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CHARACTERISATION OF MODULES OVER PATH ALGEBRA

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Abstract Let K be a field, $Q = (Q_0, Q_1)$ be a quiver and $\overline{KQ}$ be the generalised path algebra [10]. This paper gives a characterisation for the right and left modules over the path algebras of finite acyclic quiver. The study shows that the modules over such path algebras could be written as the decomposition of $\overline{KQ}$-submodules. For $\overline{KQ}$-modules over path algebras of quiver with countably many vertices, a sequence of $\overline{KQ}$-submodules is identified which in finite case is a composition series.

1 Introduction

Many mathematical objects could be studied by describing their internal structure, which means decomposing them to simpler objects like prime factorisation of natural numbers. With rings and modules, this means direct sum decomposition. Modules over arbitrary rings can be visualised as generalizations of vector spaces and abelian groups.In the 1920s, Emmy Noether used modules as an important tool in bringing to light the connection between the study of representations of finite groups via groups of matrices and the study of rings [4]. Krull-Schmidt theorem states that any two direct sum decompositions of a module of finite length into indecomposable submodules are unique upto isomorphism. A similar study first appeared in the work of Frobenius and Stickeberger [7] in which they discussed direct sum decomposition of finite abelian groups into cyclic subgroups with prime power orders. Later in 1950, Azumaya [2] generalised Krull-Schmidt theorem to infinite direct sums of modules with local endomorphism ring.
Caldero and Keller [5] dealt with cluster category of finitely generated right modules over path algebra and proved that such an algebra of finite type can be viewed as a Hall algebra. Alamsyah et. al. [9] showed that the indecomposable simple modules over the path algebra of Dynkin quiver of type A_n and D_n are c-prime modules. Kariman [8]studied projective and hereditary modules over path algebra of cyclic quivers using representation theory. Okoh [6] characterised pure-injective modules, which are direct summands of direct products made up from finite dimensional R-modules using systems of cardinal invariants, which help to comprehend path algebras over fields and how multiplication is given by path composition.

Definition 1.1. For a K-algebra A and a vector space M together with $\cdot : A \times M \to M$, defined by $(a, m) \mapsto am$, $(M, \cdot)$ is said to be a **left A-module** if the following conditions are met:

(a) $(a_1a_2)m = a_1(a_2m)$

(b) $a(m_1 + m_2) = am_1 + am_2$

(c) $(a_1 + a_2)m = a_1m + a_2m$

(d) $1m = m$

(e) $a(km) = (ak)m = k(am)$

for all $a, a_1, a_2 \in A, m, m_1, m_2 \in M, k \in K$.

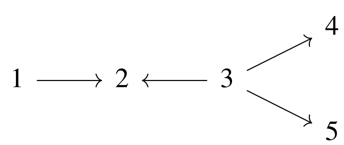
A non-empty subset N of an A-module M is a submodule if for every $a, b \in A$ and $m, n \in N$, we have that $am + bn \in N$. The quotient group of cosets of N, denoted by M/N, is an A-module.

The left A-module M is called a free module if there exists a subset $X \subseteq M$ such that each element $m \in M$ can be expressed uniquely as a finite sum $m = \sum_{i=1}^{n} a_i x_i for a_1, ..., a_n \in A$ and $x_1, ..., x_n \in X$.
Suppose M is an A-module, and M_1, M_2 are submodules of M. M is the internal direct sum of M_1 and M_2 if $M = M_1 + M_2$ and $M_1 \cap M_2 = 0$. In this case, every $m \in M$ can be written uniquely as $m = m_1 + m_2$ for $m_1 \in M_1, m_2 \in M_2$.
A quiver is a graph with directed edges with no constraints on the number of arrows between two vertices or loops or even cycles. We decide to represent each point on the quiver by an open dot, with each arrow pointing in the direction of its intended target.

