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Graph of a Rough Approximation Set

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Abstract In this paper, we introduce, graph of lower and upper approximation of a non-empty subset of a nearring with respect to an ideal. We relate the properties of these graphs with properties of ideals. We study the relationship between the connectivity of the graph and properties of the approximation set. We obtain the properties of these graphs under nearring homomorphism.

1 Introduction

Science and technology, especially the computer network progress year by year, and a huge quantity of data is generated every day. To deal with this mounting amount of data, which is in exact, uncertain or vague knowledge, the need for tools to analyze this information is becoming more and more. In 1982, to analyze the collected inconsistent, imprecise, data and knowledge Palwak [16] proposed an extension to set theory called rough set theory. This theory is a formal approach to model and process the unpredictable and imprecision that appears in the form of inexact, uncertain, or vague knowledge. It does not require any extra or pilot details of the data, it just requires two sets known as lower and upper approximation sets. This theory has a widespread range of applications in a number of real-life fields, like intelligent information processing [17], [10], banking [21], medicine [15], decision support and analysis, knowledge discovery, machine learning [18], etc.

Graph is a geometric structure that represents relationship between elements of a set. In graph the objects are called vertices which are connected by edges. It is a very significant fundamental tool in mathematics having applications in many branches of Science and Engineering. He and Shi [13] defined edge rough graph using the notion of a partition on the edge set of a graph. By enduing the edges of a rough graph with weight attribute, He, Chan, and Shi[12] extended the concept edge set of a graph to a weighted rough graph. Mathew, John and Garg [7] extended the idea of edge rough graph to vertex rough graph and defined R$-$vertex graph in terms of R$-$lower and R$-$upper approximate graph of the subgraph of the given graph.

Many authors studied the relationship between algebraic properties of an algebraic structure and properties of a graph. Beck [4] introduced the notion of zero divisor graphs of a commutative ring. Anderson and Livingston [2] generalized this notion. Further using this notion Redmond [20] found bound on the size of a ring. Redmond [19] proposed the ideal-based zero divisor graphs of a nearring. Anderson and Badawi [3] defined and studied the total graph of a commutative ring. The graph of a nearring with respect to an ideal and ideal symmetry of the graph was introduced by Bhavanari, Kuncham and Kedukodi [5]. Davaaz [9] associated rough sets with ring theory, and defined lower and upper approximation of a set with respect to an ideal of the ring.

In this paper, we find a relation between the rough approximation set and graph theory. We define the graph of an approximation set namely the graph of lower approximation of a set A with respect to an ideal E of nearring L and the graph of upper approximation of a set A with respect to an ideal E of nearring L, denoted by $C_{\underline{Apr}^L_E(A)}$ and $C_{\overline{Apr}^L_E(A)}$ respectively and provide some examples. We relate the properties of an ideal with properties of graph and show how the properties of a graph change with the properties of an ideal and vice versa. Prime numbers play a very important role in many fields such as cryptography, telecommunication to produce error-correcting codes, etc. Prime ideals are the generalization of prime numbers. When the ideal, lower (upper) approximation set are c-prime ideal, we study the connectedness of vertices in the graph of lower (upper) approximation and provide examples. Further, we prove that if the nearring is an integral nearring with only trivial ideals then the graph of lower (upper) approximation is either a star graph or a complete graph or a null graph. We show that the lower (upper) approximation graph of set $L \setminus A$ is a subgraph of the lower (upper) approximation graph of set A. We define the ideal symmetry in the graph of lower (upper) approximation and derive some results. Nearring homomorphism is a relation between the elements of two nearrings and graph homomorphism is the relation between two graphs that preserves adjacency. In the last part, we study the graph homomorphism of the graph of lower (upper) approximation. We prove that nearring homomorphism is a graph homomorphism in both lower and upper approximation graphs. The connectedness of vertices is preserved under homomorphism.

2 Preliminaries

We refer to Anderson and Fuller [1], Bhavanari and Kuncham [6] for rings, Harrary [11], Clark and Holton [8] for graph theory, Z.Suraj [22] and Davvaz [9] for rough set theory. In this paper L, L_1, L_2 represent nearrings and E represents an ideal.

Definition 2.1. (Bhavanari, Kuncham [6]) Let $(L, +, \cdot)$ be a nearring and I be a normal subgroup of $(L, +)$. Then I is called an *ideal*, if for all $n, m \in L$ and for all $i \in I$
(i) $i \cdot n \in I$ (ii) $n \cdot (m + i) - n \cdot m \in I$

Definition 2.2. (Davvaz [9]) Let I be an ideal of a ring R and X be a non $-$ empty subset of R. Then the sets $\underline{Apr}_I(X) = \{x \in R : x + I \subseteq X\}$ and $\overline{Apr}_I(X) = \{x \in R : (x + I) \cap X \neq \emptyset\}$ are called, respectively lower and upper approximations of the set X with respect to the ideal I.

Definition 2.3. (Davvaz [9]) Let I be an ideal of a ring R and $Apr_I(X) = (\underline{Apr}_I(X), \overline{Apr}_I(X))$ a rough set in the approximation space (R, I). If $\underline{Apr}_I(X)$ and $\overline{Apr}_I(X)$ are ideals (resp.subrings) of R , then we call $Apr_I(X)$ a rough ideal (resp.subring) .

Definition 2.4. (Harrary [11]) A *graph* $G = (V, F)$ consists of a set of objects $V = \{v_1, v_2, ...\}$ called *vertices* (or points) and another set $F = \{e_1, e_2, ..\}$ whose elements are called *edges* (or lines) such that each edge e is identified with an unordered pair (v_i, v_j) of vertices. The vertices v_i and v_j are called *end vertices* of e.

Definition 2.5. (Harrary [11]) A *vertex cover* of graph G is a subset K of V such that if (u, v) is an edge of G, then $u \in K$ or $v \in K$ or both $u \in K$ and $v \in K$.

Definition 2.6. (Hell and Nesetril [14]) Let $G_1 = (V_1, F_1)$ and $G_2 = (V_2, F_2)$ be two graphs. A *graph homomorphism* of G_1 to G_2 is a mapping $g : V_1 \to V_2$ such that $(g(u), g(v)) \in F_2$ whenever $(u, v) \in F_1$.

Definition 2.7. (Bhavanari, Kuncham [6]) Let $L_1 and L_2$ be two nearrings. Then $\varsigma : L_1 \to L_2$ is called a *nearring homomorphism* if for all $x, y \in L_1$,
(i) $\varsigma(x + y) = \varsigma(x) + \varsigma(y)$ (ii) $\varsigma(xy) = \varsigma(x)\varsigma(y)$.

Definition 2.8. (Bhavanari, Kuncham [6]) An ideal I of a nearring L is called *c-prime* (completely prime) if $x, y \in L$ and $x \cdot y \in I$ implies $x \in I$ or $y \in I$.

Definition 2.9. (Bhavanari, Kuncham [6]) Let L be a nearring.
(i) A nonzero element r is said to be a right zero divisor, if there exists a nonzero element $a \in L$ such that $ar = 0$.
(ii)A nonzero element r is said to be a left zero divisor, if there exists a nonzero element $a \in L$ such that $ra = 0$.
(iii) L is said to be *integral nearring* if it has no zero divisors.

Definition 2.10. (Kedukodi, Jagadeesha , Kuncham , Juglal [5]) Let I be an ideal of a ring R. Let $C_I(R)$ be the graph with vertex set R and the pair of distinct vertices x and y are adjacent if and only if $x \cdot y \in I$ or $y \cdot x \in I$. The graph $C_I(R)$ is called as *c-prime graph of R with respect to an ideal I*.

Definition 2.11. (Kedukodi, Jagadeesha , Kuncham , Juglal [5]) The graph $C_I(R)$ is said to be ideal symmetric if for every pair of vertices $x, y \in C_I(R)$ with an edge between them, either $deg(x) = deg(0)$ or $deg(y) = deg(0)$.

