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CATEGORY OF CHAINS OF STOCHASTIC MATRICES

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Abstract In this paper, we describe categories whose objects are chains of stochastic vectors/ stochastic matrices with morphisms that are appropriate maps between such objects with functors in this category are chain maps. The chain categories are interesting and significant in several situations such as Markov chains, stochastic processes, and queuing theory. Here we provide examples of chain categories and discuss many interesting properties of such categories.

1 Introduction

In recent times category theory has taken on a significant role as a generalization and a tool for exploring many mathematical systems. In [5] we defined a category chain bundles $\mathfrak{CB}_{\mathcal{C}}$ in a category $\mathcal{C}$ with zero, whose objects are a sequence of objects in $\mathcal{C}$ such that for any two objects X and Y in $\mathcal{C}$ any subset of $Hom(X,Y)$ constitute morphisms in chain bundle. With appropriate maps which are functors between chain bundles as morphisms the chain bundles together with these functors constitute the category of chain bundles $\mathfrak{CB}_{\mathcal{C}}$. In particular, if chain bundles $\mathfrak{CB}_{\mathcal{C}}$ is a preorder it is termed as a chain. There are several natural examples of chains, such as chain complexes, chains of ideals of rings, Markov chains, ergodic Markov chains, unichains, and the like. In [6] we described the category of chains whose objects are certain chains and whose morphisms are chain maps with a grant aim to view all such chains in a categorical setting.

We offer the category theory concepts and findings required for the sequel in the first section, and the second section describes the categories of chain bundles and chains. Further, the chain of stochastic vectors as well as the chain of stochastic matrices are discussed in third section, we also present a category of chains of stochastic vectors and stochastic matrices in this section. Here it is also noted that the stochastic [doubly stochastic] matrices forms a semigroup which is realised as polytopes.

2 Preliminaries

In the following, we briefly recall some basic notions related to category theory and go through category with subobjects.

Definition 1. *A category $\mathcal{C}$ consists of the following data*

(i) a class $\nu\mathcal{C}$ called the class of vertices or objects

(ii) a class $\mathcal{C}$ of disjoint sets $\mathcal{C}(a,b)$, one for each pair $(a,b) \in \nu\mathcal{C} \times \nu\mathcal{C}$, an element $f \in \mathcal{C}(a,b)$ called a morphism (arrow) from a to b, written $f : a \to b$; $a = dom\, f$ and $b = cod\, f$

(iii) For $a, b, c \in \nu\mathcal{C}$, a map

$$\circ : \mathcal{C}(a,b) \times \mathcal{C}(b,c) \to \mathcal{C}(a,c)$$

$$(f, g) \to f \circ g$$

called the composition of morphisms in $\mathcal{C}$

(iv) For each $a \in \nu\mathcal{C}$, a unique $1_a \in \mathcal{C}(a, a)$ is called the identity morphism on a

These must satisfy the following axioms:

Cat 1 The compostion is associative: for $f \in \mathcal{C}(a, b)$, $g \in \mathcal{C}(b, c)$ and $h \in \mathcal{C}(c, d)$ we have

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

Cat 2 For each $a \in \nu\mathcal{C}$, $f \in \mathcal{C}(a, b)$, $g \in \mathcal{C}(c, a)$, $\quad 1_a \circ f = f \, and \, g \circ 1_a = g$.

We may identify $\nu\mathcal{C}$ as a subclass of $\mathcal{C}$ and with this identification, categories may regard in terms of morphisms(arrows) alone. The category $\mathcal{C}$ is said to be small if the class $\mathcal{C}$ is a set. For any category $\mathcal{C}$ an opposite category denoted as $\mathcal{C}^{op}$ may be defined as follows

$$\nu\mathcal{C}^{op} = \nu\mathcal{C}, \mathcal{C}^{op}(a, b) = \mathcal{C}(b, a) \,\text{for all}\, a, b \in \nu\mathcal{C}$$

and the composition $*$ in $\mathcal{C}^{op}$ is given by

$$g * h = h \circ g \,\text{for all}\, g, h \in \mathcal{C}^{op} = \mathcal{C}$$

for which $h \circ g$ is defined. For examples see [9].