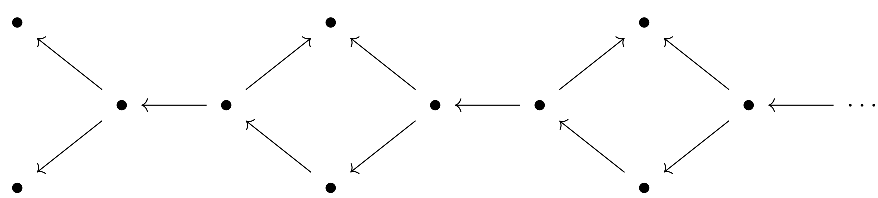
Definition 1.2. A **quiver** [1, page 41] consists of two sets, Q_0 and Q_1, together with two mappings, s and t, such that Q_0 is the set of elements called vertices, Q_1 is the set of elements called arrows and $s, t : Q_1 \to Q_0$ which maps each edge α to its source, $s(\alpha)$ and target, $t(\alpha)$ respectively.

Throughout, we denote an arrow from a to b as $\alpha : a \to b$ or simply α_{ab} and a quiver as (Q_0, Q_1, s, t) or (Q_0, Q_1).

Example 1.1. (i)



(ii)



A quiver is said to be finite if Q_O and Q_1 are finite sets. A path of length $l \geq 1$ with source a and target b is a sequence $(a|\alpha_1, \alpha_2, ..., \alpha_l|b)$ where $\alpha_k \in Q$ for all $1 \leq k \leq l$ and $s(\alpha_1) = a, t(\alpha_k) = \alpha_{k+1}$ for $1 \leq k \leq l$ and $t(\alpha_l) = b$. The composition of paths in a quiver is used to define an algebra called path algebra.

Let $Q = (Q_0, Q_1)$ be a quiver. The K- vector space generated by the set of all paths in Q forms a K-algebra called **path algebra**. The product of two paths is the concatenation of paths, that is,

$$(a|\beta_1 \cdots \beta_n|b)(c|\gamma_1 \cdots \gamma_m|d) = \delta_{bc}(a|\beta_1 \cdots \beta_n \gamma_1 \cdots \gamma_m)$$

where $\delta_{bc} = 1$ only when $b = c$ and 0, otherwise. Thus the product of two paths is zero if the target of first path does not coincide with the source of second path and if it does, the product is the composed path. This product is then extended to all of KQ by using law of distribution.

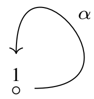
Definition 1.3 (**Path Algebra- A Generalised definition**). [10] Let Q be a quiver, and let P be the set of all paths in Q. A Path Algebra $\overline{KQ}$ of Q is defined as

$$\left\{ \sum_{\alpha \in P} c_\alpha \alpha \mid c_\alpha \in K, \alpha \in P \right\}$$

Addition and scalar multiplication is defined componentwise.

The generalised definition and the previous definition of path algebra are equivalent if Q is a finite acyclic quiver. The set of all the paths in a finite acyclic quiver Q will act as the basis for KQ.

Example 1.2. a. Let Q be the quiver consisting of a single point and a single loop.



The defining basis of $\overline{KQ}$ is $\varepsilon_1, \alpha, \alpha^2, ..., \alpha^l, ...$ and the multiplication is given by

$$\varepsilon_1\alpha^l = \alpha^l\varepsilon_1 = \alpha^l \forall l \geq 0$$

$$\alpha^l\alpha^k = \alpha^{l+k} \forall l, k \geq 0$$

where $\alpha^0 = \varepsilon_1$. Thus $\overline{KQ}$ is isomorphic to the polynomial algebra $K[t]$ in one indeterminate t.

b. Consider a quiver with two vertices $1, 2$ and an arrow α from 1 to 2.

$$\overset{1}{\circ} \xrightarrow{\alpha} \overset{2}{\circ}$$

The basis of the path algebra of this quiver is $\{\epsilon_1, \epsilon_2, \alpha\}$. The product of all basis elements is given by the table Thus, the associated matrix algebra for the above quiver is given by

