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Square Power Graph of Dihedral Group of order 2n for odd number n

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Abstract Let D_n be dihedral group of order $2n$ with identity element e then its square power graph is a undirected, finite, simple graph in which pair of distinct vertices a, b have edge iff $ab = c^2$ or $ba = c^2$ for any $c \in D_n$ where $c^2 \neq e$. In this research paper we have studied various structural properties such as connectedness, vertex degree, girth, clique, chromatic number and laplacian spectrum of square power graph of dihedral group D_n of order $2n$ for odd number n.

1 Introduction

Studying graphs associated with groups is an growing and interesting topic among researchers. Depending upon different group properties we have various types of graphs such as power graph, co-prime graph, additive power graph, square element graph, square power graph, cubic power graph, k^{th}-power graph, etc. A. Sehgal and SN Singh researched the degree of a vertex in the power graph of a finite abelian group [1]. L. Lu and others studied the Integral Cayley graphs over dihedral groups [2]. R. S. Gupta and M. K. Sen studied square element graph in association with ring in [3]. B. Biswas and other researchers studied connectedness of square element graph of arbitrary rings in [4], properties of square element graph of square-subtract rings in [5] and over semigroups in [6]. R. R. Prathap and T. T. Chelvam researched cubic power graph in [7] and complement of square power graph in [8] for finite abelian group. P. Rana and others studied degree of a vertex in $k^{th}-$power graph of a finite abelian group [9]. A. Siwach and others researched square power graph of Z_n and $2-group$ in [10]. A. Sehgal and others explored the structural characteristics of Co-Prime Order graph for a finite abelian Group and Dihedral Group in [11]. F. Ali and others studied the properties of commuting graph of dihedral group [12]. Girth, diameter, completeness, bipartiteness, chromatic number and traversability of line graphs associated to the unit graphs of rings are studied in [13]. In [14] laplacian spectra for power graphs of abelian groups is studied.
In this paper we have studied square power graph of dihedral group. Square power graph of a finite group G with identity element e, $\Gamma_{sq}(G)$ is undirected, simple, finite graph with vertex set G and two distinct vertices $a, b \in G$ are adjacent if and only if $ab = c^2$ or $ba = c^2$ for any $c \in G$ and $c^2 \neq e$.

2 Preliminaries

Let us recall some important results about graphs and dihedral group which are necessary and used in our study. D_n be a dihedral group of order $2n$, with n reflection elements and n rotation elements i.e $D_n = \{x^i y^j : x^2 = e, y^n = e, xy = y^{-1}x; i = 0, 1; j = 0, 1, 2, \cdots, n-1\}$.
Throughout this paper we have used
$D_n = \{R_0, R_{\frac{360}{n}}, R_{\frac{2\times 360}{n}}, R_{\frac{3\times 360}{n}}, \cdots R_{\frac{(n-1)\times 360}{n}}, F_{a_1}, F_{a_2}, F_{a_3}, \cdots, F_{a_n}\}$ here $R_{\frac{i\times 360}{n}}$ are rotation elements for $0 \leq i \leq n-1$ and $R_{360} = R_0$; and F_{a_j} are reflection elements all with order 2 for $1 \leq j \leq n$. We have two distinct rotation elements or two distinct reflection elements always combine to form rotation element i.e $R_{\frac{i\times 360}{n}} \times R_{\frac{j\times 360}{n}} = R_{\frac{k\times 360}{n}}$ where $0 \leq i, j, k \leq n-1$ and $F_{a_i} \times F_{a_j} = R_{\frac{k\times 360}{n}}$ where $1 \leq i, j \leq n$ and $0 \leq k \leq n-1$. We also have one

rotation and one reflection element combine to form reflection element i.e $R_{\frac{i\times 360}{n}} \times F_{a_j} = F_{a_k}$ where $0 \leq i \leq n-1$ and $1 \leq j,k \leq n$. When n is odd we have n number of elements of order 2 in D_n. $D_n^2 = \{R_0^2, R^2_{\frac{360}{n}}, R^2_{\frac{2\times 360}{n}}, R^2_{\frac{3\times 360}{n}}, \cdots R^2_{\frac{(n-1)\times 360}{n}}, F^2_{a_1}, F^2_{a_2}, F^2_{a_3}, \cdots, F^2_{a_n}\} = \{R_0, R_{\frac{360}{n}}, R_{\frac{2\times 360}{n}}, R_{\frac{3\times 360}{n}}, \cdots, R_{\frac{(n-1)\times 360}{n}}\}$ when n is odd number.