3 Graph of Rough Approximation Set

Definition 3.1. Let E be an ideal and A be a nonempty subset of a nearring L.
i) Let $C_{\underline{Apr}^L_E(A)}$ be the graph with vertex set L and two vertices $a \neq y, (a, y) \in F[C_{\underline{Apr}^L_E(A)}]$ if and only if $a \cdot y \in \underline{Apr}^L_E(A)$ or $y \cdot a \in \underline{Apr}^L_E(A)$. Then the graph $C_{\underline{Apr}^L_E(A)}$ is called the graph of lower approximation of A with respect to the ideal E of L.
ii) Let $C_{\overline{Apr}^L_E(A)}$ be the graph with vertex set L and two vertices $a \neq y, (a, y) \in F[C_{\overline{Apr}^L_E(A)}]$ if and only if $a \cdot y \in \overline{Apr}^L_E(A)$ or $y \cdot a \in \overline{Apr}^L_E(A)$. Then the graph $C_{\overline{Apr}^L_E(A)}$ is called the graph of upper approximation of A with respect to the ideal E of L.

Example 3.2. Let L be the ring of integer modulo 6, $E = \{0, 2, 4\}$ and $A = \{1, 2, 3, 5\}$. Then $\underline{Apr}^L_E(A) = \{1, 3, 5\}$ and $\overline{Apr}^L_E(A) = \{0, 1, 2, 3, 4, 5\}$. The graph of lower and upper approximation of A with respect to the ideal E of nearring L is shown in Figure 1 and 2 respectively.

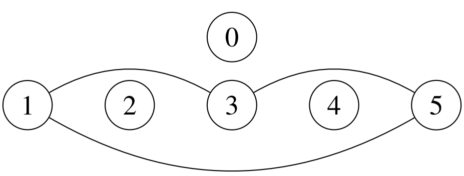

Figure 1. $C_{\underline{Apr}^L_E(A)}$

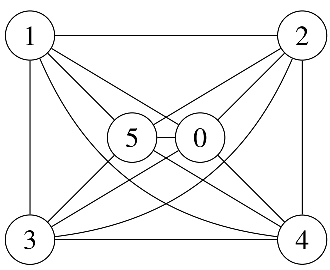

Figure 2. $C_{\overline{Apr}^L_E(A)}$

Example 3.3. Consider the nearring $L = \{0, r, s, t\}$, defined as in Table 1.

$+$	0	r	s	t
0	0	r	s	t
r	r	0	t	s
s	s	t	0	r
t	t	s	r	0

$\cdot$	0	r	s	t
0	0	0	0	0
r	r	r	r	r
s	0	r	s	t
t	r	0	t	s

Table 1. Nearring for Example 3.3

Let $E = \{0, r\}$ and $A = \{0, r, s\}$. Then $\underline{Apr}^L_E(A) = \{0, r\}$ and $\overline{Apr}^L_E(A) = \{0, r, s, t\}$. Graph of lower and upper approximation of A with respect to the ideal E of nearring L is shown Figure 3 and 4 respectively.

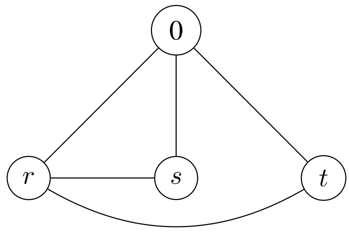

Figure 3. $C_{\underline{Apr}^L_E(A)}$

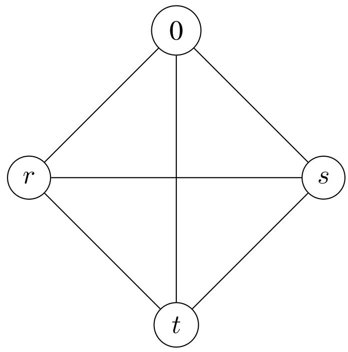

Figure 4. $C_{\overline{Apr}^L_E(A)}$

Remark 3.4. Let E be an ideal of a nearring L and A be a non empty subset of L. If $\mid A \mid < \mid E \mid$ then $C_{\underline{Apr}^L_E(A)}$ is an empty graph.

Example 3.5. Let $L = Z_6$, be the ring of integer modulo 6, $E = \{0, 2, 4\}$ and $A = \{2, 4\}$. Then $\underline{Apr}^L_E(A) = \emptyset$. Therefore the graph of lower approximation of A with respect to the ideal E is an empty graph. We observe that $\mid A \mid < \mid E \mid$.

Proposition 3.6. *Let E be an ideal of a nearring L and A be a non-empty subset of L. Let $a \in V[C_{\underline{Apr}^L_E(A)}]$.*
(i) If E is a c-prime ideal of L and a is connected to all other vertices of $C_{\underline{Apr}^L_E(A)}$ then $a \notin E$.
(ii) If $\underline{Apr}^L_E(A)$ is a c-prime ideal of L and a is connected to all other vertices of $C_{\underline{Apr}^L_E(A)}$ then $a \in \underline{Apr}^L_E(A)$.

Proof. (i) Let E be a c-prime ideal of L and $a \in V[C_{\underline{Apr}^L_E(A)}]$ be such that a is connected to all other vertices of $C_{\underline{Apr}^L_E(A)}$. Then for all $y \in L$, $(a, y) \in F[C_{\underline{Apr}^L_E(A)}] \implies ay \in \underline{Apr}^L_E(A)$ or $ya \in \underline{Apr}^L_E(A)$. Let $ay \in \underline{Apr}^L_E(A)$. Then $ay + E \subseteq A$. Hence $ay \in A$ and $ay \notin E$. As E is a c-prime ideal of L, $a \notin E$.
The proof is similar for $ya \in \underline{Apr}^L_E(A)$.
(ii) Let $\underline{Apr}^L_E(A)$ be a c-prime ideal of L and $a \in V[C_{\underline{Apr}^L_E(A)}]$ be such that a is connected to all other vertices of $C_{\underline{Apr}^L_E(A)}$. Then for all $y \in L \setminus \underline{Apr}^L_E(A), (a, y) \in F[C_{\underline{Apr}^L_E(A)}] \implies ay \in \underline{Apr}^L_E(A)$ or $ya \in \underline{Apr}^L_E(A)$. Let $ay \in \underline{Apr}^L_E(A)$. As $\underline{Apr}^L_E(A)$ is a c-prime ideal of L, we get $a \in \underline{Apr}^L_E(A)$. The proof is similar for $ya \in \underline{Apr}^L_E(A)$. □

Now we give an example to show that the Proposition 3.6(i) is not true if we exclude the assumptions.

Example 3.7. Consider the nearring $L = \{0, c, d, e\}$, defined as in Table 2.

$+$	0	c	d	e
0	0	c	d	e
c	c	0	e	d
d	d	e	0	c
e	e	d	c	0

$\cdot$	0	c	d	e
0	0	0	0	0
c	0	c	0	c
d	d	d	d	d
e	d	e	d	e

Table 2. Nearring for Example 3.7

Let $E = \{0\}$ and $A = \{0, e\}$. Then $\underline{Apr}_E^L(A) = \{0, e\}$. The graph of $C_{\underline{Apr}_E^L(A)}$ is shown in Figure 5.

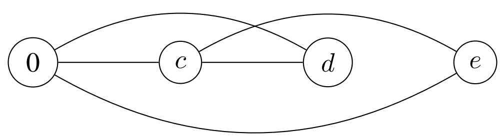

Figure 5. $C_{\underline{Apr}_E^L(A)}$

We observe that $\{0\}$ is not a c-prime ideal ,($c \cdot d = 0 \in E$ however $c \notin E$ and $d \notin E$) and 0 is a vertex in $C_{\underline{Apr}_E^L(A)}$ connected to all other vertices but $0 \in E$.

Now we provide an example to show that the Proposition 3.6(ii) is not true if we exclude the assumptions.

Example 3.8. Let L be the ring of integers modulo 6, $E = \{0, 2, 4\}$ and $A = \{0, 1, 2, 4, 5\}$ then $\underline{Apr}_E^L(A) = \{0, 2, 4\}$. The graph of $C_{\underline{Apr}_E^L(A)}$ is shown in Figure 6.

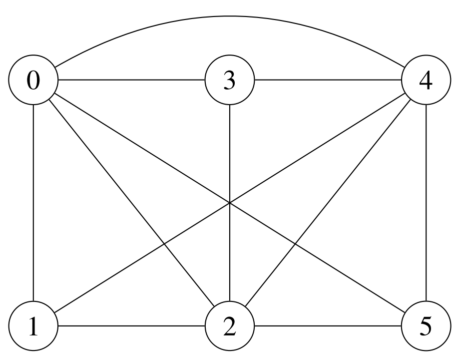

Figure 6. $C_{\underline{Apr}_E^L(A)}$

Note that $\underline{Apr}_E^L(A) = \{0, 2, 4\}$ a c-prime ideal of L , $3 \in V[C_{\underline{Apr}_E^L(A)}]$, which is not connected to all other vertices of $C_{\underline{Apr}_E^L(A)}$ and $3 \notin \underline{Apr}_E^L(A)$.

