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CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING HURWITZ-LERCH ZETA FUNCTION

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Abstract Making use of the Hurwitz - Lerch zeta operator, we introduce a new subclass of analytic functions defined in the open unit disk and investigate its various characteristics. Further we obtain some usual properties of the geometric function theory such as coefficient bounds, extreme points, closure theorems radius of starlikness and convexity, partial sums and neighbourhood results belonging to the class.

1 Introduction

Let A denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions.

A function $u \in A$ is a starlike function of the order $\xi,\ 0 \leq \xi < 1$, if it satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \xi, \quad z \in U. \tag{1.2}$$

We denote this class with $S^*(\xi)$.

A function $u \in A$ is a convex function of the order $\xi,\ \ 0 \leq \xi < 1$, if it fulfils

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \xi, \quad z \in U. \tag{1.3}$$

We denote this class with $K(\xi)$.

Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in U respectively. For $f \in A$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.4}$$

their convolution (or Hadamard product), denoted by $(u * g)$, is defined as

$$(u * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * u)(z), \quad (z \in U). \tag{1.5}$$

Note that $u * g \in A$. Let T denotes the class of functions analytic in U that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \ \ (z \in U) \tag{1.6}$$

and let $T^*(\xi) = T \cap S^*(\xi)$, $C(\xi) = T \cap K(\xi)$. The class $T^*(\xi)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [17].

The study of operators is essential in geometric function theory and its associated topics. In recent years, there has been a surge in interest in issues involving evaluations of various families of series associated with the Riemann and Hurwitz zeta functions, as well as their extensions and generalisations such as the Hurwitz-Lerch zeta function. These functions ascend naturally in many branches of analytic function theory and their studies have plentiful important applications in mathematics [1]. As a overview of both Riemann and Hurwitz zeta functions, the so-called Hurwitz-Lerch zeta function is defined in [7]. Hurwitz- Lerch Zeta function $\Phi(z, s, a)$ defined in [20] given by

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.7}$$

$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}; \Re(s) > 1)$ and $|z| < 1$ where, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}$, $(\mathbb{Z} := \{\pm 0, \pm 1. \pm 2, \pm 3, ...\})$. It is clear that Φ is an analytic function in both variables s and z in a suitable region and it reduces to the ordinary Lerch zeta function when $z = e^{2\pi i \lambda}$. Morever, Φ yields the following known result [7].

$$\Phi(z, 1, a) = a^{-1}\, {}_2F_1(a, 1; a+1, z),$$

where ${}_2F_1$ is the Gaussian hypergeometric function. Several interesting properties and characteristics of the Hurwitz- Lerch Zerch function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [6], and (also see[11]) the reference stated therein.The double zeta function of Barnes [3]is defined by

$$\zeta(x, a, \sigma) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (m + a + n\sigma)^{-x},$$

where $a \neq 0$ and σ is a non zero complex number with $|arg(\sigma)| < \pi$. Bin- Saad [5] posed a generalized double zeta function of the form

$$\zeta_\sigma^\nu(z, s, a) = \sum_{n=0}^{\infty} (\nu)_n \Phi(z, s, a + n\sigma) \frac{z^n}{n!}$$

where $\sigma \in \mathbb{C} \setminus \{0\}; \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ a \in \mathbb{C} \setminus \{-(m + \sigma n)\}, n, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, |s| < 1;\ |z| < 1$ and Φ is the Hurwitz-Lerch zeta function distinct by (1.7) and $(\nu)_n$ is the Pochhammer symbol defined by

$$(\nu)_n = \begin{cases} 1, & n = 0 \\ \nu(\nu+1)(\nu+2)...(\nu+n-1), & n \in \mathbb{N} \end{cases} \tag{1.8}$$

In [15],Rabhaw and Darus defined a function as follows:

$$\Theta_n(z, s, a) = \frac{\Phi(z, s, a + n\sigma)}{\Phi(z, s, a)}, \qquad n \in \mathbb{N}_0 \tag{1.9}$$

It is clear that $\Theta_0(z, s, a) = 1$. Now consider the function

$$\Upsilon_\nu(z, s, a) = \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} \Theta_n(z, s, a) z^n, \tag{1.10}$$

which implies

$$z\Upsilon_\nu(z, s, a) = z + \sum_{n=2}^{\infty} \frac{(\nu)_{n-1}}{(n-1)!} \Theta_{n-1}(z, s, a) z^n$$