Square power graph of a finite group G with identity element e, $\Gamma_{sq}(G)$ is undirected, simple, finite graph with vertex set G and two distinct vertices $a, b \in G$ are adjacent if and only if $ab = c^2$ or $ba = c^2$ for any $c \in G$ and $c^2 \neq e$. It can be clearly seen that any pair of vertices $X, Y \in D_n$ are adjacent in square power graph $\Gamma_{sq}(D_n)$ if and only if $XY \in D_n^2 \setminus \{R_0\}$ or $YX \in D_n^2 \setminus \{R_0\}$. When two distinct vertices are adjacent we say they have edge between them. We have used $d(X, Y)$ for distance between vertices X and Y vertices, shortest path length between vertices X and Y. Girth $gr(\Gamma_{sq}(G))$ of square power graph is the length of shortest cycle in $\Gamma_{sq}(G)$. Maximal complete subgraph of $\Gamma_{sq}(G)$ is known as clique of $\Gamma_{sq}(G)$ and number of vertices in largest clique is denoted as $\omega(\Gamma_{sq}(G))$ and known by clique number of square power graph. Number of vertices with which vertex X is adjacent in $\Gamma_{sq}(G)$ is known as degree of vertex X in square power graph $\Gamma_{sq}(G)$, denoted as $deg_{\Gamma_{sq}(G)}(X)$. K_n is the complete graph with n vertices. If there are n disjoint components $C_1, C_2, \cdots, C_n$ of graph C then we denote them as $C = [C_1] \cup [C_2] \cup \cdots \cup [C_n]$. Complement of any component of graph means pairs of vertices which were having edge in that component is not having edge in its complement and pair of vertices which were not having edge in that component are having edge in complement of that component. For complement of any component C_1 of C we have used $\overline{C_1}$. So we have $\overline{[K_n] \cup [K_n] \cup \cdots \cup [K_n]} = nK_n$, $\overline{[K_n]} = nK_1$ We have two disjoint components $\overline{K_1 \cup \frac{n-1}{2} K_2}$ and K_n in $\Gamma_{cpg}(G)$ when $G = D_n$ with odd number n. Thus we have used $\Gamma_{cpg}(G) = \overline{[K_1 \cup \frac{n-1}{2} K_2]} \cup [K_n]$.

3 Properties of $\Gamma_{sq}(D_n)$

Theorem 3.1. *Let $\Gamma_{sq}(G)$ be square power graph of D_n and n is odd number then $\Gamma_{sq}(G)$ is disconnected graph.*

Proof. For $G = D_n$, $\Gamma_{sq}(G)$ be square power graph. We have $X, Y \in D_n$ if and only if $XY \in D_n^2$ or $YX \in D_n^2$. We have $D_n^2 = \{R_{\frac{i\times 360}{n}} : 0 \leq i \leq n-1\}$ when n is odd number. When one of X, Y is rotation element and another is reflection element then we have XY and YX both are reflection elements. Thus we have $XY \notin D_n^2$ and $YX \notin D_n^2$. So we have no edge between any reflection element and rotation element vertex in square power graph of dihedral group. We have no path between reflection and rotation vertices. Hence square power graph of dihedral group $\Gamma_{sq}(D_n)$ for odd n is disconnected graph. □

Theorem 3.2 ([6]). *Let $\Gamma_{sq}(G)$ be square power graph of D_n then $\Gamma_{sq}(G) = \overline{[K_1 \cup \frac{n-1}{2} K_2]} \cup [K_n]$ if n is odd number.*