Proposition 3.9. *Let E be an ideal of a nearring L and A be a nonempty subset of L. Let $a \in V[C_{\overline{Apr}_E^L(A)}]$.*
(i) If $\overline{Apr}_E^L(A)$ is a c-prime ideal of L and a is connected to all other vertices of $C_{\overline{Apr}_E^L(A)}$ then $a \in \overline{Apr}_E^L(A)$.
(ii) If $\overline{Apr}_E^L(A)$ is a c-prime ideal of L and $a \in \overline{Apr}_E^L(A)$ then a is connected to all other vertices of $C_{\overline{Apr}_E^L(A)}$.

Proof. (i) Let $\overline{Apr}^L_E(A)$ be a c-prime ideal of L and $a \in V[C_{\overline{Apr}^L_E(A)}]$ such that a is connected to all other vertices of $C_{\overline{Apr}^L_E(A)}$. Then for all $y \in L \setminus \overline{Apr}^L_E(A), (a, y) \in F[C_{\overline{Apr}^L_E(A)}] \implies ay \in \overline{Apr}^L_E(A)$ or $ya \in \overline{Apr}^L_E(A)$. Let $ay \in \overline{Apr}^L_E(A)$. As $\overline{Apr}^L_E(A)$ is c-prime ideal $a \in \overline{Apr}^L_E(A)$. The proof is similar for $ya \in \overline{Apr}^L_E(A)$.
(ii) Let $a \in \overline{Apr}^L_E(A)$, a c-prime ideal of L. Assume that a is an isolated vertex in $C_{\overline{Apr}^L_E(A)}$. Then for all $y \in L$, $(a, y) \notin F[C_{\overline{Apr}^L_E(A)}] \implies ay \notin \overline{Apr}^L_E(A)$ and $ya \notin \overline{Apr}^L_E(A)$. As $\overline{Apr}^L_E(A)$ is c-prime ideal , we get $a \notin \overline{Apr}^L_E(A)$. - a contradiction. Therefore a is connected to all other vertices of $C_{\overline{Apr}^L_E(A)}$. □

Proposition 3.10. *Let E be an ideal of a nearring L and A be a nonempty subset of L. Let $a \in V[C_{\overline{Apr}^L_E(A)}]$. If $\overline{Apr}^L_E(A)$ is a c-prime ideal of L then a is connected to all other vertices of $C_{\overline{Apr}^L_E(A)}$ if and only if $a \in \overline{Apr}^L_E(A)$.*

Proof. Proof follows from Proposition 3.9 (i) and (ii) □

Now we give an example to prove that the Proposition 3.9(i) is not true if we exclude the assumptions.

Example 3.11. Let $L = Z_8$ be the ring of integers modulo 8, $E = \{0, 4\}$ and $A = \{0, 2\}$. Then $\overline{Apr}^L_E(A) = \{0, 2, 4, 6\}$. The graph of $C_{\overline{Apr}^L_E(A)}$ is shown in Figure 7.

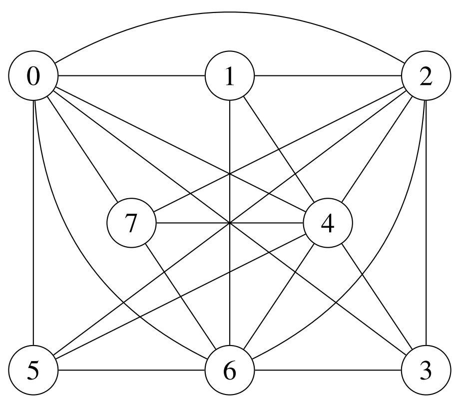

Figure 7. $C_{\overline{Apr}^L_E(A)}$

We observe that $\overline{Apr}^L_E(A)$ is a c-prime ideal of L, 1 is not connected to all other vertices of $C_{\overline{Apr}^L_E(A)}$ and $1 \notin \overline{Apr}^L_E(A)$.

Proposition 3.12. *Let E be an ideal and A be a non-empty subset of nearring L. Let $a \in V[C_{\overline{Apr}^L_E(A)}]$.*
(i) If E is a c-prime ideal of L and $a \in E$ then a is an isolated vertex $C_{\underline{Apr}^L_E(A)}$.
(ii) If $\underline{Apr}^L_E(A)$ is a c-prime ideal of L and $a \in \underline{Apr}^L_E(A)$ then a is connected to all other vertices of $C_{\underline{Apr}^L_E(A)}$.

Proof. (i) Let E be a c-prime ideal of L and $a \in E$ be such that a is conneceted to a vertex $y \in L$. Then $(a, y) \in F[C_{\underline{Apr}^L_E(A)}]$. Hence $ay \in \underline{Apr}^L_E(A)$ or $ya \in \underline{Apr}^L_E(A)$. Let $ay \in \underline{Apr}^L_E(A)$. Then $ay + E \subseteq A$. Therefore $ay \in A$ and $ay \notin E$ - a contradiction to the fact that E is a c - prime ideal and $a \in E$. Therefore a is an isolated vertex in $C_{\underline{Apr}^L_E(A)}$. The proof is similar for $ya \in \underline{Apr}^L_E(A)$.
(ii) Let $a \in \underline{Apr}^L_E(A)$, a c-prime ideal of L. Assume that a is an isolated vertex in $C_{\underline{Apr}^L_E(A)}$. Then for all $y \in L$, $(a, y) \notin F[C_{\underline{Apr}^L_E(A)}] \implies ay \notin \underline{Apr}^L_E(A)$ and $ya \notin \underline{Apr}^L_E(A)$. - a

contradiction to $a \in \underline{Apr}_E^L(A)$, and $\underline{Apr}_E^L(A)$ is a c-prime ideal of L. Therefore a is connected to all other vertices of $C_{\underline{Apr}_E^L(A)}$. □

Now we provide an example to show that Proposition 3.12(i) is not true if we exclude the assumptions.

Example 3.13. Let $L = Z_6$ be the ring of integers modulo 6, $E = \{0\}$ and $A = \{0, 1, 3\}$. Then $\underline{Apr}_E^L(A) = \{0, 1, 3\}$. The graph of $C_{\underline{Apr}_E^L(A)}$ is shown in Figure 8.

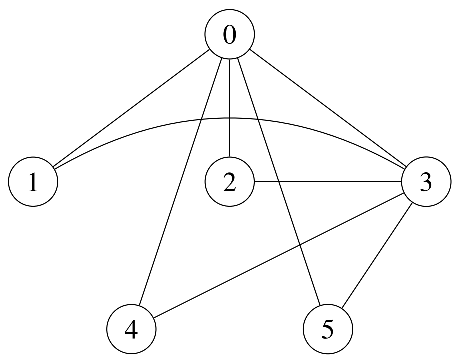

Figure 8. $C_{\underline{Apr}_E^L(A)}$

We observe that $\{0\}$ is not a c-prime ideal of L ($3 \cdot 4 = 0 \in E$ however $3 \notin E$ and $4 \notin E$) and $0 \in E$ but 0 is not a isolated vertex in $C_{\underline{Apr}_E^L(A)}$.

Now we provide an example to show that Proposition 3.12(ii) is not true if we exclude the assumptions.

Example 3.14. Let $L = Z_6$ be the ring of integers modulo 6, $E = \{0, 2, 4\}$ and $A = \{0, 1, 2, 4, 5\}$. Then $\underline{Apr}_E^L(A) = \{0, 2, 4\}$. The graph of $C_{\underline{Apr}_E^L(A)}$ is shown in Figure 9.

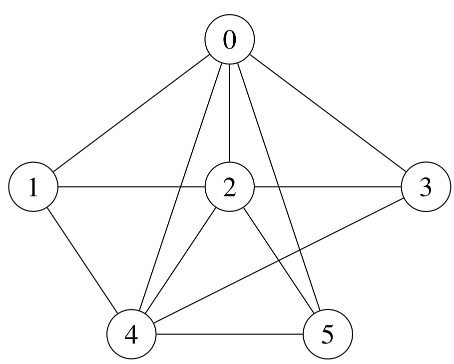

Figure 9. $C_{\underline{Apr}_E^L(A)}$

Note that $\underline{Apr}_E^L(A) = \{0, 2, 4\}$ is a c-prime ideal of L , $3 \notin \underline{Apr}_E^L(A)$ and 3 is not connected to all other vertices of $C_{\underline{Apr}_E^L(A)}$.