Thus,

$$z\Upsilon_\nu(z, s, a) * (z\Upsilon_\nu(z, s, a))^{-1} = \frac{z}{(1-z)^\delta} = z + \sum_{n=2}^{\infty} \frac{(\delta)_{n-1}}{(n-1)!} z^n, \quad \delta > -1$$

poses a linear operator

$$\mathfrak{J}_{\nu}^{\delta}(z,s,a)u(z) = (z\Upsilon_{\nu}(z,s,a))^{-1} * u(z) = z + \sum_{n=2}^{\infty} \frac{(\delta)_{n-1}}{(\nu)_{n-1}\Theta_{n-1}(z,s,a)} a_n z^n \tag{1.11}$$

where $\sigma \in \mathbb{C} \setminus \{0\}; \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; a \in \mathbb{C} \setminus \{-(m+\sigma n)\}, n, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, |s| < 1; |z| < 1$ and $\Theta_n(z,s,a)$ is defined in (1.9). It is clear that

$$\mathfrak{J}_{\nu}^{\delta}u(z) = \mathfrak{J}_{\nu}^{\delta}(z,s,a)u(z) = z + \sum_{n=2}^{\infty} \Psi_n a_n z^n \tag{1.12}$$

where

$$\Psi_n = \frac{(\delta)_{n-1}}{(\nu)_{n-1}\Theta_{n-1}(z,s,a)}$$

Now, by making use of the Hurwitz - Lerch zeta operator, we define a new subclass of functions motivated by the recent work of Thirupathi Reddy and Venkateswarlu [23], Venkateswarlu et al [24, 25] and Niranjan et al [13].

Definition 1.1. For $-1 \le \upsilon < 1$, $0 \le \tau < 1$ and $\varrho \ge 0$, we let $S(\tau, \upsilon, \varrho)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\Re\left\{\frac{z(\mathfrak{J}_{\nu}^{\delta}u(z))' + \tau z^2(\mathfrak{J}_{\nu}^{\delta}u(z))''}{(1-\tau)\mathfrak{J}_{\nu}^{\delta}u(z) + \tau z(\mathfrak{J}_{\nu}^{\delta}u(z))'} - \upsilon\right\} \ge \varrho\left|\frac{z(\mathfrak{J}_{\nu}^{\delta}u(z))' + \tau z^2(\mathfrak{J}_{\nu}^{\delta}u(z))''}{(1-\tau)\mathfrak{J}_{\nu}^{\delta}u(z) + \tau z(\mathfrak{J}_{\nu}^{\delta}u(z))'} - 1\right|, \tag{1.13}$$

for $z \in U$.

2 Coefficient bounds

In this sectin we obtain a necessary and sufficient condition for function $u(z)$ is in the classes $S(\tau, \upsilon, \varrho)$ and $TS(\tau, \upsilon, \varrho)$.

Theorem 2.1. *The function u defined by (1.1) is in the class $S(\tau, \upsilon, \varrho)$ if*

$$\sum_{n=2}^{\infty}[1+\tau(n-1)][n(1+\varrho)-(\upsilon+\varrho)]\Psi_n|a_n| \le 1-\upsilon, \tag{2.1}$$

where $-1 \le \upsilon < 1$, $0 \le \tau \le 1, \varrho \ge 0$.

Proof. It suffices to show that

$$\varrho\left|\frac{z(\mathfrak{J}_{\nu}^{\delta}u(z))' + \tau z^2(\mathfrak{J}_{\nu}^{\delta}u(z))''}{(1-\tau)\mathfrak{J}_{\nu}^{\delta}u(z) + \tau z(\mathfrak{J}_{\nu}^{\delta}u(z))'} - 1\right| - \Re\left\{\frac{z(\mathfrak{J}_{\nu}^{\delta}u(z))' + \tau z^2(\mathfrak{J}_{\nu}^{\delta}u(z))''}{(1-\tau)\mathfrak{J}_{\nu}^{\delta}u(z) + \tau z(\mathfrak{J}_{\nu}^{\delta}u(z))'} - 1\right\}$$
$$\le 1-\upsilon.$$