Proof. When n is odd number, we have $D_n^2 = \{R_{\frac{i\times 360}{n}} : 0 \leq i \leq n-1\}$ and we have two vertices $X, Y \in D_n$ are adjacent in $\Gamma_{sq}(D_n)$ if and only if $XY \in D_n^2 \setminus \{R_0\}$ or $YX \in D_n^2 \setminus \{R_0\}$. When X and Y both are rotation elements and $X^{-1} \neq Y$ then we have XY and YX both belongs to $D_n^2 \setminus \{R_0\}$ and so are adjacent. In this case we have only R_0 element which is self inverse so we have R_0 element vertex having edge with all other rotation vertices in $\Gamma_{sq}(D_n)$.
When both of X and Y are both are reflection elements then we have XY and YX both belongs to D_n^2. So we have edge between every pair of reflection element vertices in square power graph. When one of X and Y is reflection element and another is rotation element then we have both XY and YX reflection elements and so we have both XY and YX does not belong to D_n^2. Thus we have no edge between any reflection element vertex and rotation element vertex. Hence we have two components in $\Gamma_{sq}(D_n)$, one with n rotation element vertices and another with n reflection element vertices. In component with rotation element vertices we have R_0 having edge with every other rotation element vertex and every rotation element other than R_0 having edge with every other rotation other than its inverse. Hence we have $\overline{[K_1 \cup \frac{n-1}{2} K_2]}$ structure of this component. In second component with n reflection element vertices we have edge between

every pair of distinct reflection vertices and so we have K_n structure of this component. Hence we have $\Gamma_{sq}(D_n) = \overline{[K_1 \cup \frac{n-1}{2} K_2]} \cup [K_n]$ when n is odd number. □

Corollary 3.3. *Let $\Gamma_{sq}(G)$ be square power graph of D_n then number of components in $\Gamma_{sq}(G)$ is two when n is odd number.*

Proof. From Theorem 3.2 we have two disjoint components in $\Gamma_{sq}(G)$ when n is odd number.□

Corollary 3.4. *Let $\Gamma_{sq}(G)$ be square power graph of D_n and n is odd number then*

$$(i) d(R_i, R_j) = \begin{cases} 1 \text{ if } R_i^{-1} \neq R_j \\ 2 \text{ if } R_i^{-1} = R_j \end{cases}$$

$$(ii) d(F_{a_i}, F_{a_j}) = 1$$

Proof. Using Theorem 3.2 we have the required result. □

Theorem 3.5. *Let $\Gamma_{sq}(G)$ be square power graph of D_n and n is odd number then girth*

$$gr(\Gamma_{sq}(G)) = \begin{cases} \infty \text{ when } n = 1 \\ 3 \text{ otherwise} \end{cases}$$

Proof. When $n = 1$ then by using Theorem 3.2 we have $\Gamma_{sq}(G) = \overline{[K_1]} \cup [K_1] = K_1 \cup K_1$. Hence we have no cycle in $\Gamma_{sq}(G)$ and so $gr(\Gamma_{sq}(G)) = \infty$ when $n = 1$.
When n is odd number other than 1 then by using Theorem 3.2 we have one component K_n of $\Gamma_{sq}(G)$. Hence we have cycle of length three in $\Gamma_{sq}(G)$ for odd number n other than 1. Hence $gr(\Gamma_{sq}(G)) = 3$ when n is odd number other than 1. □

Theorem 3.6. *Let $\Gamma_{sq}(G)$ be square power graph of D_n then clique number $\omega(\Gamma_{sq}(G)) = n$ when n is odd number.*

