implies $(v^{-1})(U) \subseteq ker\ f$.
Let $k^{''} \in K^{''}$. As v is a surjective map, there exists $k \in K$ such that $v(k) = k^{''}$. Now define $g : K^{''} \to N$ as $g(k^{''}) = g(v(k)) = f(k)$. To see g is well defined, let ${k_1}^{''}, {k_2}^{''} \in K^{''}$ be such that ${k_1}^{''} = {k_2}^{''}$. Then there exist $k_1, k_2 \in K$ such that $v(k_1) = {k_1}^{''}$ and $v(k_2) = {k_2}^{''}$. Then $k_1 - k_2 \in ker\ v \subseteq v^{-1}(U) = ker\ f$, we get $g({k_1}^{''}) = g({k_2}^{''})$. Let $x \in U$. Then, there exists $y \in K$ such that $x = v(y)$ implies $y \in v^{-1}(x) \subseteq v^{-1}(U)$. Now, $g(x) = g(v(y)) = f(y) \in f(v^{-1}(x)) \subseteq f(v^{-1}(U)) = 0_N$. Hence $g(U) = 0_N$ and so $g \in U^{'}$.
Now, $\bar{v}(x) = (g \circ v)(x) = g(v(x)) = f(x)$, for every $x \in K$. That is, $\bar{v}(g) = f$ implies $f = \bar{v}(g) \in \bar{v}(U^{'})$. Therefore, $ker\ \bar{u} \subseteq \bar{v}(U^{'})$.
To prove $\bar{v}(U^{'}) \subseteq ker\ \bar{u}$, let $f \in \bar{v}(U^{'})$. Then there exists $h \in U^{'}$ such that $f = \bar{v}(h)$. Now $\bar{u}(f)(K^{'}) = \bar{u}(\bar{v}(h))(K^{'}) = \bar{u}(h \circ v)(K^{'}) = (h \circ v \circ u)(K^{'}) = (h \circ v)(u(K^{'})) = (h \circ v)(v^{-1}(U)) = h(v \circ v^{-1})(U) = h(U) = 0_N$, since $h \in U^{'}$. Therefore, $\bar{v}(U^{'}) \subseteq ker\ \bar{u}$.
Hence the sequence is U'-coexact at $\mathcal{H}(K, N)$.
Conversely, suppose that the sequence-2 is $U^{'}$-exact at $\mathcal{H}(K, N)$ for every R-module N.
To prove sequence-1 is U-exact, write $N = \frac{K^{''}}{Im(v)}$. By converse hypothesis, the sequence,

$$(0) \to \mathcal{H}(K^{''}, \frac{K^{''}}{Im(v)}) \xrightarrow{\bar{v}} \mathcal{H}(K, \frac{K^{''}}{Im(v)}) \xrightarrow{\bar{u}} \mathcal{H}(K^{'}, \frac{K^{''}}{Im(v)})$$

is $U^{'}$-coexact, where $U^{'} = \{h \in \mathcal{H}(K^{''}, \frac{K^{''}}{Im(v)}) : h(U) \subseteq Im(v)\}$.
Consider the map $\pi : K^{''} \to \frac{K^{''}}{Im(v)}$ defined as $\pi(k^{''}) = k^{''} + Im(v)$. Then $\pi \in \mathcal{H}(K^{''}, \frac{K^{''}}{Im(v)})$.

Let $x \in K$. Now $(\bar{v}(\pi))(x) = (\pi \circ v)(x) = \pi(v(x)) = v(x) + Im(v) = v(x) + v(K)$, a zero element in $\frac{K''}{Im(v)}$, as $v(x) \in v(K)$. Therefore, $\bar{v}(\pi) = 0$, so $\pi \in ker\ \bar{v} = \{0\}$, since sequence-2 is U'-coexact. Now let $k'' \in K''$. Since π is a zero mapping in $\mathcal{H}(K'', \frac{K''}{Im(v)})$, we get $\pi(k'') = 0 + Im(v)$, implies $k'' \in Im(v)$ and we get $K'' \subseteq v(K)$. Therefore, v is onto.
To prove $v^{-1}(U) \subseteq Im(u)$, put $N = \frac{K}{Im(u)}$. Clearly, N is an R-module, and by the converse hypothesis, the sequence,