we have

$$\varrho\left|\frac{z(\mathfrak{J}_{\nu}^{\delta}u(z))' + \tau z^2(\mathfrak{J}_{\nu}^{\delta}u(z))''}{(1-\tau)\mathfrak{J}_{\nu}^{\delta}u(z) + \tau z(\mathfrak{J}_{\nu}^{\delta}u(z))'} - 1\right| - \Re\left\{\frac{z(\mathfrak{J}_{\nu}^{\delta}u(z))' + \tau z^2(\mathfrak{J}_{\nu}^{\delta}u(z))''}{(1-\tau)\mathfrak{J}_{\nu}^{\delta}u(z) + \tau z(\mathfrak{J}_{\nu}^{\delta}u(z))'} - 1\right\}$$
$$\le (1+\varrho)\left|\frac{z(\mathfrak{J}_{\nu}^{\delta}u(z))' + \tau z^2(\mathfrak{J}_{\nu}^{\delta}u(z))''}{(1-\tau)\mathfrak{J}_{\nu}^{\delta}u(z) + \tau z(\mathfrak{J}_{\nu}^{\delta}u(z))'} - 1\right|$$
$$\le \frac{(1+\varrho)\sum_{n=2}^{\infty}(n-1)[1+\tau(n-1)]\Psi_n|a_n|}{1-\sum_{n=2}^{\infty}[1+\tau(n-1)]\Psi_n|a_n|}.$$

This last expression is bounded above by $(1-\upsilon)$ by

$$\sum_{n=2}^{\infty}[1+\tau(n-1)][n(1+\varrho)-(\upsilon+\varrho)]\Psi_n|a_n| \le 1-\upsilon,$$

and hence the proof is complete.

Theorem 2.2. *A necessary and sufficient condition for $u(z)$ of the form (1.6) to be in the class $TS(\tau, \upsilon, \varrho), -1 \leq \upsilon < 1, 0 \leq \tau \leq 1, \varrho \geq 0$ is that*

$$\sum_{n=2}^{\infty}[1+\tau(n-1)][n(1+\varrho)-(\upsilon+\varrho)]\Psi_n|a_n| \leq 1-\upsilon. \tag{2.2}$$

Proof. □

In view of Theorem 2.1, we need only to prove the necessity.
If $u \in TS(\tau, \upsilon, \varrho)$ and z is real then

$$\frac{1-\sum\limits_{n=2}^{\infty} n[1+\tau(n-1)]\Psi_n a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}[1+\tau(n-1)]\Psi_n a_n z^{n-1}} - \upsilon \geq \varrho \left| \frac{\sum\limits_{n=2}^{\infty}(n-1)[1+\tau(n-1)]\Psi_n|a_n|}{1-\sum\limits_{n=2}^{\infty}[1+\tau(n-1)]\Psi_n|a_n|} \right|.$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[1+\tau(n-1)][n(1+\varrho)-(\upsilon+\varrho)]\Psi_n|a_n| \leq 1-\upsilon.$$

□

Theorem 2.3. *The class $TS(\tau, \upsilon, \varrho)$ is convex set.*

Proof. Let the function

$$u_j = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \ a_{n,j} \geq 0, \ j = 1, 2, \tag{2.3}$$

be in the class $TS(\tau, \upsilon, \varrho)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \zeta u_1(z) + (1-\zeta)u_2(z), \ \ 0 \leq \zeta \leq 1,$$

is in the class $TS(\tau, \upsilon, \varrho)$ then

$$h(z) = z - \sum_{n=2}^{\infty}[\zeta a_{n,1} + (1-\zeta)a_{n,2}]z^n.$$

By simple computation with the aid of Theorem 2.2 gives,

$$\begin{aligned}
&\sum_{n=2}^{\infty}[1+\tau(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\Psi_n \zeta a_{n,1} \\
&\quad + \sum_{n=2}^{\infty}[1+\tau(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\Psi_n(1-\zeta)a_{n,2} \\
&\leq \zeta(1-\upsilon) + (1-\zeta)(1-\upsilon) \\
&\leq 1-\upsilon,
\end{aligned}$$

which implies that $g \in TS(\tau, \upsilon, \varrho)$.
Hence $TS(\tau, \upsilon, \varrho)$ is convex. □

3 Extreme points

The proof of Theorem 3.1, follows on lines similar to the proof of the theorem on extreme points given in Silverman [23].

Theorem 3.1. *Let* $u_1(z) = z$ *and*

$$u_n(z) = z - \frac{1-\upsilon}{[1+\tau(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\Psi_n} z^n, \tag{3.1}$$

for $n = 2, 3, \cdots$ *. Then* $u(z) \in TS(\tau, \upsilon, \varrho)$ *if and only if* $u(z)$ *can be expressed in the form* $u(z) = \sum_{n=2}^{\infty} \zeta_n u_n(z)$, *where* $\zeta_n \geq 0$ *and* $\sum_{n=1}^{\infty} \zeta_n = 1$.