Proof. When $n = 1$ then by using Theorem 3.2 we have $\Gamma_{sq}(G) = \overline{[K_1]} \cup [K_1] = K_1 \cup K_1$. Thus we have K_1 complete subgraph of $\Gamma_{sq}(G)$ with maximum vertices. Hence $\omega(\Gamma_{sq}(G)) = 1$ when $n = 1$.
When n is odd number ≥ 3 then from Theorem 3.2 we have two components each with n vertices and one of them is K_n. Thus K_n is maximal complete subgraph of $\Gamma_{sq}(G)$. Hence $\omega(\Gamma_{sq}(G)) = n$ when n is odd number ≥ 3.
Hence the required result. □

Theorem 3.7 ([6]). *Let $\Gamma_{sq}(G)$ be square power graph of D_n then chromatic number $\chi(\Gamma_{sq}(G)) = n$ when n is odd number.*

Proof. Using Theorem 3.2 and Corollary 3.3 we have two components in $\Gamma_{sq}(G)$, one of which is of form K_n. Assign n different colours to n vertices in K_n component. Now using the $1 + \frac{n-1}{2}$ colours out of n colours already used in K_n component, we get the proper colouring i.e adjacent vertices assigned different colours. Hence $\chi(\Gamma_{sq}(G)) = n$. □

Theorem 3.8. *Let $\Gamma_{sq}(G)$ be square power graph of D_n and n is odd number then $\Gamma_{sq}(G)$ is weakly perfect.*

Proof. Using Theorem 3.6 and Theorem 3.7 we get $\omega(\Gamma_{sq}(G)) = n = \chi(\Gamma_{sq}(G))$. Thus $\Gamma_{sq}(G)$ is weakly perfect. □

Theorem 3.9. *Let $\Gamma_{sq}(G)$ be square power graph of D_n and n is odd number then degree of any vertex X*

$$deg_{\Gamma_{sq}(G)}(X) = \begin{cases} n-1 \text{ if } X = R_0 \\ n-2 \text{ if } X \text{ is any rotation element vertex other than } R_0 \\ n-1 \text{ if } X \text{ is any reflection element vertex} \end{cases}$$

Proof. From Theorem 3.2 we have two components in $\Gamma_{sq}(G)$. In one component with n rotation element vertices, R_0 is adjacent with all other $n-1$ rotation element vertices. Any rotation element other than R_0 is not self inverse in D_n when n is odd number. Every rotation element vertex

in this component other than R_0 is adjacent with $n-2$ rotation element vertices which excludes only their inverse rotation element vertex. So $deg_{\Gamma_{sq}(G)}(R_0) = n-1$ and $deg_{\Gamma_{sq}(G)}(X) = n-2$ if X is any rotation element vertex other than R_0.
In second component of $\Gamma_{sq}(G)$, we have n reflection elements vertices with every pair of distinct vertices having edge. Hence $deg_{\Gamma_{sq}(G)}(X) = n-1$ if X is any reflection element vertex.□

Theorem 3.10. *Let $\Gamma_{sq}(G)$ be square power graph of D_n, dihedral group of order $2n$ and n is odd number then independent number,* $\beta(\Gamma_{sq}(G)) = \begin{cases} 2 \text{ if } n = 1, \\ 3 \text{ otherwise.} \end{cases}$

Proof. Let $\Gamma_{sq}(G)$ be square power graph of D_n, dihedral group of order $2n$ and n is odd number. Then by using Theorem 3.2 we have two disjoint components in $\Gamma_{sq}(G)$, one of which is K_n and another is $\overline{[K_1 \cup \frac{n-1}{2}K_2]}$.
Case 1. When $n = 1$
In this case we have $\Gamma_{sq}(G) = 2K_1$. Hence $\beta(\Gamma_{sq}(G)) = 2$.
Case 2. When n is odd number other than 1
In this case we have one vertex from K_n component and two vertices of a, a^{-1} type from $\overline{[K_1 \cup \frac{n-1}{2}K_2]}$ component forming the maximal independent set. Thus $\beta(\Gamma_{sq}(G)) = 3$. □

Theorem 3.11. *Let $\Gamma_{sq}(G)$ be square power graph of D_n, dihedral group of order $2n$ and n is odd number then matching number, $\mu(\Gamma_{sq}(G)) = n-1$.*