$$(0) \to \mathcal{H}(K'', \frac{K}{Im(u)}) \xrightarrow{\bar{v}} \mathcal{H}(K, \frac{K}{Im(u)}) \xrightarrow{\bar{u}} \mathcal{H}(K', \frac{K}{Im(u)})$$

is U'-coexact, where $U' = \{h \in \mathcal{H}(K'', \frac{K}{Im(u)}) | h(U) \subseteq Im(u)\}$.
Consider the natural map $\pi : K \to \frac{K}{Im(u)}$ defined by $\pi(k) = k + Im(u)$. Let $k' \in K'$. Then $(\bar{u}(\pi))(k') = (\pi \circ u)(k') = \pi(u(k')) = u(k') + Im(u) = 0 + Im(u)$, as $u(k') \in Im(u)$. Now, $\bar{u}(\pi) = 0$ and since sequence-2 is U'-coexact, we have $\pi \in \bar{v}(U')$. Then $\pi = \bar{v}(h) = h \circ v$, for some $h \in U'$. For any $x \in v^{-1}(U)$, $\pi(x) = h(v(x)) \in h(U) \subseteq Im(u)$, since $h \in U'$. So $x + Im(u) = 0 + Im(u)$, and hence $x \in Im(u)$.
To prove $Im(u) \subseteq v^{-1}(U)$, put $N = \frac{K''}{U}$. Then the sequence

$$(0) \to \mathcal{H}(K'', \frac{K''}{U}) \xrightarrow{\bar{v}} \mathcal{H}(K, \frac{K''}{U}) \xrightarrow{\bar{u}} \mathcal{H}(K', \frac{K''}{U})$$

is U'-coexact, where $U' = \{h \in \mathcal{H}(K'', \frac{K''}{U}) | h(U) \subseteq U\}$.
Define $g : K'' \to \frac{K''}{U}$ by $g(k'') = k'' + U$ for every $k'' \in K''$.
Clearly $g \in \mathcal{H}(K'', \frac{K''}{U})$.
Now $g(U) = 0$ in $\frac{K''}{U}$, so that $g \in U'$. Since sequence-2 is U'-coexact, $\bar{u}(g) \in \bar{v}(U') = ker(\bar{u})$, implies that $\bar{u}(\bar{v}(g)) = 0$ in $\frac{K''}{U}$. That is, $g \circ v \circ u = 0$. Now, $0 + U = (g \circ v \circ u)(K') = (g \circ v)(u(K'))$.
This implies $g(v(u(K'))) = v(u(K')) + U$, and so $v(u(K')) \subseteq U$. Therefore, $u(K') \subseteq v^{-1}(U)$, which shows that the sequence-1 is U-exact at K. □

In view of this theorem, we conclude that from the following corollary, an injective module [23] can be characterized in terms of U-exact and U'-coexact sequences.

Corollary 2.5. *Let L be an R-module. Then the folowing statements are equivalent.*

(i) L is injective.

(ii) Given an U-exact sequence,

$$(0) \to K' \xrightarrow{u} K \xrightarrow{v} K'' \to (0), \text{ then}$$

the sequence

$$(0) \to \mathcal{H}(K'', L) \xrightarrow{\bar{v}} \mathcal{H}(K, L) \xrightarrow{\bar{u}} \mathcal{H}(K', L) \to (0)$$

is U'-coexact.

The following examples explicitly illustrate the existence of U-exact and the corresponding U'-coexactness stated in the theorem 2.4.