Next we prove the following closure theorem.

4 Closure theorem

Theorem 4.1. *Let the function* $u_j(z), j = 1, 2, \cdots, l$ *defined by* (2.3) *be in the classes* $TS(\tau, \upsilon_j, \varrho),\ j = 1, 2, \cdots, l$ *respectively. Then the function* $h(z)$ *defined by*

$$h(z) = z - \frac{1}{l} \sum_{n=2}^{\infty} \left(\sum_{j=1}^{l} a_{n,j} \right) z^n$$

is in the class $TS(\tau, \upsilon, \varrho)$, *where* $\upsilon = \min_{1 \leq j \leq l} \{\upsilon_j\}$, $-1 \leq \upsilon_j \leq 1$.

Proof. Since $u_j(z) \in TS(\tau, \upsilon_j, \varrho),\ j = 1, 2, \cdots, l$ by applying Theorem 2.2 to (2.3), we observe that

$$\begin{aligned}
&\sum_{n=2}^{\infty} [1+\tau(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\Psi_n \left(\frac{1}{l} \sum_{j=1}^{l} a_{n,j} \right) \\
=&\frac{1}{l} \sum_{j=1}^{l} \left(\sum_{n=2}^{\infty} [1+\tau(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\Psi_n a_{n,j} \right) \\
\leq&\frac{1}{l} \sum_{j=1}^{l} (1-\upsilon_j) \\
\leq&1-\upsilon
\end{aligned}$$

which in view of Theorem 2.2, again implies that $h(z) \in TS(\tau, \upsilon, \varrho)$ and so the proof is complete. □

Theorem 4.2. *Let* $u \in TS(\tau, \upsilon, \varrho)$. *Then*

(1). u *is starlike of order* ω, $0 \leq \omega < 1$, *in the disc* $|z| < r_1$
i.e., $\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \omega$, $|z| < r_1$, *where*

$$r_1 = \inf_{n \geq 2} \left\{ \left(\frac{1-\omega}{n-\omega} \right) \frac{[1+\tau(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\Psi_n}{1-\upsilon} \right\}^{\frac{1}{n-1}}.$$

(2). u *is convex of order* ω, $0 \leq \omega < 1$, *in the disc* $|z| < r_1$
i.e., $\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \omega$, $|z| < r_2$, *where*

$$r_2 = \inf_{n \geq 2} \left\{ \left(\frac{1-\omega}{n-\omega} \right) \frac{[1+\tau(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\Psi_n}{1-\upsilon} \right\}^{\frac{1}{n}}.$$

Each of these results are sharp for the extremal function $u(z)$ *given by* (3.1).

Proof. Given $u \in A$ and u is starlike of order ω, we have

$$\left|\frac{zu'(z)}{u(z)} - 1\right| < 1 - \omega. \tag{4.1}$$

For the left hand side (4.1), we have

$$\left|\frac{zu'(z)}{u(z)} - 1\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n|z|^{n-1}}.$$

The last expression is less than $1 - \omega$ if

$$\sum_{n=2}^{\infty} \frac{n-\omega}{1-\omega} a_n|z|^{n-1} < 1.$$

Using the fact, that $u \in TS(\tau, \upsilon, \varrho)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[1+\tau(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\Psi_n}{1-\upsilon} a_n < 1.$$

We can say (4.1) is true if

$$\frac{n-\omega}{1-\omega}|z|^{n-1} < \frac{[1+\tau(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\Psi_n}{1-\upsilon}$$

Or equivalently,

$$|z|^{n-1} < \frac{(1-\omega)[1+\tau(n-1)][n(\varrho+1)-(\upsilon+\varrho)]\Psi_n}{(n-\omega)(1-\upsilon)}$$

which yields the starlikeness of the family.

(2). Using the fact that u is convex if and only if zu' is starlike, we can prove (2), on lines similar to the proof of (1). □

5 Partial Sums

Following the earlier works by Silverman [18] and Silvia [19] on partial sums of analytic functions. We consider in this section partial sums of functions in this class $S(\tau, \upsilon, \varrho)$ and obtain sharp lower bounds for the ratios of real part of $u(z)$ to $u_q(z)$ and $u'(z)$ to $u'_q(z)$.