Proposition 3.15. *Let E be an ideal and A be a non-empty subset of a nearring L. Let $a \in V[C_{\underline{Apr}_E^L(A)}]$. If $\underline{Apr}_E^L(A)$ is a c-prime ideal of L then $a \in \underline{Apr}_E^L(A)$ if and only if a is connected to all other vertices of $C_{\underline{Apr}_E^L(A)}$.*

Proof. Proof follows from the Proposition 3.6(ii) and Proposition 3.12(ii). □

Proposition 3.16. *Let L be an integral nearring with only ideals $\{0\}$ and L. Let E and A be the ideals of L. Then $C_{\underline{Apr}_E^L(A)}$ is either a star graph or a complete graph or a null graph.*

Proof. Let L be an integral nearring with only ideals $\{0\}$ and L. Let E and A be the ideals of L.
Case(i): When $E = L$ and $A = L$, we get $\underline{Apr}_E^L(A) = L$. Therefore $C_{\underline{Apr}_E^L(A)}$ is a complete graph.
Case(ii): When $E = \{0\}$ and $A = \{0\}$. Let $a, y \in L$ such that $a \neq 0 \neq y$. Suppose $(a, y) \in F[C_{\underline{Apr}_E^L(A)}]$ then $ay \in \underline{Apr}_E^L(A)$ or $ya \in \underline{Apr}_E^L(A)$. Let $ay \in \underline{Apr}_E^L(A)$. Then $ay + E \subseteq A$. Therefore $ay \in A = \{0\} \implies ay = 0$. As L is a integral nearring $a = 0$ or $y = 0$. Contradiction to $a \neq 0 \neq y$. Hence $C_{\underline{Apr}_E^L(A)}$ is a star graph.
Case(iii): When $E = \{0\}$ and $A = L$. Then $\underline{Apr}_E^L(A) = L$. Therefore $C_{\underline{Apr}_E^L(A)}$ is a complete graph.
Case(iv): When $E = L$ and $A = \{0\}$. Then $\underline{Apr}_E^L(A) = \emptyset$. Therefore $C_{\underline{Apr}_E^L(A)}$ is a null graph. □

Now we provide an example to show that Proposition 3.16 is not true if we exclude the assumptions.

Example 3.17. Let $L = Z_5$ be the ring of integers modulo 5, $E = \{0\}$ and $A = \{0, 1, 3\}$ then $\underline{Apr}_E^L(A) = \{0, 1, 3\}$. The graph of $C_{\underline{Apr}_E^L(A)}$ is shown in Figure 10

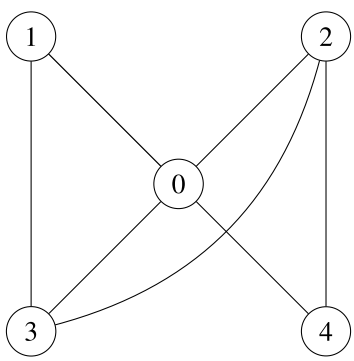

Figure 10. $C_{\underline{Apr}_E^L(A)}$

We observe that L is an integral nearring with only ideals $\{0\}$ and L, E is an ideal of L but A is not an ideal of L. Then $C_{\underline{Apr}_E^L(A)}$ is neither a complete graph nor a star graph.

Proposition 3.18. *Let L be an integral nearring with only ideals $\{0\}$ and L. Let E and A be the ideals of L. Then $C_{\overline{Apr}_E^L(A)}$ is either a star graph or a complete graph.*

Proof. Let L be an integral nearring with only ideals $\{0\}$ and L. Let E and A be the ideals of L.
Case(i): When $E = L$ and $A = L$, we get $\overline{Apr}_E^L(A) = L$. Therefore $C_{\overline{Apr}_E^L(A)}$ is a complete graph.
Case(ii): When $E = \{0\}$ and $A = \{0\}$. Let $a, y \in L$ such that $a \neq 0 \neq y$. Suppose $(a, y) \in F[C_{\overline{Apr}_E^L(A)}]$ then $ay \in \overline{Apr}_E^L(A)$ or $ya \in \overline{Apr}_E^L(A)$. Let $ay \in \overline{Apr}_E^L(A)$. Then $(ay+E) \cap A \neq \emptyset$. Let $z \in (ay + E) \cap A$. Then $z \in A = \{0\}$. Therefore $z = 0 \implies 0 \in ay + E$. As $E = \{0\}, ay = 0$. As L is Integral nearring $a = 0$ or $y = 0$ - a contradiction to $a \neq 0 \neq y$. Therefore $C_{\overline{Apr}_E^L(A)}$ is a star graph.
Case(iii): $E = \{0\}$ and $A = L$. Then $\overline{Apr}_E^L(A) = L$. Therefore $C_{\overline{Apr}_E^L(A)}$ is a complete graph.
Case(iv): When $E = L$ and $A = \{0\}$, we get $\overline{Apr}_E^L(A) = \{0\}$. Therefore $C_{\overline{Apr}_E^L(A)}$ is a star graph. □

Now we provide an example to show that Proposition 3.18 is not true if we exclude the assumptions.

Example 3.19. Let $L = Z_5$ be the ring of integers modulo 5, $E = \{0\}$ and $A = \{0, 2, 3\}$ then $\overline{Apr}^L_E(A) = \{0, 2, 3\}$. The graph of $C_{\overline{Apr}^L_I(A)}$ is shown in Figure 11.

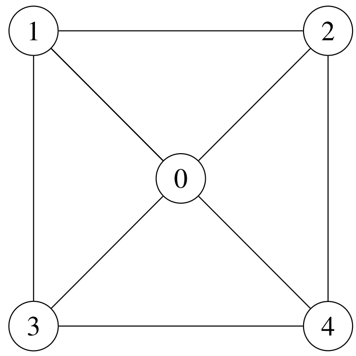

Figure 11. $C_{\overline{Apr}^L_E(A)}$

We observe that E is an ideal of a integral nearring L with only ideals $\{0\}$ and L, A is not an ideal of L. Then $C_{\overline{Apr}^L_E(A)}$ is neither a complete graph nor a star graph.

Proposition 3.20. *Let E be an ideal of nearring L.*
(i) If $\underline{Apr}^L_E(A)$ is a c-prime ideal then $\underline{Apr}^L_E(A)$ is a vertex cover of $C_{\underline{Apr}^L_E(A)}$.
(ii) If E is c-prime ideal then $\underline{Apr}^L_E(A)$ is not a vertex cover of $C_{\underline{Apr}^L_E(A)}$.

Proof. (i) Let $\underline{Apr}^L_E(A)$ be a c-prime ideal of L. Let $a, y \in L$ be such that $(a, y) \in F[C_{\underline{Apr}^L_E(A)}]$. Then $ay \in \underline{Apr}^L_E(A)$ or $ya \in \underline{Apr}^L_E(A)$. Let $ay \in \underline{Apr}^L_E(A)$. Then as $\underline{Apr}^L_E(A)$ is c- prime ideal, $a \in \underline{Apr}^L_E(A)$ or $y \in \underline{Apr}^L_E(A)$. Therefore $\underline{Apr}^L_E(A)$ is a vertex cover of $C_{\underline{Apr}^L_E(A)}$.
(ii) Let E be a c-prime ideal of L. Let $a, y \in L$ be such that $(a, y) \in F[C_{\underline{Apr}^L_E(A)}]$. Then $ay \in \underline{Apr}^L_E(A)$ or $ya \in \underline{Apr}^L_E(A)$. Let $ay \in \underline{Apr}^L_E(A)$. Then $ay + E \subseteq A \implies ay \in A - E$. Therefore $ay \notin E$. As E is c-prime ideal, $a \notin E$ and $y \notin E$. Hence $\underline{Apr}^L_E(A)$ is not a vertex cover of $C_{\underline{Apr}^L_E(A)}$. □

Proposition 3.21. *Let E be an ideal of nearring L. If $\overline{Apr}^L_E(A)$ is a c-prime ideal then $\overline{Apr}^L_E(A)$ is a vertex cover of $C_{\overline{Apr}^L_E(A)}$.*