	ϵ_1	ϵ_2	α
ϵ_1	ϵ_1	0	α
ϵ_2	0	ϵ_2	0
α	0	α	0

$$\begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$$

It is clear that the path algebra and the associated matrix algebra have isomorphic correspondence using the linear mapping,

$$\epsilon_1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \epsilon_2 \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \alpha \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Theorem 1.1. *[10] Let Q be a quiver and $\overline{KQ}$ be the corresponding path algebra.Then,*

(a) $\overline{KQ}$ is an associative algebra

(b) The element $\sum_{a \in Q_0} \epsilon_a$ is the identity in $\overline{KQ}$.

(c) $\overline{KQ}$ is finite dimensional if and only if Q is finite and acyclic.

The above theorem characterises the path algebra of quivers including infinite dimensional cases using the generalized definition. Another result that is necessary for this study on modules over path algebras is the following theorem which characterises simple R-modules.

Theorem 1.2. *[3, page 74] Let M be a left A-module. M is simple if and only if $Rm = M$ for all nonzero $m \in M$.*

2 Characterisation of Modules Over Path algebra

Theorem 2.1. *Let Q be a finite acyclic quiver with n vertices. Any left module M over $\overline{KQ}$ will have the following properties:*

(a) M can be written as

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$$

where M_i are submodules of M

(b) ϵ_i acts as an identity operator on M_i, that is $\epsilon_i M_i = M_i$ for each i.

(c) If α is a path from i to j, then $\alpha M_k = 0$ for $k \neq j$ and $\alpha M_j \subseteq M_i$

Proof. Let Q be an acyclic quiver with n vertices and M be a left $\overline{KQ}$-module.
For each $i = 1, 2, \cdots, n$, define $M_i = \epsilon_i M$.To prove (a), we need to first prove (b).
Consider an arbitrary element $m_i \in M_i$. Then there exists $m \in M$ such that $m_i = \epsilon_i m$.

$$\epsilon_i m_i = \epsilon_i(\epsilon_i m) = \epsilon_i^2 m = \epsilon_i m = m_i$$

for all $i = 1, 2, \cdots, n$. This proves ϵ_i acts as an identity operator on each M_i.

$$\epsilon_i M_j = \epsilon_i(\epsilon_j M) = 0 \quad (\because \epsilon_i \epsilon_j = 0; i \neq j)$$

That is, ϵ_i annihilates M_j for $i \neq j$.
We have $\left(\sum_{i=1}^n \epsilon_i\right) M = M$ which expands to

$$\epsilon_1 M + \epsilon_2 M + \cdots + \epsilon_n M = M$$

$$M_1 + M_2 + \cdots + M_n = M$$

We claim $M_i \cap M_j = 0$ for $i \neq j$.
Let $m \in M_i \cap M_j$ for $i \neq j$. Then m is in M_i and M_j. Since ϵ_i acts as an identity operator on M_i and annihilates M_j, we get

$$m = \epsilon_i m = 0$$

Consider a path α from i to j.

$$\alpha M_k = \alpha \epsilon_k M$$

Now, $\alpha \epsilon_k$ is zero only when $k \neq j$. That is, $\alpha M_k = 0$ for $k \neq j$

$$\begin{aligned} \alpha M_j &= \alpha \epsilon_j M \\ &= \epsilon_i \alpha \epsilon_j M \\ &\subseteq \epsilon_i(\alpha M) \\ &\subseteq \epsilon_i M = M_i \end{aligned}$$

□

Remark 2.2. The above theorem also holds for any right module M over $\overline{KQ}$. In that case, if α is a path from i to j, then $M_k \alpha = 0$ for $k \neq i$ and $M_i \alpha \subseteq M_j$

Proposition 2.3. *Let M be a left $\overline{KQ}$-module. The modules over finite dimensional path algebra are faithful.*