Theorem 5.1. *Let $u(z) \in S(\tau, \upsilon, \varrho)$. Define the partial sums $u_1(z)$ and $u_q(z)$ by*

$$u_1(z) = z \text{ and } u_q(z) = z + \sum_{n=2}^{q} a_n z^n, \ (q \in \mathbb{N} \setminus \{1\}). \tag{5.1}$$

Suppose that $\sum\limits_{n=2}^{\infty} d_n|a_n| \leq 1$,

$$\text{where } d_n = \frac{[1+\tau(n-1)][n(1+\varrho)-(\upsilon+\varrho)]\Psi_n}{1-\upsilon} \tag{5.2}$$

Then $u \in S(\tau, \upsilon, \varrho)$.

$$\text{Further more, } \Re\left[\frac{u(z)}{u_q(z)}\right] > 1 - \frac{1}{d_{q+1}}, \ (z \in E, \ q \in \mathbb{N} \setminus \{1\}) \tag{5.3}$$

$$\text{and } \Re\left[\frac{u_q(z)}{u(z)}\right] > \frac{d_{q+1}}{1 + d_{q+1}}. \tag{5.4}$$

Proof. For the coefficients d_n given by (5.2) it is not difficult to verify that

$$d_{n+1} > d_n > 1. \tag{5.5}$$

$$\text{Therefore we have } \sum_{n=2}^{q} |a_n| + d_{q+1} \sum_{n=q+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} d_n |a_n| \leq 1 \tag{5.6}$$

by using the hypothesis (5.2). By setting

$$\begin{aligned} g_1(z) &= d_{q+1}\left[\frac{u(z)}{u_q(z)} - \left(1 - \frac{1}{d_{q+1}}\right)\right] \\ &= 1 + \frac{d_{q+1} \sum\limits_{n=q+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=2}^{q} a_n z^{n-1}} \end{aligned} \tag{5.7}$$

and applying (5.6), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_{q+1} \sum\limits_{n=q+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=2}^{q} |a_n| - d_{q+1} \sum\limits_{n=q+1}^{\infty} |a_n|} \leq 1 \tag{5.8}$$

which readily yields the assertion (5.3) of Theorem 5.1. In order to see that

$$u(z) = z + \frac{z^{q+1}}{d_{q+1}} \text{ gives sharp result, we observe that for } z = re^{\frac{i\pi}{q}} \text{ that} \tag{5.9}$$

$$\frac{u(z)}{u_q(z)} = 1 + \frac{z^q}{d_{q+1}} \;\rightarrow\; 1 - \frac{1}{d_{q+1}} \text{ as } z \rightarrow 1^-.$$

Similarly, if we take

$$\begin{aligned} g_2(z) &= (1 + d_{q+1})\left(\frac{u_q(z)}{u(z)} - \frac{d_{q+1}}{1 + d_{q+1}}\right) \\ &= 1 - \frac{(1 + d_{n+1}) \sum\limits_{n=q+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} a_n z^{n-1}} \end{aligned} \tag{5.10}$$

and making use of (5.6), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1 + d_{q+1}) \sum\limits_{n=q+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=2}^{q} |a_n| - (1 - d_{q+1}) \sum\limits_{n=q+1}^{\infty} |a_n|}$$

which leads is immediately to the assertion (5.4) of Theorem 5.1.

The bound in (5.4) is sharp for each $q \in \mathbb{N}$ with the external function $u(z)$ given by (5.9). The proof of the Theorem 5.1 is thus complete. □

Theorem 5.2. *If $u(z)$ of the form (1.1) satisfies the condition (2.1) then*

$$\Re\left[\frac{u'(z)}{u'_q(z)}\right] \geq 1 - \frac{q+1}{d_{q+1}}. \tag{5.11}$$

Proof. By setting

$$g(z) = d_{q+1}\left[\frac{u'(z)}{u_q'(z)}\right] - \left(1 - \frac{q+1}{d_{q+1}}\right)$$
$$= \frac{1 + \frac{d_{q+1}}{q+1}\sum\limits_{n=q+1}^{\infty} n a_n z^{n-1} + \sum\limits_{n=2}^{\infty} n a_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} n a_n z^{n-1}} = 1 + \frac{\frac{d_{q+1}}{q+1}\sum\limits_{n=q+1}^{\infty} n a_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} n a_n z^{n-1}}$$

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\frac{d_{q+1}}{q+1}\sum\limits_{n=q+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{q} n|a_n| - \frac{d_{q+1}}{q+1}\sum\limits_{n=q+1}^{\infty} n|a_n|}. \tag{5.12}$$

$$\text{Now } \left|\frac{g(z)-1}{g(z)+1}\right| \leq 1 \text{ if } \sum_{n=2}^{q} n|a_n| + \frac{d_{q+1}}{q+1}\sum_{n=q+1}^{\infty} n|a_n| \leq 1. \tag{5.13}$$