Definition 2. *A functor $F : \mathcal{C} \to \mathcal{D}$ consists of a vertex map $\nu F : \nu\mathcal{C} \to \nu\mathcal{D}$ which assigns to each $a \in \nu\mathcal{C}$, a vertex $\nu F(a) \in \nu\mathcal{D}$ and a morphism map F which assigns to each morphism $f : a \to b$ in $\mathcal{C}$, a morphism $F(f) : F(a) \to F(b) \in \mathcal{D}$ such that*

(Fn. 1) $F(1_a) = 1_{F(a)} \; \forall a \in \nu\mathcal{C}$

(Fn .2) $F(f)F(g) = F(fg)$ for all morphisms $f, g \in \mathcal{C}$ for which the composite fg exists.

Definition 3. *A category $\mathcal{D}$ is a subcategory of a category $\mathcal{C}$ if the class $\mathcal{D}$ is a subclass of $\mathcal{C}$ and the composition in $\mathcal{D}$ is the restriction of the composition in $\mathcal{C}$ to $\mathcal{D}$. In this case, the inclusion $\mathcal{D} \subseteq \mathcal{C}$ preserves composition and identities and so represents a functor of $\mathcal{D}$ to $\mathcal{C}$ which is called the Inclusion functor of $\mathcal{D}$ into $\mathcal{C}$.*

A preorder $\mathcal{P}$ is a category such that for any $p, p' \in v\mathcal{P}$, the hom-set $Hom(p, p')$ contains at most one morphism. In this case, the relation $\subseteq$ on the class $v\mathcal{P}$ of objects of $\mathcal{P}$ defined by $p \subseteq p'$ if $HomP(p, p') \neq \emptyset$ is a quasi- order and $\mathcal{P}$ is said to be a strict preorder if $\subseteq$ is a partial order.

Definition 4. *Let $\mathcal{C}$ be a small category and $\mathcal{P}$ be a subcategory of $\mathcal{C}$ such that $\mathcal{P}$ is a strict preorder with $v\mathcal{P} = v\mathcal{C}$. Then $(\mathcal{C}, \mathcal{P})$ is a category with sub objects if*

(1) *every $f \in \mathcal{P}$ is a monomorphism in $\mathcal{C}$*

(2) *if $f = hg$ for $f, g \in \mathcal{P}$ then $h \in \mathcal{P}$.*

In the category $(\mathcal{C}, \mathcal{P})$ with subobjects, morphisms in $\mathcal{P}$ are called inclusions and for an inclusion $c' \to c$, we write $c' \subseteq c$ and denotes this inclusion by $j^c_{c'}$.

3 Category of chain bundles and chains

Next, we recall the category of chain bundles and the category of chains which we described in cf. [6]. Here it should be noted that our definition of the bundle is not exactly the same as that of the usual sequence of fibre bundles or algebra bundles (see [3]). However, it is easy to see that our definition is much more general and it includes bundles in the classical sense.

Definition 5. *Let $\mathcal{C}$ be category with zero. A chain bundle is a sequence in the category $\mathcal{C}$ of the form*

$$\cdots M_3 \overset{S_3}{\to} M_2 \overset{S_2}{\to} M_1 \overset{S_1}{\to} M_0 = \mathbf{0}$$

where $M_i \in \nu\mathcal{C}$ and S_i is a subset of the morphism set $Hom(M_{i+1}, M_i)$ for all i, which also include homsets of the form $Hom(M_i, M_i)$ and all possible composite of morphisms.