Example 2.6. The sequence $2\mathbb{Z} \xrightarrow{u} \mathbb{Z} \xrightarrow{v} \frac{\mathbb{Z}}{4\mathbb{Z}} \to (0)$ is $\frac{2\mathbb{Z}}{4\mathbb{Z}}$-exact at $\mathbb{Z}$, where u is the inclusion map and v is the canonical map. Let $N = \frac{\mathbb{Z}}{4\mathbb{Z}}$. Then the sequence,

$$(0) \to \mathcal{H}(\frac{\mathbb{Z}}{4\mathbb{Z}}, \frac{\mathbb{Z}}{4\mathbb{Z}}) \xrightarrow{\bar{v}} \mathcal{H}(\mathbb{Z}, \frac{\mathbb{Z}}{4\mathbb{Z}}) \xrightarrow{\bar{u}} \mathcal{H}(2\mathbb{Z}, \frac{\mathbb{Z}}{4\mathbb{Z}})$$

is U'-coexact at $\mathcal{H}\big(\frac{\mathbb{Z}}{4\mathbb{Z}}, \frac{\mathbb{Z}}{4\mathbb{Z}}\big)$ where $U' = \{f \in \mathcal{H}\big(\frac{\mathbb{Z}}{4\mathbb{Z}}, \frac{\mathbb{Z}}{4\mathbb{Z}}\big) : f\big(\frac{2\mathbb{Z}}{4\mathbb{Z}}\big) = 0 \text{ in } \frac{\mathbb{Z}}{4\mathbb{Z}}\}$.
Clearly, $\bar{v}$ is one-one.
$\mathcal{H}\big(\frac{\mathbb{Z}}{4\mathbb{Z}}, \frac{\mathbb{Z}}{4\mathbb{Z}}\big) = \{id, f_0, f_1\}$ where id is an identity map, f_0 is a zero map and f_1 is defined by $f_1\{\bar{0}, \bar{2}\} = \bar{0}$ and $f_1\{\bar{1}, \bar{3}\} = \bar{2}$. Clearly, $U' = \{f_0, f_1\}$, and $\bar{v}(U') = \{h_0, h_1\}$ where h_0 is a zero mapping and h_1 is defined by

$$h(a) = \begin{cases} \bar{0} \text{ if } a = 2x, \ x \in \mathbb{Z} \\ \bar{2} \text{ if } a = 2x+1, \ x \in \mathbb{Z} \end{cases}$$

Now, $\mathcal{H}\big(\mathbb{Z}, \frac{\mathbb{Z}}{4\mathbb{Z}}\big) = \{h', h_0, h_1\}$ where h' is the canonical map. Now $ker\ \bar{u} = \{f \in \mathcal{H}\big(\mathbb{Z}, \frac{\mathbb{Z}}{4\mathbb{Z}}\big) : \bar{u}(f) = 0 \text{ in } \mathcal{H}\big(2\mathbb{Z}, \frac{\mathbb{Z}}{4\mathbb{Z}}\big)\}$. Clearly, $h' \notin ker\ \bar{u}$ and $ker\ \bar{u} = \{h_0, h_1\}$. Therefore, $ker\ \bar{u} = \bar{v}(U')$ and hence the sequence is U'-coexact.

Example 2.7. Consider the sequence of $\mathbb{Z}$-modules

$$\mathbb{Z}_2 \times \mathbb{Z}_3 \xrightarrow{u} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \xrightarrow{v} \mathbb{Z}_2 \times \mathbb{Z}_2 \to (0),$$

which is $((0) \times \mathbb{Z}_2)$-exact at $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, where $u(l,m) = (0,l,m)$ for all $(l,m) \in \mathbb{Z}_2 \times \mathbb{Z}_3$ and $v(l,m,n) = (l,m)$ for all $(l,m,n) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.
Let $N = \mathbb{Z}_2$ be a $\mathbb{Z}$-module. Then the sequence

$$(0) \to \mathcal{H}\big(\mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2\big) \xrightarrow{\bar{v}} \mathcal{H}\big(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_2\big) \xrightarrow{\bar{u}} \big(\mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_2\big)$$

is U'-exact at $\mathcal{H}\big(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_2\big)$ where

$$U' = \{f \in \mathcal{H}\big(\mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2\big) : f((0) \times \mathbb{Z}_2) = 0 \text{ in } \mathbb{Z}_2\}.$$