Proof. Let $\overline{Apr}^L_E(A)$ be a c-prime ideal of L. Let $a, y \in L$ such that $(a, y) \in F[C_{\overline{Apr}^L_E(A)}]$. Then $ay \in \overline{Apr}^L_E(A)$ or $ya \in \overline{Apr}^L_E(A)$. Let $ay \in \overline{Apr}^L_E(A)$. As $\overline{Apr}^L_E(A)$ is c- prime ideal, $a \in \overline{Apr}^L_E(A)$ or $y \in \overline{Apr}^L_E(A)$. Therefore $\overline{Apr}^L_E(A)$ is a vertex cover of $C_{\overline{Apr}^L_E(A)}$. □

Proposition 3.22. *Let E be an ideal of a nearring L and A be a nonempty subset of L. Then*
(i) $C_{\underline{Apr}^L_E(L\backslash A)}$ is a subgraph of $C_{\underline{Apr}^L_E(L)}$.
(ii) $C_{\overline{Apr}^L_E(L\backslash A)}$ is a subgraph of $C_{\overline{Apr}^L_E(L)}$.
(iii) If $A = E$ then $C_{\underline{Apr}^L_E(L\backslash E)}$ is a subgraph of $C_{\underline{Apr}^L_E(L)}$.
(iv) If $A = E$ then $C_{\overline{Apr}^L_E(L\backslash E)}$ is a subgraph of $C_{\overline{Apr}^L_E(L)}$.

Proof. (i) We have $V[C_{\underline{Apr}^L_E(L\backslash A)}] = L = V[C_{\underline{Apr}^L_E(A)}]$. If $A = L$ then $L\backslash A = L\backslash L = \emptyset$. Hence $\underline{Apr}^L_E(L\backslash A) = \emptyset \implies C_{\underline{Apr}^L_E(L\backslash A)}$ is an empty graph. Therefore $F[C_{\underline{Apr}^L_E(L\backslash A)}] \subseteq F[C_{\underline{Apr}^L_E(L)}]$. Let $A \neq L$ and $a, y \in L \setminus A$ be such that $(a, y) \in F[C_{\underline{Apr}^L_E(L\backslash A)}]$. Then $ay \in \underline{Apr}^L_E(L \setminus A)$ or $ya \in \underline{Apr}^L_E(L \setminus A)$. Let $ay \in \underline{Apr}^L_E(L \setminus A)$. Then $ay + E \subseteq (L \setminus A)$. As $L \setminus A \subseteq L$, $ay + E \subseteq L \implies ay \in \underline{Apr}^L_E(L)$. Hence $(a, y) \in F[C_{\underline{Apr}^L_E(L)}]$. Therefore $F[C_{\underline{Apr}^L_E(L\backslash A)}] \subseteq F[C_{\underline{Apr}^L_E(L)}]$. The proof is similar for $ya \in \underline{Apr}^L_E(L \setminus A)$. Thus, $C_{\underline{Apr}^L_E(L\backslash A)}$ is a subgraph of $C_{\underline{Apr}^L_E(L)}$.

(ii) We have $V[C_{\overline{Apr}^L_E(L\setminus A)}] = L = V[C_{\overline{Apr}^L_E(L)}]$. If $A = L$ then $L \setminus A = L \setminus L = \emptyset$. Hence $\overline{Apr}^L{}_E(L\setminus A) = \emptyset \implies C_{\overline{Apr}^L_E(L\setminus A)}$ is a empty graph. Therefore $F[C_{\overline{Apr}^L_E(L\setminus A)}] \subseteq F[C_{\overline{Apr}^L_E(L)}]$. Let $A \neq L$ and $a, y \in L \setminus A$ such that $(a, y) \in F[C_{\overline{Apr}^L_E(L\setminus A)}]$. Then $ay \in \overline{Apr}^L_E(L \setminus A)$ or $ya \in \overline{Apr}^L_E(L \setminus A)$. Let $ay \in \overline{Apr}^L_E(L \setminus A)$. Then $(ay + E) \cap (L \setminus A) \neq \emptyset$. As $L \setminus A \subseteq L$, $(ay + E) \cap L \neq \emptyset$. Hence, $ay \in \overline{Apr}^L_E(L)$. Therefore, $(a, y) \in F[C_{\overline{Apr}^L_E(L)}]$. Therefore $F[C_{\overline{Apr}^L_E(L\setminus A)}] \subseteq F[C_{\overline{Apr}^L_E(L)}]$. The proof is similar for $ya \in \overline{Apr}^L{}_E(L \setminus A)$. Thus $C_{\overline{Apr}^L_E(L\setminus A)}$ is a subgraph of $C_{\overline{Apr}^L_E(L)}$.
Proof of (iii) and (iv) are similar. □

Definition 3.23. Let $(a, y) \in C_{\underline{Apr}^L_E(A)}$, then the graph $C_{\underline{Apr}^L_E(A)}$ is said to be *ideal symmetric* if and only if either a is connected to all other vertices of L or y is connected to all other vertices of L. Let $(a, y) \in C_{\overline{Apr}^L_E(A)}$, then the graph $C_{\overline{Apr}^L_E(A)}$ is said to be *ideal symmetric* if and only if either a is connected to all other vertices of L or y is connected to all other vertices of L.

Proposition 3.24. *Let E be an ideal of L and A be a nonempty subset of L.*
(i)If $\underline{Apr}^L_E(A)$ is a c-prime ideal of L then $C_{\underline{Apr}^L_E(A)}$ is ideal symmetric.

(ii) Suppose

(a) $\underline{Apr}^L_E(A)$ is c-semiprime
(b) $C_{\underline{Apr}^L_E(A)}$ is ideal symmetric
(c) For every $a \in L$, a is connected to all other vertices of L in $C_{\underline{Apr}^L_E(A)}$ $\implies a \in \underline{Apr}^L_E(A)$.

Then $\underline{Apr}^L_E(A)$ is c-prime.

Proof. (i) Let $\underline{Apr}^L_E(A)$ be a c-prime ideal of L. Let $a, y \in L$ be such that $(a, y) \in F[C_{\underline{Apr}^L_E(A)}]$. Then $ay \in \underline{Apr}^L_E(A)$ or $ya \in \underline{Apr}^L_E(A)$. Let $ay \in \underline{Apr}^L_E(A)$. Then, as $\underline{Apr}^L_E(A)$ is c- prime ideal, $a \in \underline{Apr}^L_E(A)$ or $y \in \underline{Apr}^L_E(A)$. Hence from Proposition 3.12(ii), a is connected to all other vertices of $C_{\underline{Apr}^L_E(A)}$ or y is connected to all other vertices of $C_{\underline{Apr}^L_E(A)}$. Hence $C_{\underline{Apr}^L_E(A)}$ is ideal symmetric.
(ii) Let $a, y \in L$ such that $ay \in \underline{Apr}^L_E(A)$. If $a = y$ then $a \in \underline{Apr}^L_E(A)$, as $\underline{Apr}^L_E(A)$ is c-semiprime. Let $a \neq y$. Then there is an edge between a and y in $C_{\underline{Apr}^L_E(A)}$. As $C_{\underline{Apr}^L_E(A)}$ is ideal symmetric , a is connected to all other vertices of $C_{\underline{Apr}^L_E(A)}$ or y is connected to all other vertices of $C_{\underline{Apr}^L_E(A)}$. From (c), $a \in \underline{Apr}^L_E(A)$ or $y \in \underline{Apr}^L_E(A)$. Therefore $\underline{Apr}^L_E(A)$ is c-prime. □

Proposition 3.25. *Let E be an ideal of L and A be a nonempty subset of L.*
(i)If $\overline{Apr}^L_E(A)$ is a c-prime ideal of L then $C_{\overline{Apr}^L_E(A)}$ is ideal symmetric.

(ii) Suppose

(a) $\overline{Apr}^L_E(A)$ is c-semiprime
(b) $C_{\overline{Apr}^L_E(A)}$ is ideal symmetric
(c) For every $a \in L$, a is connected to all other vertices of L in $C_{\overline{Apr}^L_E(A)}$ $\implies a \in \overline{Apr}^L_E(A)$.

Then $\overline{Apr}^L_E(A)$ is c-prime.