A chain bundle map between two chain bundles $\cdots M_2 \overset{S_2}{\to} M_1 \overset{S_1}{\to} M_0 = \mathbf{0}$, $\cdots N_2 \overset{S_2'}{\to} N_1 \overset{S_1'}{\to} N_0 = \mathbf{0}$ in $\mathcal{C}$ is a functor F between the two whose vertex map is the collection $\nu F = \{f_i : M_i \to N_i\}$ of morphisms in $\mathcal{C}$ and morphism map is a map between homsets of $\mathcal{C}$, such that diagram commutes

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M_3 & \xrightarrow{S_3} & M_2 & \xrightarrow{S_2} & M_1 & \xrightarrow{S_1} & \mathbf{0} \\
 & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
\cdots & \longrightarrow & N_3 & \xrightarrow{S_3'} & N_2 & \xrightarrow{S_2'} & N_1 & \xrightarrow{S_1'} & \mathbf{0}
\end{array}$$

where $S_i \circ f_i = \{x_i \circ f_i : x_i \in S_i\}$.

Definition 6. *Let $\mathcal{C}$ be a category with zero. Category whose objects are chain bundles and morphism between chain bundles are chain bundle maps is called the category of chain bundles and is written as $\mathfrak{CB}_{\mathcal{C}}$.*

A chain bundle $\mathfrak{CB}_{\mathcal{C}}$ whose morphism set is a strict preorder (ie., exactly one morphism in each hom set) is a chain.

Definition 7. *A chain map between two chains in $\mathfrak{CB}_{\mathcal{C}}$ is a functor F between the two whose vertex map $\nu F = \{f_i : M_i \to N_i\}$ is a sequence of morphisms in $\mathcal{C}$ and morphism map is a map between homsets of $\mathcal{C}$, such that diagram commutes*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M_3 & \xrightarrow{s_3} & M_2 & \xrightarrow{s_2} & M_1 & \xrightarrow{s_1} & \boldsymbol{0} \\
 & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
\cdots & \longrightarrow & N_3 & \xrightarrow{s_3'} & N_2 & \xrightarrow{s_2'} & N_1 & \xrightarrow{s_1'} & \boldsymbol{0}
\end{array}$$

Definition 8. *Let $\mathcal{C}$ be a category with zero object. The category of chains in $\mathcal{C}$ is a subcategory of the category of chain bundles $\mathfrak{CB}_{\mathcal{C}}$ whose objects are chains*

$$\cdots M_3 \overset{s_3}{\to} M_2 \overset{s_2}{\to} M_1 \overset{s_1}{\to} M_0 = \boldsymbol{0}$$

with $M_i \in \nu\mathcal{C}$ and morphisms are chain maps.

Definition 9. *Consier the chain $c : \cdots M_3 \overset{s_3}{\to} M_2 \overset{s_2}{\to} M_1 \overset{s_1}{\to} M_0 = \boldsymbol{0}$, then $c' : \cdots M_3' \overset{s_3'}{\to} M_2' \overset{s_2'}{\to} M_1' \overset{s_1'}{\to} M_0' = \boldsymbol{0}$ with M_i' a subobject of M_i and $(s_i')^0 = (j^{M_i}_{M_{i'}} s_i)^0$ is called a subchain of c.*

4 Chain of stochastic vectors/matrices

A stochastic n- vector is an n-tuple of non negative entries that add up to 1.

Definition 10. *A $n \times m$ matrix $[a_{ij}]$ is said to be stochastic if all row vectors of $[a_{ij}]$ are stochastic vectors.*

Also it can be seen that the product of two stochastic matrices is again a stochastic matrix.

Example 1. *The matrix* $A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1 & 0 & 0 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$ *is stochastic.*

Definition 11. *A $n \times m$ matrix $[a_{ij}]$ is said to be doubly stochastic if all row vectors and column vectors are of $[a_{ij}]$ are stochastic vectors.*

It is easy to note that the product of two doubly stochastic matrices is again a doubly stochastic matrix.

Example 2. *The matrix* $\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ *is doubly stochastic.*

We write $St_n[D_n]$ to denote the set of all $n \times n$ stochastic [doubly stochastic] matrices. Note that the transpose of a stochastic [doubly stochastic] matrix is also stochastic [doubly stochastic] and the collection of all stochastic [doubly stochastic] matrices of order n, St_n $[D_n]$ forms a semigroup. (cf.[8]).