Clearly, $\bar{v}$ is one-one.
$\mathcal{H}\big(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2\big) = \{f_0, f_1, f_2\}$ where f_0 is a zero mapping and $f_1(l,m) = l$ and $f_2(l,m) = m$ for all $(l,m) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. It can be seen that $U' = \{f_0, f_1\}$, and $\bar{v}(U') = \{h_0, h_1\}$ where h_0 is a zero mapping and $h_1(l,m,n) = l$ for all $(l,m,n) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.
Now, $\mathcal{H}\big(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_2\big) = \{h_0, h_1, h_2\}$, where $h_2(l,m,n) = m$ for all $(l,m,n) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.
Now $ker\ \bar{u} = \{h \in \mathcal{H}\big(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_2\big) : \bar{u}(h) = 0 \text{ in } \big(\mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_2\big)\}$. It can be verified that $h_2 \notin ker\ \bar{u}$ and $ker\ \bar{u} = \{h_0, h_1\}$. Therefore, $ker\ \bar{u} = \bar{v}(U')$ and hence the sequence is U'-coexact.

Note 2.8. [16] Let $\begin{array}{ccc} K & \xrightarrow{u} & K' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ N & \xrightarrow{v} & N' \end{array}$ be a commutative diagram of R-modules and R-homomorphisms.
Then

(i) $u(ker\ f) \subseteq ker\ f'$

(ii) $v(Im\ f) \subseteq Im\ f'$

Note 2.9. [16] The above u and v give rise to the R-homomorphisms:
$\bar{u} : ker\ f \to ker\ f'$ defined by $\bar{u}(k) = u(k)$ for every $k \in ker\ f$
$\tilde{v} : coker\ f \to coker\ f'$ defined by $\tilde{v}(n + Im\ f) = v(n) + Im\ f'$. Then

(i) If u is one-one, then $\bar{u}$ is one-one.

(ii) If v is onto, then $\tilde{v}$ is onto.

Theorem 2.10. *Let*

$$
\begin{array}{ccccccccccc}
(0) & \longrightarrow & K' & \xrightarrow{u} & K & \xrightarrow{v} & K'' & \longrightarrow & (0), & & (U-exact) \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & & & \\
(0) & \longrightarrow & N' & \xrightarrow{u'} & N & \xrightarrow{v'} & N'' & \longrightarrow & (0), & & (U'-exact)
\end{array}
$$

$fig.1$

be a commutative diagram of R-modules and R-homomorphisms with row-1 is U-exact and row-2 is U'-exact. Then the sequences,

$$(0) \rightarrow kerf' \xrightarrow{\bar{u}} kerf \xrightarrow{\bar{v}} kerf''$$

is $(kerf'' \cap U)$-exact at $ker\ f$, and

$$cokerf' \xrightarrow{\tilde{u'}} cokerf \xrightarrow{\tilde{v'}} cokerf'' \rightarrow (0)$$

is $(\frac{U'+Im\ f''}{Im\ f''})$-exact at $coker\ f$.