Proof. (i) Let $\overline{Apr}^L_E(A)$ be a c-prime ideal of L. Let $a, y \in L$ such that $(a, y) \in F[C_{\overline{Apr}^L_E(A)}]$. Then $ay \in \overline{Apr}^L_E(A)$ or $ya \in \overline{Apr}^L_E(A)$. Let $ay \in \overline{Apr}^L_E(A)$. Then as $\overline{Apr}^L_E(A)$ is c- prime ideal, $a \in \overline{Apr}^L_E(A)$ or $y \in \overline{Apr}^L_E(A)$. Hence from Proposition 3.9(ii), a is connected to all other vertices of $C_{\overline{Apr}^L_E(A)}$ or y is connected to all other vertices of $C_{\overline{Apr}^L_E(A)}$. Therefore $C_{\overline{Apr}^L_E(A)}$ is ideal symmetric.

(ii) Let $a, y \in L$ such that $ay \in \overline{Apr}_E^L(A)$. If $a = y$ then $a \in \overline{Apr}_E^L(A)$, as $\overline{Apr}_E^L(A)$ is c-semiprime. Let $a \neq y$. Then there is an edge between a and y in $C_{\overline{Apr}_E^L(A)}$. As $C_{\overline{Apr}_E^L(A)}$ is ideal symmetric , a is connected to all other vertices of $C_{\overline{Apr}_E^L(A)}$ or y is connected to all other vertices of $C_{\overline{Apr}_E^L(A)}$. From (c), $a \in \overline{Apr}_E^L(A)$ or $y \in \overline{Apr}_E^L(A)$. Therefore $\overline{Apr}_E^L(A)$ is c-prime. □

4 Graph Homomorphism

Proposition 4.1. *Let $\varsigma : L_1 \to L_2$ be a nearring homomorphism. Let E be an ideal and A be a non empty subset of L_1. Then*
(i) ς is a graph homomorphism from $C_{\underline{Apr}_E^{L_1}(A)}$ to $C_{\underline{Apr}_{\varsigma(E)}^{L_2}\varsigma(A)}$.
(ii) ς is a graph homomorphism from $C_{\overline{Apr}_E^{L_1}(A)}$ to $C_{\overline{Apr}_{\varsigma(E)}^{L_2}\varsigma(A)}$.

Proof. Let E be an ideal and A be a non empty subset of L_1. Then $\varsigma(E)$ is an ideal of L_2.
(i) Let $(a, y) \in F[C_{\underline{Apr}_E^{L_1}(A)}]$. Then $ay \in \underline{Apr}_E^{L_1}(A)$ or $ya \in \underline{Apr}_E^{L_1}(A)$. Let $ay \in \underline{Apr}_E^{L_1}(A)$. Then $ay + E \subseteq A$. Hence, $ay \in A$ and $ay \notin E \implies \varsigma(ay) \in \varsigma(A)$ and $\varsigma(ay) \notin \varsigma(E) \implies \varsigma(ay) \in \varsigma(A) - \varsigma(E)$. As ς is a nearring homomorphism $\varsigma(a)\varsigma(y) \in \varsigma(A) - \varsigma(E)$. Therefore $\varsigma(a)\varsigma(y) + \varsigma(E) \subseteq \varsigma(A) \implies \varsigma(a)\varsigma(y) \in \underline{Apr}_{\varsigma(E)}^{L_2}\varsigma(A)$. Thus $(\varsigma(a), \varsigma(y)) \in F[C_{\underline{Apr}_{\varsigma(E)}^{L_2}\varsigma(A)}]$. The proof is similar for $ya \in \underline{Apr}_E^{L_1}(A)$. Therefore ς is a graph homomorphism from $C_{\underline{Apr}_E^{L_1}(A)}$ to $C_{\underline{Apr}_{\varsigma(E)}^{L_2}\varsigma(A)}$.
(ii) Let$(a, y) \in F[C_{\overline{Apr}_E^{L_1}(A)}]$. Then $ay \in \overline{Apr}_E^{L_1}(A)$ or $ya \in \overline{Apr}_E^{L_1}(A)$. Let $ay \in \overline{Apr}_E^{L_1}(A)$. Then $(ay + E) \cap A \neq \emptyset$. Let $w \in (ay + E) \cap A$. Then $w \in ay + E$ and $w \in A$. Therefore $\varsigma(w) \in \varsigma(ay + E)$ and $\varsigma(w) \in \varsigma(A)$. As ς is nearring homomorphism, $\varsigma(w) \in \varsigma(ay) + \varsigma(E)$ and $\varsigma(w) \in \varsigma(A) \implies \varsigma(w) \in \varsigma(a)\varsigma(y) + \varsigma(E)$ and $\varsigma(w) \in \varsigma(A)$. Hence $\varsigma(w) \in [\varsigma(a)\varsigma(y) + \varsigma(E)] \cap \varsigma(A) \implies [\varsigma(a)\varsigma(y) + \varsigma(E)] \cap \varsigma(A) \neq \emptyset$. Therefore $\varsigma(a)\varsigma(y) \in \overline{Apr}_{\varsigma(E)}^{L_2}\varsigma(A)$. Thus $(\varsigma(a), \varsigma(y)) \in F[C_{\overline{Apr}_{\varsigma(E)}^{L_2}\varsigma(A)}]$. Proof is similar for $ya \in \overline{Apr}_E^{L_1}(A)$. Therefore ς is a graph homomorphism from $C_{\overline{Apr}_E^{L_1}(A)}$ to $C_{\overline{Apr}_{\varsigma(E)}^{L_2}\varsigma(A)}$. □

Now we provide an example for the graph homomorphism in Proposition 4.1

Example 4.2. Let $L_1 = \dfrac{Z}{8Z}$ and $L_2 = \dfrac{Z}{4Z}$. Then L_1 and L_2 are commutative rings. Let $\varsigma : L_1 \to L_2$ be defined by $\varsigma(a + 8Z) = a + 4Z$. Then ς is an onto nearring homomorphism. Consider $E = \{0 + 8Z, 2 + 8Z, 4 + 8Z, 6 + 8Z\}$ an ideal of L_1 and $A = \{1 + 8Z, 2 + 8Z, 3 + 8Z, 5 + 8Z, 7 + 8Z\}$. Then $\underline{Apr}_E^{L_1}(A) = \{1 + 8Z, 3 + 8Z, 5 + 8Z, 7 + 8Z\}$ and $\overline{Apr}_E^{L_1}(A) = \{0+8Z, 1+8Z, 2+8Z, 3+8Z, 4+8Z, 5+8Z, 6+8Z, 7+8Z\}$. We have $\varsigma(E) = \{0+4Z, 2+4Z\}$ and $\varsigma(A) = \{1+4Z, 2+4Z, 3+4Z\}$. Then $\underline{Apr}_{\varsigma(E)}^{L_2}\varsigma(A) = \{1+4Z, 3+4Z\}$ and $\overline{Apr}_{\varsigma(E)}^{L_2}\varsigma(A) = \{0 + 4Z, 1 + 4Z, 2 + 4Z, 3 + 4Z\}$. The graph of $C_{\underline{Apr}_E^{L_1}(A)}$ and $C_{\underline{Apr}_{\varsigma(E)}^{L_2}\varsigma(A)}$ is shown in Figure 12 and 13 respectively.

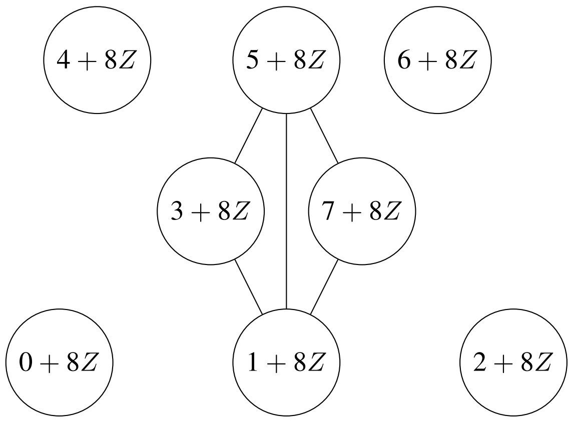

Figure 12. $C_{\underline{Apr}_E^{L_1}(A)}$

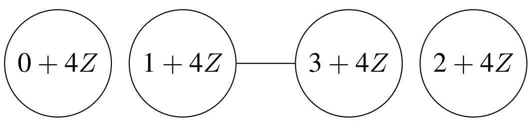

Figure 13. $C_{\underline{Apr}_{\varsigma(E)}^{L_2}\varsigma(A)}$

We observe that ς is a graph homomorphism from $C_{\underline{Apr}_E^{L_1}(A)}$ to $C_{\underline{Apr}_{\varsigma(E)}^{L_2}\varsigma(A)}$.
The graph of $C_{\overline{Apr}_E^{L_1}(A)}$ and $C_{\overline{Apr}_{\varsigma(E)}^{L_2}\varsigma(A)}$ is shown in Figures 14 and 15 respectively.