Since the left hand side of(5.13)is bounded above by $\sum\limits_{n=2}^{q} d_n|a_n|$ if

$$\sum_{n=2}^{q}(d_n - n)|a_n| + \sum_{n=q+1}^{\infty} d_n - \frac{d_{q+1}}{q+1} n|a_n| \geq 0. \tag{5.14}$$

and the proof is complete.
The result is sharp for the extremal function $u(z) = z + \frac{z^{q+1}}{d_{q+1}}$. □

Theorem 5.3. *If* $u(z)$ *of the form (1.1) satisfies the condition (2.1) then*

$$\Re\left[\frac{u_q'(z)}{u'(z)}\right] \geq \frac{d_{q+1}}{q+1+d_{q+1}}. \tag{5.15}$$

Proof. By setting

$$g(z) = [q+1+d_{q+1}]\left[\frac{u_q'(z)}{u'(z)} - \frac{d_{q+1}}{q+1+d_{q+1}}\right]$$
$$= 1 - \frac{\left(1+\frac{d_{q+1}}{q+1}\right)\sum\limits_{n=q+1}^{\infty} n a_n z^{n-1}}{1 + \sum\limits_{n=2}^{q} n a_n z^{n-1}}$$

and making use of (5.14), we deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\left(1+\frac{d_{q+1}}{q+1}\right)\sum\limits_{n=q+1}^{\infty} n|a_n|}{2-2\sum\limits_{n=2}^{q} n|a_n| - \left(1+\frac{d_{q+1}}{q+1}\right)\sum\limits_{n=q+1}^{\infty} n|a_n|} \leq 1$$

which leads us immediately to the assertion of the Theorem 5.3. □

6 Neighbouhood for the class $S^{\xi}(\tau, \upsilon, \varrho)$

In this section, we determine the neighbourhoods for the class $S^{\xi}(\tau, \upsilon, \varrho)$ which we define as follows:

Definition 6.1. A function $u \in A$ is said to be in the class $S^{\xi}(\tau, \upsilon, \varrho)$ if there exist a function $g \in S(\tau, \upsilon, \varrho)$ such that

$$\left|\frac{u(z)}{g(z)} - 1\right| < 1 - \upsilon, \quad (z \in U, 0 \leq \upsilon < 1). \tag{6.1}$$

For any function $u(z) \in A, z \in U$ and $\delta \geq 0$, we define

$$N_{n,\delta}(u) = \Big\{ g \in \Sigma : g(z) = z + \sum_{n=2}^{\infty} b_n z^n \ \text{ and } \ \sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta \Big\} \tag{6.2}$$

which is the $(n, \delta)-$neighbourhood of $u(z)$.

The concept of neighbourhoods was first introduced by Goodman [8] and generalized by Ruscheweyh [16].

Theorem 6.2. *If* $g \in S(\tau, \upsilon, \varrho)$ *and*

$$\xi = 1 - \frac{\delta(1-\upsilon)}{2[(1-\upsilon) - (1+\tau)(2+\varrho-\upsilon)\phi(\mu, s, 2)]} \tag{6.3}$$

then $\ N_{n,\delta}(g) \subset S^{\xi}(\tau, \upsilon, \varrho)$.

Proof. Suppose $u \in N_{n,\delta}(g)$. We then find from (6.2) that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta \tag{6.4}$$

which yields the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \frac{\delta}{2} \ \ (n \in \mathbb{N}). \tag{6.5}$$

Next, since $g \in S(\tau, \upsilon, \varrho)$, we have

$$\sum_{n=2}^{\infty} b_n \leq \frac{(1+\tau)(2+\varrho-\upsilon)\phi(\mu, s, 2)}{1-\upsilon}. \tag{6.6}$$

So that

$$\begin{aligned} \left| \frac{u(z)}{g(z)} - 1 \right| &< \frac{\sum\limits_{n=2}^{\infty} |a_n - b_n|}{1 - \sum\limits_{n=2}^{\infty} b_n} \\ &= \frac{\delta(1-\upsilon)}{2[(1-\upsilon) - (1+\tau)(2+\varrho-\upsilon)\phi(\mu, s, 2)]} \\ &= 1 - \xi \end{aligned}$$

provided ξ is given by (6.3). Thus the proof of the is completed. □

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