Proof. Annihilator of M is the set of elements in $\overline{KQ}$ such that $am = 0$ for all $m \in M$. That is

$$Ann(M) = \{a = \sum_{\alpha \in Q_0} a_\alpha \alpha \in \overline{KQ} : (\sum_{\alpha \in Q_0} a_\alpha \alpha) m = 0, \forall m\}$$

If α is from i to j, by theorem, we have $\alpha M_j \subseteq M_i$. That is, $\alpha m = 0$ for all m if and only if $\alpha = 0$. As a consequence of the above theorem, we can observe that $Ann(M)$ must be zero. Hence M is a faithful module. □

Example 2.4. a. Lets begin with a simple example to understand the consequences of above theorem. Consider the quiver,

$$\overset{1}{\circ} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \overset{2}{\circ}$$

The path algebra $\overline{KQ}$ is given by

$$\overline{KQ} = \begin{bmatrix} K & K^2 \\ 0 & K \end{bmatrix}$$

It is easy to verify that the module $M = K^2$ is a left module over $\overline{KQ}$. Here, $M_1 = \begin{pmatrix} K \\ 0 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 \\ K \end{pmatrix}$. So $M = M_1 \oplus M_2$.

For $m_1 \in M_1, m_2 \in M_2$,

$$\epsilon_1 m_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ 0 \end{pmatrix} = \begin{pmatrix} k_1 \\ 0 \end{pmatrix}$$

$$\epsilon_2 m_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ k_2 \end{pmatrix}$$

ϵ_i acts as an identity operator on each M_i.

Consider $\alpha = \begin{pmatrix} 0 & (1,0) \\ 0 & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 0 & (0,1) \\ 0 & 0 \end{pmatrix}$ from 1 to 2.

$$\alpha M_2 = \begin{pmatrix} 0 & (1,0) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ K \end{pmatrix} = \begin{pmatrix} K \\ 0 \end{pmatrix} \subseteq M_1$$

$$\beta M_2 = \begin{pmatrix} 0 & (0,1) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ K \end{pmatrix} = \begin{pmatrix} K \\ 0 \end{pmatrix} \subseteq M_1$$

Hence the theorem is verified.

b. Let $n \in \mathbb{N}$ and $n < \infty$. Consider the quiver

$$\overset{1}{\circ} \xrightarrow{\alpha} \overset{2}{\circ} \xrightarrow{\beta} \cdots \longrightarrow \overset{n}{\circ} \longrightarrow \cdots$$

The path algebra $\overline{KQ}$ is given by the $n \times n$ upper triangular matrix

$$\begin{pmatrix} K & K & \cdots & K \\ 0 & K & \cdots & K \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K \end{pmatrix}$$

Clearly, $M = K^n$ is a left $\overline{KQ}$-module. Each submodule M_i is the $n \times 1$ matrix with an element from K in the i^{th} row and all other entries 0. Then, $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$.

Consider $m_1 = \begin{pmatrix} k_1 & 0 & \cdots & 0 \end{pmatrix}^t \in M_1$

$$\epsilon_1 m_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} k_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = m_1$$

This also holds true for $m_i \in M_i$ for each $1 \leq i \leq n.$ That is, ϵ_i acts as an identity operator on each M_i. Consider the path $p = 2\alpha\beta$ from 1 to 3. We need to show that $pM_3 \subseteq 1$.

$$pM_3 = \begin{pmatrix} 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ K \\ \cdot \\ 0 \end{pmatrix} = \begin{pmatrix} 2K \\ 0 \\ 0 \\ \cdot \\ 0 \end{pmatrix} \subseteq M_1$$

This could be verified for any path in $\overline{KQ}$. Hence the theorem is verified.