Proof. By hypothesis, $Im\ u = v^{-1}(U)$, v is onto and u is one-one.
Consider the sequence $(0) \rightarrow kerf' \rightarrow kerf \rightarrow kerf''$. Since u is injective, we have $\bar{u}$ is injective (by note 2.9). It remains to show that $Im(\bar{u}) = \bar{v}^{-1}(U \cap ker\ f'')$.
To prove $Im(\bar{u}) \subseteq \bar{v}^{-1}(U \cap ker\ f'')$. Let $k \in Im(\bar{u})$. Then there exists $k' \in ker\ f'$ such that $\bar{u}(k') = k$. By 2.8 and from the fact that row-1 is U-exact, it follows that $\bar{v}(k) \in U$. That is, $k \in \bar{v}^{-1}(U)$. Now since $k \in Im(\bar{u}) \subseteq ker\ f$. we have $v(k) \in ker\ f''$, and by note 2.9. $\bar{v}(k) = v(k) \in ker\ f''$, shows that $k \in \bar{v}^{-1}(ker\ f'')$ and concludes $Im(\bar{u}) \subseteq \bar{v}^{-1}(U \cap ker\ f'')$.
On the other hand, let $k \in \bar{v}^{-1}(U \cap ker\ f'')$. Then $\bar{v}(k) \in U \cap ker\ f''$. Since $k \in v^{-1}(U \cap ker\ f'') \subseteq v^{-1}(ker\ f'') \subseteq ker\ f$, we have $v(k) \in U \cap ker\ f''$. Now $k \in v^{-1}(U \cap ker\ f'') \subseteq v^{-1}(U) = Im(u)$, there exists $k' \in K'$ such that $u(k') = k$.
Now we show that $k' \in ker\ f'$, so that $k \in Im(\bar{u})$. For this, since $k \in ker\ f$, it follows that $u'(f'(k')) = (u' \circ f')(k') = (f \circ u)(k') = f(k) = 0$. Since u' is one-one, $f'(k') \in ker\ u' = 0$. Therefore, $k' \in ker\ f'$ and hence $\bar{u}(k') = u(k') = k$, proves $v^{-1}(U \cap ker\ f'') \subseteq Im(\bar{u})$.
Consider the sequence: $coker\ f' \xrightarrow{\tilde{u'}} coker\ f \xrightarrow{\tilde{v'}} coker\ f'' \rightarrow (0)$. Since v is onto, by the note 2.9, we have $\tilde{v'}$ is onto. Therefore, it remains to show that $Im(\tilde{u'}) = \tilde{v'}^{-1}(\frac{U'+Im\ f''}{Im\ f''})$.
Let $n + Im\ f \in Im(\tilde{u'})$. Then there exists $n' + Im\ f' \in coker\ f'$ such that $\tilde{u'}(n' + Im\ f') = n + Im\ f$. Now, $\tilde{v'}(n + Im\ f) = \tilde{v'}(\tilde{u'}(n' + Im\ f')) = \tilde{v'}(u'(n') + Im\ f) = v'(u'(n')) + Im\ f''$(note 2.9) $\in v'(Im(u')) + Im\ f''$. Since row-2 is U'-exact, it follows that $\tilde{v'}(n + Im\ f) = v'(v'^{-1}(U')) + Im\ f'' = U' + Im\ f''$. Therefore, $\tilde{v'}(n + Im\ f) = x + Im\ f'$ for some $x \in U'$, which shows that $\tilde{v'}(n + Im\ f) \in (\frac{U'+Im\ f''}{Im\ f''})$.
On the other hand, let $n + Im\ f \in \tilde{v'}^{-1}(\frac{U'+Im\ f''}{Im\ f''})$, implying that $v'(n) + Im\ f' = x + Im\ f'$ for some $x \in U'$. Now $v'(n) - x \in Im\ f'$ and so $v'(n) - x = f''(k'')$ for some $k'' \in K''$. Since v is onto, there exists $k \in K$ such that $v(k) = k''$. Now we have, $v'(n) - x = f''(v(k)) = v'(f(k))$ since $f'' \circ v = v' \circ f$. Therefore, $n - f(k) \in v'^{-1}(U') = Im(u')$, and hence $n - f(k) = u'(n')$ for some $n' \in N'$. Now $(n - f(k)) + Im\ f = u'(n') + Im\ f$, implying $n + Im\ f \in Im(\tilde{u'})$, as desired.

□

3 Conclusion

In this paper, we have continued the notions U-exact, V-coexact sequences which was introduced in [20]. We have proved further results on these notions. The study can be extended to flat modules and tensor products of modules in terms of U-exact sequences. One can also study similar exactness of sequences of ideals of modules over nearrings (for comprehensive literature on module over nearrings, one may refer to [18, 19]).

Acknowledgement

The authors express their gratitude to the referee(s)/editor(s) for their valuable suggestions that have improved the paper. Author[1] acknowledges Dr. T M A Pai fellowship, MAHE, Manipal. All the authors acknowledge Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal for the kind encouragement.