Proof. Let $\Gamma_{sq}(G)$ be square power graph of D_n, dihedral group of order $2n$ and n is odd number. Then using Theorem 3.2 we have two disjoint components in $\Gamma_{sq}(G)$, one of which is K_n and another is $\overline{[K_1 \cup \frac{n-1}{2}K_2]}$. So $\frac{n-1}{2}$ edges from K_n component and $\frac{n-1}{2}$ edges from $\overline{[K_1 \cup \frac{n-1}{2}K_2]}$ component forms the maximal independent edge set. Thus $\mu(\Gamma_{sq}(G)) = \frac{n-1}{2} + \frac{n-1}{2} = n-1$. □

4 Laplacian Spectrum of $\Gamma_{sq}(D_n)$

Theorem 4.1. *Let $\Gamma_{sq}(G)$ be square power graph of D_n where n is odd number then we have Laplacian matrix* $L = \begin{bmatrix} U & O \\ O & V \end{bmatrix}_{2n \times 2n}$. *Characteristic polynomial of L is $C(x) = x^2(n-x)^{\frac{3(n-1)}{2}}(n-2-x)^{\frac{n-1}{2}}$ having laplacian spectrum 0 with multiplicity 2, $n-2$ with multiplicity $\frac{n-1}{2}$ and n with multiplicity $\frac{3(n-1)}{2}$.*

where $U = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & n-2 & -1 & \cdots & -1 & -1 & 0 \\ -1 & -1 & n-2 & \cdots & -1 & 0 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & 0 & \cdots & -1 & n-2 & -1 \\ -1 & 0 & -1 & \cdots & -1 & -1 & n-2 \end{bmatrix}_{n \times n}$,

$$V = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & n-1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & -1 & n-1 & \cdots & -1 & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & \cdots & n-1 & -1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & n-1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & n-1 \end{bmatrix}_{n \times n}$$

and O is $n \times n$ zero matrix.

Proof. Using Theorem 3.9 and Corollary 3.4 we get the $2n \times 2n$ laplacian matrix,
$L = \begin{bmatrix} U & O \\ O & V \end{bmatrix}_{2n \times 2n}$ and so we have Characteristic polynomial of L,

$C(x) = det(L - xI) = \begin{bmatrix} U - xI & O \\ O & V - xI \end{bmatrix}$. So, $C(x) = det(U - xI)det(V - xI)$. We have

$$U - xI = \begin{bmatrix} n-1-x & -1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & n-2-x & -1 & \cdots & -1 & -1 & 0 \\ -1 & -1 & n-2-x & \cdots & -1 & 0 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & 0 & \cdots & -1 & n-2-x & -1 \\ -1 & 0 & -1 & \cdots & -1 & -1 & n-2-x \end{bmatrix}_{n\times n} \quad \text{and}$$

$$V - xI = \begin{bmatrix} n-1-x & -1 & -1 & \cdots & -1 & -1 & -1 \\ -1 & n-1-x & -1 & \cdots & -1 & -1 & -1 \\ -1 & -1 & n-1-x & \cdots & -1 & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & \cdots & n-1-x & -1 & -1 \\ -1 & -1 & -1 & \cdots & -1 & n-1-x & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & n-1-x \end{bmatrix}_{n\times n}$$

Applying $R_1 \to \sum_{i=1}^{n} R_i$ and then taking out $-x$ common from first row in $det(U - xI)$, we get $det(U - xI) = -xdet(U_1)$

$$\text{where } U_1 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & n-2-x & -1 & \cdots & -1 & -1 & 0 \\ -1 & -1 & n-2-x & \cdots & -1 & 0 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & 0 & \cdots & -1 & n-2-x & -1 \\ -1 & 0 & -1 & \cdots & -1 & -1 & n-2-x \end{bmatrix}_{n\times n}$$