Theorem 2.5. *Let Q be an acyclic quiver with countable vertices and M be a left $\overline{KQ}$-module. Then there exists a sequence of submodules $M \supset M_1 \supset M_2 \supset \cdots$ which has the following properties*

(a) M_i/M_{i+1} is a left $\overline{KQ}$–module on which ϵ_{i+1} acts as an identity operator for each i.

(b) M_i/M_{i+1} is a simple $\overline{KQ}$-module for each i.

(c) M/M_i is a left $\overline{KQ}$–module on which $\sum_{j=1}^{i} \epsilon_j$ acts as an identity operator.

If $\overline{KQ}$ is finite dimensional, the sequence terminates for some i.

Proof. Define $M_k = (\sum_{i=k+1}^{\infty} \epsilon_i)M$ for $k = 1, 2, \cdots$
Recall that for cosets of a ring H, $aH = bH$ when $a - b \in H$
To prove ϵ_{i+1} acts as an identity operator on M_i/M_{i+1}:
By definition,

$$M_i/M_{i+1} = \{m + M_{i+1} : m \in M_i\} \ and$$
$$\epsilon_{i+1}(m + M_{i+1}) = \epsilon_{i+1}m + M_{i+1}$$

Since m is in M_i, $m = (\sum_{j=i+1}^{\infty} \epsilon_j)m'$ for some $m' \in M.$

$$\epsilon_{i+1}m = \epsilon_{i+1}(\sum_{j=i+1}^{\infty} \epsilon_j)m' = \epsilon_{i+1}m'$$

Then, $m - \epsilon_{i+1}m = (\sum_{j=i+2}^{\infty} \epsilon_j)m' \in M_{i+1}$ which implies

$$m + M_{i+1} = \epsilon_{i+1}m + M_{i+1}$$

This proves (a).
Now to prove (c), we will show that

$$m + M_i = (\sum_{j=1}^{i} \epsilon_j)(m + M_i)$$

or equivalently,

$$m - (\sum_{j=1}^{i} \epsilon_j)m \in M_i$$

Since $\sum_{j=1}^{\infty} \epsilon_j$ acts as an identity operator on M,

$$m - (\sum_{j=1}^{i} \epsilon_j)m = \sum_{j=1}^{\infty} \epsilon_j m - \sum_{j=1}^{i} \epsilon_j m = (\sum_{j=i+1}^{\infty} \epsilon_j)m \in M_i$$

M_i/M_{i+1} is a simple $\overline{KQ}$-module iff $\overline{KQ}(m+M_{i+1}) = M_i/M_{i+1}$ for all $m+M_{i+1} \in M_i/M_{i+1}$.

Clearly, $\overline{KQ}(m + M_{i+1}) \subseteq M_i/M_{i+1}$. To prove the converse, consider an arbitrary element $m' + M_{i+1} \in M_i/M_{i+1}$ where $m' \in M_i$.
Since $m \in M_i$ and $M_i = \Big(\sum_{j=i+1}^{\infty} \epsilon_j\Big)M$, there exists an element $m \in M$ such that

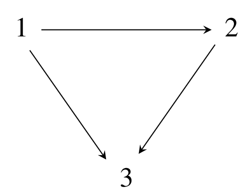
$$m' = \Big(\sum_{j=i+1}^{\infty} \epsilon_j\Big)m$$

Hence

$$\begin{aligned} m' + M_{i+1} &= \Big(\sum_{j=i+1}^{\infty} \epsilon_j\Big)m + M_{i+1} \\ &= \Big(\sum_{j=i+1}^{\infty} \epsilon_j\Big)(m + M_{i+1}) \\ &\in \overline{KQ}(m + M_{i+1}) \end{aligned}$$

Since $m' + M_{i+1}$ is arbitrary, this is true for all $m' + M_{i+1} \in M_i/M_{i+1}$. This proves the factor modules M_i/M_{i+1} are simple $\overline{KQ}$-modules. □