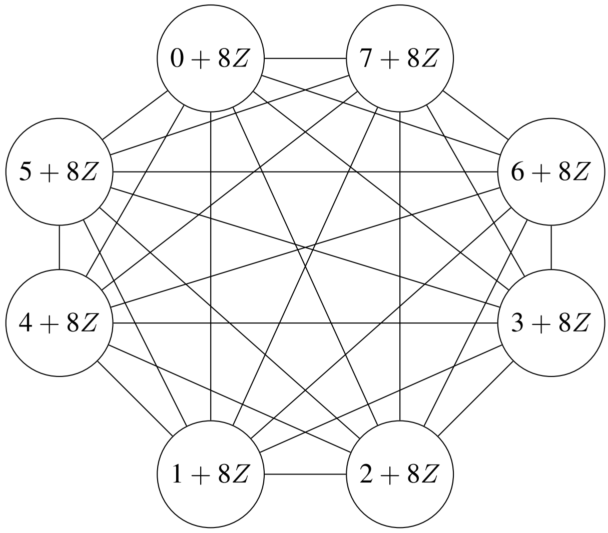

Figure 14. $C_{\overline{Apr}^{L_1}_{E}(A)}$

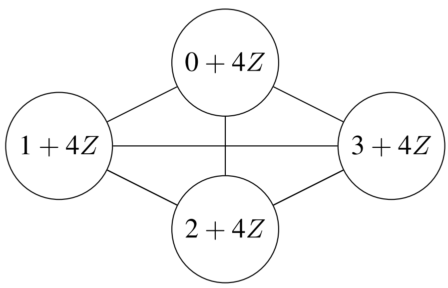

Figure 15. $C_{\overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$

We observe that ς is a graph homomorphism from $C_{\overline{Apr}^{L_1}_{E}(A)}$ to $C_{\overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$.

Lemma 4.3. *Let $\varsigma : L_1 \to L_2$ be an onto nearring homomorphism. Let E be an ideal and A be a non empty subset of L_1. If $a \in L_1$ and $\varsigma(a) \notin \varsigma(E)$ then $a \notin E$.*

Proof. Let E be an ideal and A be a non empty subset of L_1. Then $\varsigma(E)$ is an ideal of L_2. Let $a \in L_1$ be such that $\varsigma(a) \notin \varsigma(E)$. Suppose $a \in E$. Then $\varsigma(a) \in \varsigma(E)$, - a contradiction to $\varsigma(a) \notin \varsigma(E)$. Therefore $a \notin E$.

□

Proposition 4.4. *Let $\varsigma : L_1 \to L_2$ be an onto nearring homomorphism. Let E be an ideal and A be a non empty subset of L_1.*
(i) If $a \in L_1$ and $\varsigma(a)$ is connected to all other vertices of $C_{\underline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$ then a is connected to all other vertices of $C_{\underline{Apr}^{L_1}_{E}(A)}$.
(ii) If $a \in L_1$ with $\varsigma(a)$ is connected to all other vertices of $C_{\overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$ then a is connected to all other vertices of $C_{\overline{Apr}^{L_1}_{E}(A)}$.

Proof. (i) Let $a \in L_1$ be such that $\varsigma(a)$ is connected to all other vertices of $C_{\underline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$. Then for all, $y \in L_1$ there exists $\varsigma(y) \in L_2$ such that $(\varsigma(a), \varsigma(y)) \in F[C_{\underline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}]$. Therefore $\varsigma(a)\varsigma(y) \in \underline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)$ or $\varsigma(y)\varsigma(a) \in \underline{Apr}^{L_2}_{\varsigma(I)}\varsigma(A)$. Let $\varsigma(a)\varsigma(y) \in \underline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)$. Then $\varsigma(a)\varsigma(y) + \varsigma(E) \subseteq \varsigma(A) \implies \varsigma(a)\varsigma(y) \in \varsigma(A) - \varsigma(E)$. As ς is nearring homomorphism $\varsigma(ay) \in \varsigma(A) - \varsigma(E) \implies \varsigma(ay) \in \varsigma(A)$ and $\varsigma(ay) \notin \varsigma(E)$. Hence $ay \in A$ and from

Lemma 4.3, $ay \notin E$. Therefore $ay \in A - E$ and $ay + E \subseteq A \implies ay \in \underline{Apr}_E^{L_1}(A)$ and $(a, y) \in F[C_{\underline{Apr}_E^{L_1}(A)}]$.
The proof is similar for $\varsigma(y)\varsigma(a) \in \underline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)$.
Thus if $a \in L_1$ and $\varsigma(a)$ is connected to all other vertices of $C_{\underline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$ then a is connected to all other vertices of $C_{\underline{Apr}_E^{L_1}(A)}$.
(ii) Let $a \in L_1$ with $\varsigma(a)$ is connected to all other vertices of $C_{\overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$. Then for all $y \in L_1$, there exists $\varsigma(y) \in L_2$ such that $(\varsigma(a), \varsigma(y)) \in F[C_{\overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}]$. Therefore $\varsigma(a)\varsigma(y) \in \overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)$ or $\varsigma(y)\varsigma(a) \in \overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)$. Let $\varsigma(a)\varsigma(y) \in \overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)$. Then $[\varsigma(a)\varsigma(y)+\varsigma(E)]\cap \varsigma(A) \neq \emptyset$. Therefore there exists $\varsigma(q) \in L_2$ such that $\varsigma(q) \in \varsigma(a)\varsigma(y) + \varsigma(E)$ and $\varsigma(q) \in \varsigma(A)$. As ς is a onto nearring homomorphism , $\varsigma(q) \in \varsigma(ay) + \varsigma(E)$ and $\varsigma(q) \in \varsigma(A) \implies$
$\varsigma(q) \in \varsigma(ay + E)$ and $\varsigma(q) \in \varsigma(A)$. Hence $q \in ay + E$ and $q \in A \implies q \in (ay + E) \cap A \implies (ay + E) \cap A \neq \emptyset$. Therefore $ay \in \overline{Apr}_E^{L_1}(A)$ and $(a, y) \in F[C_{\overline{Apr}_E^{L_1}(A)}]$. The proof is similar for $\varsigma(y)\varsigma(a) \in \overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)$. Thus if $a \in L_1$ with $\varsigma(a)$ is connected to all other vertices of $C_{\overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$ then a is connected to all other vertices of $C_{\overline{Apr}_E^{L_1}(A)}$. □

Now we provide an example for the Proposition 4.4

Example 4.5. Let $L_1 = \dfrac{Z}{8Z}$ and $L_2 = \dfrac{Z}{4Z}$. Then L_1 and L_2 are commutative rings. Let $\varsigma : L_1 \to L_2$ be defined by $\varsigma(a + 8Z) = a + 4Z$. Then ς is an onto nearring homomorphism. Consider $E = \{0 + 8Z, 2 + 8Z, 4 + 8Z, 6 + 8Z\}$, an ideal of L_1.
Let $A = \{0+8Z, 1+8Z, 2+8Z, 4+8Z, 5+8Z, 6+8Z\}$. Then $\underline{Apr}_E^{L_1}(A) = \{0+8Z, 2+8Z, 4+8Z, 6+8Z\}$ and $\overline{Apr}_E^{L_1}(A) = \{0+8Z, 1+8Z, 2+8Z, 3+8Z, 4+8Z, 5+8Z, 6+8Z, 7+8Z\}$.
We have $\varsigma(E) = \{0 + 4Z, 2 + 4Z\}$ and $\varsigma(A) = \{0 + 4Z, 1 + 4Z, 2 + 4Z\}$.
$\underline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A) = \{0+4Z, 2+4Z\}$ and $\overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A) = \{0+4Z, 1+4Z, 2+4Z, 3+4Z\}$. Graphs of $C_{\underline{Apr}_E^L(A)}$ and $C_{\underline{Apr}^L_{\varsigma(E)}\varsigma(A)}$ are shown in Figures 16 and 17 respectively.