Applying $R_i \to R_i + R_1$ for $2 \leq i \leq n$ on $det(U_1)$, we get
$det(U - xI) = -xdet(U_1) = -xdet(U_2)$ where

$$U_2 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & n-1-x & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & n-1-x & \cdots & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & \cdots & 0 & n-1-x & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & n-1-x \end{bmatrix}_{n\times n}$$

Solving U_2 along first coloumn we get
$det(U - xI) = -xdet(U_1) = -xdet(U_2) = -xdet(U_3)$ where

$$U_3 = \begin{bmatrix} n-1-x & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & n-1-x & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & n-1-x & \cdots & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & 0 & n-1-x & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & n-1-x \end{bmatrix}_{(n-1)\times(n-1)}$$

Now applying $R_1 \to \sum_{i=1}^{n-1} R_i$, and taking $(n-x)$ common from first row we get, $det(U-xI) = -xdet(U_1) = -xdet(U_2) = -xdet(U_3) = -x(n-x)det(U_4)$

$$\text{where } U_4 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & n-1-x & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & n-1-x & \cdots & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & 0 & n-1-x & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & n-1-x \end{bmatrix}_{(n-1)\times(n-1)}$$

Now applying $R_{n-1} \to R_{n-1} - R_1$, on $det(U_4)$ and then solving it along first coloumn we get,

$det(U - xI) = -xdet(U_1) = -xdet(U_2) = -xdet(U_3) = -x(n-x)det(U_4) = -x(n-x)det(U_5)$

where $U_5 = \begin{bmatrix} n-1-x & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & n-1-x & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & n-1-x & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 0 & n-1-x & 0 \\ -1 & -1 & -1 & \cdots & -1 & -1 & n-2-x \end{bmatrix}_{(n-2)\times(n-2)}$

Now solving $det(U_5)$ along $(n-2)^{th}$ column we get,
$det(U - xI) = -x(n-x)det(U_5) = -x(n-x)(n-2-x)det(U_6)$

where $U_5 = \begin{bmatrix} n-1-x & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & n-1-x & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & n-1-x & \cdots & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & 0 & n-1-x & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & n-1-x \end{bmatrix}_{(n-3)\times(n-3)}$

On repeating the same process in last we get,
$det(U - xI) = -x(n-x)^{\frac{n-1}{2}}(n-2-x)^{\frac{n-1}{2}}$
Now let us find $det(V - xI)$, applying $R_1 \to \sum_{i=1}^{n} R_i$ on $det(V - xI)$, then taking out $-x$ common from first row, we get $det(V - xI) = -xdet(V_1)$

where $V_1 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & n-1-x & -1 & \cdots & -1 & -1 & -1 \\ -1 & -1 & n-1-x & \cdots & -1 & -1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & -1 & \cdots & -1 & n-1-x & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & n-1-x \end{bmatrix}_{n\times n}$

Applying $R_i \to R_i + R_1$ for $2 \le i \le n$ on $det(V_1)$, and then solving along first column we get $det(V - xI) = -xdet(V_1) = -xdet(V_2)$ where

$V_2 = \begin{bmatrix} n-x & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & n-x & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & n-x & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & n-x & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & n-x \end{bmatrix}_{(n-1)\times(n-1)}$

Thus we get, $det(V - xI) = -xdet(V_1) = -xdet(V_2) = -x(n-x)^{n-1}$
We get $C(x) = det(U - xI)det(V - xI) = -x(n-x)^{\frac{n-1}{2}}(n-2-x)^{\frac{n-1}{2}}(-x)(n-x)^{n-1} = x^2(n-x)^{\frac{3(n-1)}{2}}(n-2-x)^{\frac{n-1}{2}}$.
Hence Characteristic polynomial of L is $C(x) = x^2(n-x)^{\frac{3(n-1)}{2}}(n-2-x)^{\frac{n-1}{2}}$ and so laplacian spectrum 0 with multiplicity 2, $n-2$ with multiplicity $\frac{n-1}{2}$ and n with multiplicity $\frac{3(n-1)}{2}$. □

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