Example 2.6. a. Let $M = K^3$. Consider the following quiver



The path algebra $\overline{KQ}$ is given by

$$\overline{KQ} = \begin{bmatrix} K & K & K^2 \\ 0 & K & K \\ 0 & 0 & K \end{bmatrix}$$

By the definition of M_i from above proof,

$$M_1 = (\sum_{j=2}^{3} \epsilon_j)M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} K \\ K \\ K \end{pmatrix} = \begin{pmatrix} 0 \\ K \\ K \end{pmatrix}$$

$$M_2 = (\sum_{j=3}^{3} \epsilon_j)M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} K \\ K \\ K \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ K \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The quotient modules are given by

$$M_0/M_1 = \{M_1, (K, 0, 0) + M_1\} = \{M_1, \epsilon_1 M + M_1\}$$

$$M_1/M_2 = \{M_2, \epsilon_2 M + M_2\}$$

$$M_2/M_3 = \{M_3, \epsilon_3 M + M_3\}$$

Consider M_0/M_1. Any element in the quotuent module M_0/M_1 is either the identity coset or a coset of the form $\epsilon_1 m + M$ for some $m \in M_0 = M$.

$$\epsilon_1(0 + M) = \epsilon_1 m + M = 0 + M$$

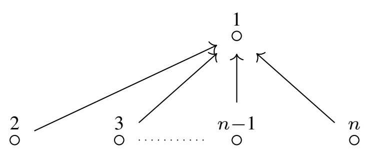
$$\epsilon_1(\epsilon_1 m + M) = \epsilon_1^2 m + M = \epsilon_1 m + M$$

This holds true for M_1/M_2 and M_2/M_3. Hence ϵ_{i+1} acts as an identity operator on M_i/M_{i+1} for each i.
Also,

$$M/M_2 = \{M_2, \epsilon_1 M + M_2, \epsilon_2 M + M_2\}, M/M_3 = M$$

It is clear that $\sum_{j=1}^{i} \epsilon_j$ acts as an identity operator on M/M_i for each i.
Since each M_{i+1} is maximal in M_i, M_i/M_{i+1} is a simple $\overline{KQ}$-module.

b. Consider the quiver



The path algebra $\overline{KQ}$ is given by the $n \times n$ lower triangular matrices

$$\overline{KQ} = \begin{pmatrix} K & 0 & 0 & \cdots & 0 \\ K & K & 0 & \cdots & 0 \\ K & 0 & K & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ K & 0 & 0 & \cdots & K \end{pmatrix}$$

$M = K^n$ is a left $\overline{KQ}$-module. The submodules M_i are given by $M_0 = M$,

$$M_1 = \begin{pmatrix} 0 \\ K \\ K \\ \vdots \\ K \end{pmatrix}, M_2 = \begin{pmatrix} 0 \\ 0 \\ K \\ \vdots \\ K \end{pmatrix}, \cdots, M_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Using similar calculations as in previous example, one could verify the results of the theorem 2.5.

3 Conclusion

The work gives a characterisation of certain modules over path algebra of quivers. The first main theorem establishes a direct sum decomposition of left modules over path algebras of finite acyclic quivers such that the stationary path at each vertex acts as the identity operator on its corresponding constituent submodules and each non zero path α from i to j in $\overline{KQ}$ corresponds to a relation $\alpha M_j \subseteq M_i$. It is also shown that the modules over finite dimensional path algebras are faithful.

The second theorem can be used to form a sequence of submodules of left modules over path algebras of acyclic quivers with countable vertices. This sequence holds the property that the stationary path at vertex $i + 1$ acts as the identity operator on the simple factor module

M_i/M_{i+1}. The sum of different stationary paths($\sum_{j=1}^{i} \epsilon_j$) acts as identity operator on different submodules(M/M_i). If finite quivers are considered, then this sequence is a composition series for the module under consideration.

The work discussed in this paper could be developed further by exploring the characterisation of modules over path algebra of infinite quivers.

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