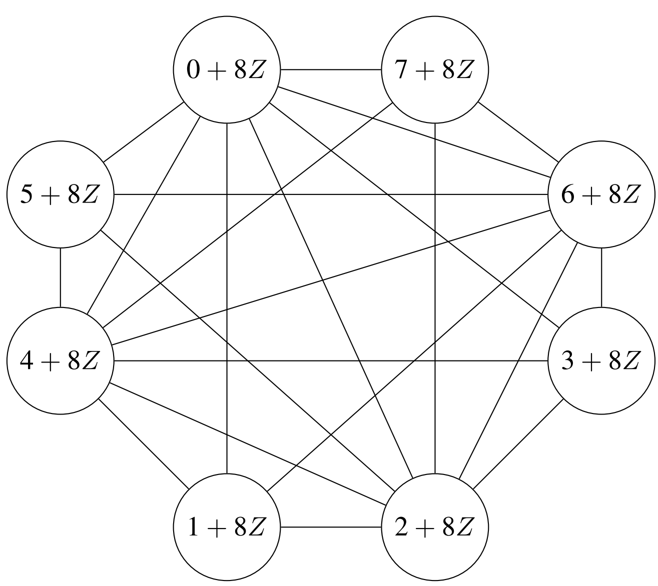

Figure 16. $C_{\underline{Apr}_E^L(A)}$

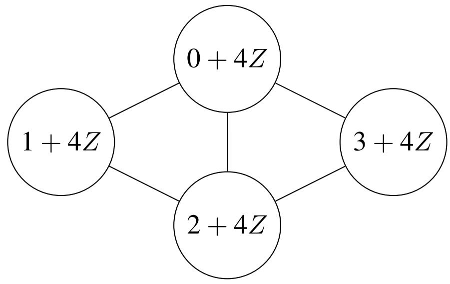

Figure 17. $C_{\underline{Apr}^{L}_{\varsigma(E)}\varsigma(A)}$

We observe that if $a \in L_1$ with $\varsigma(a)$ is connected to all other vertices of in $C_{\underline{Apr}^{L}_{\varsigma(E)}\varsigma(A)}$ then a is connected to all other vertices of $C_{\underline{Apr}^{L}_{E}(A)}$.
The graph of $C_{\overline{Apr}^{L_1}_{E}(A)}$ and $C_{\overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$ is same as in Figures 14 and 15 respectively.
We observe that if $a \in L_1$ with $\varsigma(a)$ is connected to all other vertices of $C_{\overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$ then a is connected to all other vertices of $C_{\overline{Apr}^{L_1}_{E}(A)}$.

Lemma 4.6. *Let $\varsigma : L_1 \to L_2$ be an one−to−one and onto nearring homomorphism. Let E an ideal and A be a non empty subset of L_1.Then $\varsigma(A - E) = \varsigma(A) - \varsigma(E)$.*

Proof. Let $t \in L_1$ be such that $\varsigma(t) \in \varsigma(A - E)$. Then $t \in A - E \Rightarrow t \in A$ and $t \notin E$. Hence $\varsigma(t) \in \varsigma(A)$ and $\varsigma(t) \notin \varsigma(E) \implies \varsigma(t) \in \varsigma(A) - \varsigma(E)$. Therefore $\varsigma(A - E) \subseteq \varsigma(A) - \varsigma(E)$. Now, let $y \in L_1$ be such that $\varsigma(y) \in \varsigma(A) - \varsigma(E)$. Then $\varsigma(y) \in \varsigma(A)$ and $\varsigma(y) \notin \varsigma(E) \implies y \in A$ and from Lemma 4.3, $y \notin E$. Hence $y \in A - E \implies \varsigma(y) \in \varsigma(A - E)$. Therefore $\varsigma(A) - \varsigma(E) \subseteq \varsigma(A - E)$. Thus $\varsigma(A - E) = \varsigma(A) - \varsigma(E)$ □

Proposition 4.7. *Let $\varsigma : L_1 \to L_2$ be an one−to−one and onto nearring homomorphism. Consider an ideal E and a non empty subset A of nearring L_1 .*
(i) If $a \in E$ and $\varsigma(a)$ is an isolated vertex in $C_{\underline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$ then a is an isolated vertex in $C_{\underline{Apr}^{L_1}_{E}(A)}$.
(ii)If $a \in E$ and $\varsigma(a)$ is an isolated vertex in $C_{\overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$ then a is an isolated vertex in $C_{\overline{Apr}^{L_1}_{E}(A)}$.

Proof. (i) Let $a \in E$ and $\varsigma(a)$ be an isolated vertex in $C_{\underline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$. Assume that a is not an isolated vertex in $C_{\underline{Apr}^{L_1}_{E}(A)}$. Then there exists $y \in L_1$ such that $(a, y) \in F[C_{\underline{Apr}^{L_1}_{E}(A)}]$. Therefore $ay \in \underline{Apr}^{L_1}_{E}(A)$ or $ya \in \underline{Apr}^{L_1}_{E}(A)$. Let $ay \in \underline{Apr}^{L_1}_{E}(A)$. Then $ay + E \subseteq A$ or $ay \in A - E$. Hence $\varsigma(ay) \in \varsigma(A - E)$, from Lemma 4.6, $\varsigma(ay) \in \varsigma(A) - \varsigma(E)$. Therefore $\varsigma(ay) \in \varsigma(A)$ and $\varsigma(ay) \notin \varsigma(E)$. As ς is nearring homomorphism $\varsigma(a)\varsigma(y) \in \varsigma(A)$ and $\varsigma(a)\varsigma(y) \notin \varsigma(E)$. Hence $\varsigma(a)\varsigma(y) \in \varsigma(A) - \varsigma(E) \implies \varsigma(a)\varsigma(y) + \varsigma(E) \subseteq \varsigma(A)$. Therefore $\varsigma(a)\varsigma(y) \in \underline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A) \implies (\varsigma(a), \varsigma(y)) \in F[C_{\underline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}]$, -a contradiction to $\varsigma(a)$ is an isolated vertex in $C_{\underline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$. Thus a is an isolated vertex in $C_{\underline{Apr}^{L_1}_{E}(A)}$. The proof is similar for $ya \in \underline{Apr}^{L_1}_{E}(A)$.
(ii) Let $a \in E$ and $\varsigma(a)$ be an isolated vertex in $C_{\overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$. Assume that a is not an isolated vertex in $C_{\overline{Apr}^{L_1}_{E}(A)}$. Then there exists $y \in L_1$ such that $(a, y) \in F[C_{\overline{Apr}^{L_1}_{E}(A)}]$. Therefore $ay \in \overline{Apr}^{L_1}_{E}(A)$ or $ya \in \overline{Apr}^{L_1}_{E}(A)$. Let $ay \in \overline{Apr}^{L_1}_{E}(A)$. Then $(ay+E)\cap A \neq \emptyset$. Let $p \in (ay+E)\cap A$. Then $p \in ay+E$ and $p \in A$. Therefore $\varsigma(p) \in \varsigma(ay+E)$ and $\varsigma(p) \in \varsigma(A)$. As ς is nearring homomorphism, $\varsigma(p) \in \varsigma(ay) + \varsigma(E)$ and $\varsigma(p) \in \varsigma(A) \implies \varsigma(p) \in \varsigma(a)\varsigma(y) + \varsigma(E)$ and $\varsigma(p) \in \varsigma(A)$. Hence $\varsigma(p) \in [\varsigma(a)\varsigma(y) + \varsigma(E)] \cap \varsigma(A)$. Therefore $[\varsigma(a)\varsigma(y) + \varsigma(E)] \cap \varsigma(A) \neq \emptyset \implies \varsigma(a)\varsigma(y) \in \overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)$. -a contradiction to $\varsigma(a)$ is an isolated vertex in $C_{\overline{Apr}^{L_2}_{\varsigma(E)}\varsigma(A)}$. Hence a is an isolated vertex in $C_{\overline{Apr}^{L_1}_{E}(A)}$. The proof is similar for $ya \in \overline{Apr}^{L_1}_{E}(A)$. □

Conclusions: In this paper we defined two types of graphs namely lower approximation graph and upper approximation graph. We studied the connectivity property of graph of lower (upper) approximation. We related the properties of an ideal with properties of graph. Further we defined ideal symmetry of graph of lower (upper) approximation. We proved the graph homomorphism in lower (upper) approximation graphs.

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Compliance with Ethical Standards

Conflict of Interest:

The authors declare that they have no conflict of interest.

Ethical approval:

The article does not contain any studies with human participants or animals performed by any of the authors.

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