References

[1] K. E. Aubert, On the ideal theory of commutative semi-groups, *Math. Scand.* **1**, 39–54 (1953).

[2] G. D. Bullington, On the expected number of generators of a submodule of a free module over a finite special principal ideal ring, *Rend. Circ. Mat. Palermo* **51**, 5–50 (2002).

[3] L. Fuchs, A remark on the Jacobson radical, *Acta Sci. Math. (Szeged)* **14**, 167–168 (1952).

[4] R. Gilmer, *Commutative Semigroup Rings*, Chicago Lectures in Math., Univ. Chicago Press, Chicago (1984).

[5] K. W. Gruenberg, *Cohomological Topics in Group Theory*, Lecture Notes in Math. **143**, Springer-Verlag, Berlin (1970).

[6] M. Hall, Jr., *The Theory of Groups*, Macmillan, New York (1959).

[7] N. Jacobson, *Lectures in Abstract Algebra*, volume I, Van Nostrand, Princeton (1951).

[8] I. Kaplansky, *Commutative Rings* (Revised Edn), Univ. Chicago Press, Chicago (1974).

[9] R. L. Kruse and D. T. Price, *Nilpotent Rings*, Gordon and Breach, New York (1969).

[10] E. S. Ljapin, *Semigroups*, Translations Math. Monographs **3**, Amer. Math. Soc., Providence (1974).

[11] W. R. Scott, *Group Theory*, Prentice-Hall, Englewood Cliffs (1964).

[12] O. Zariski and P. Samuel, *Commutative Algebra*, volume I, Van Nostrand, Princeton (1958).

[13] F. W. Anderson and K. R. Fuller,*Rings and Categories of Modules*, Springer Verlag, New York, Inc. (1992).

[14] S. M. Anvariyeh and B. Davvaz, U-Split-Exact Sequences, Far East J. Math. Sci.(FJMS) **4**, no. 2, 209-219, (2002).

[15] S. M. Anvariyeh and B. Davvaz, On Quasi-Exact Sequences, Bull. Korean Math. Soc. **42**, no. 1, 149-155, (2005).

[16] M. F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, (1969).

[17] S. Bhavanari, The Injective Hull of a Module with FGD, Indian J. Pure and Appl.Math. **20**, 874-883, (1989).

[18] S. Bhavanari and S. P. Kuncham, *Nearrings, Fuzzy ideals and Graph theory*, CRC Press(Taylor and Francis), (2013).

[19] S. P. Kuncham, B. S. Kedukodi, P. K. Harikrishnan and S. Bhavanari(Editors), *Nearrings, Nearfields and Related Topics*(Review volume), World Scientific, (2017).

[20] B. Davvaz and Y. A. Parnian-Garamaleky, A Note on Exact Sequences, Bull. Malays. Math. Sci. Soc. **22**, no. 1, 53-56, (1999).

[21] B. Davvaz and H. Shabani-Solt, A generalization of homological algebra, Journal of Korean Math. Soc. **39**, no. 6, 881-898, (2002).

[22] S. Rajani, S. Tapatee, P.K. Harikrishnan, B.S. Kedukodi, Kuncham S.P.: Superfluous ideals of N-groups. Rend. Circ. Mat. Palermo, II. Ser (2023), https://doi.org/10.1007/s12215-023-00888-2.

[23] D. W. Sharpe and P. Vamos, Injective Modules, Cambridge Univ. Press, (1972).

[24] S.Tapatee, B.Davvaz, Panackal H., B. S. Kedukodi, S. P.Kuncham , *Relative essential ideals in N-groups*, Tamkang Journal of Mathematics, **54**, n0. 1, 69–82, (2023), https://doi.org/10.5556/j.tkjm.54.2023.4136.

Author information

Rajani S. and Kedukodi B.S. and Harikrishnan P. and Kuncham S.P.*, Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India.
E-mail: `rajanisalvankar@gmail.com`, `babushrisrinivas.k@manipal.edu`, `pk.harikrishnan@manipal.edu`, `syamprasad.k@manipal.edu`, `kunchamsyamprasad@gmail.com`