Now it is easy to observe that the collection $\mathcal{CS}t_n$ whose objects are stochastic $n \times n$ matrices with morphisms between two such stochastic matrices P_1 and P_2 are those stochastic matrices M for which $P_1 M = P_2$ and the composite of morphisms as the multiplication of matrices forms a category. The identity identity matrix which is obviously a stochastic matrix is an identity morphism. The axioms of categories can easily be seen.

Example 3. *Consider $\mathcal{CS}t_2$. Let* $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Then there is only one morphism from P_1 to P_2 which is $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example 4. *Consider $\mathcal{CS}t_3$. Let* $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ *and* $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Then $Hom(P_1, P_2)$ comsists of matrices of the form $M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ a & b & 1-a-b \end{bmatrix}$.

4.1 Category of chains of stochastic matrices and vectors

A Markov chain or Markov process is a stochastic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event. A Markov chain is a sequence of probability vectors $x_0, x_1, x_2, \cdots$ together with a stochastic matrix P, such that

$$x_1 = x_0 P, x_2 = x_1 P, x_3 = x_2 P, \cdots$$

In the following we consider chain of stochastic/probability vectors in $\mathbb{R}^n$ as well as stochastic matrices in $M_n(\mathbb{R})$ and discuss their chains categories.

Definition 12. *A chain of stochastic vectors and stochastic matrices is a sequence of the form*

$$x_1 \xrightarrow{P_1} x_2 \xrightarrow{P_2} x_3 \xrightarrow{P_3} \cdots$$

where x_i's are stochastic vectors and and P_i's are stochastic matrices.

Example 5. $\begin{bmatrix} 1 & 0 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} 0 & 1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 0 & 1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \cdots$

Example 6. $[1 \quad 0] \xrightarrow{\begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}} [1/2 \quad 1/2] \xrightarrow{\begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}} [5/8 \quad 3/8] \xrightarrow{\begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}} \cdots$

Note 1. *If all matrices in a chain are same then chain becomes Markov chain and if the stochastic matrix is regular then steady state occurs.*

Definition 13. ([10]) *A matrix A is called a $J-potent$ matrix if there exists a number p such that $A^p = A^{p+1} = \cdots$.*

Thus if the stochastic matrix in a Markov chain is a $J-potent$ matrix, then the chain attains steady state.

Definition 14. *Consider two stochastic chains. A stochastic chain map between the two is a sequence of stochastic matrices (M_i) such that following diagram commutes.*

$$
\begin{array}{ccccccccc}
x_1 & \xrightarrow{P_1} & \cdots & \longrightarrow & x_i & \xrightarrow{P_i} & x_{i+1} & \xrightarrow{P_{i+1}} & \cdots \\
\downarrow{\scriptstyle M_1} & & & & {\scriptstyle M_i}\downarrow & & \downarrow{\scriptstyle M_{i+1}} & & \\
y_1 & \xrightarrow{T_1} & \cdots & \longrightarrow & y_i & \xrightarrow{T_i} & y_{i+1} & \xrightarrow{T_{i+1}} & \cdots
\end{array}
$$

Example 7. *Consider the two stochastic chains:*

$c_1 : [a \quad 1-a] \xrightarrow{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} [1 \quad 0] \xrightarrow{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} [1 \quad 0] \xrightarrow{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} \cdots$ *and*

$c_2 : [a \quad 1-a] \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} [0 \quad 1] \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} [0 \quad 1] \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} \cdots$

A stochastic chain map from c_1 to c_2 is a sequence (M_i) where $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $M_2 = M_3 = \cdots = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Choosing objects as stochastic chains and morphisms as stochastic chain map we obtain category of stochastic vectors and stochastic matrices.

Remark 2. *Consider Markov chains from $\mathcal{ST}_2$. We note the subsequent finding. Any Markov chain whose stochastic matrix is a vertex of the convex polytope St_2 will either achieve steady state or loop occurs.*

$$[a \quad 1-a] \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} [a \quad 1-a] \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \cdots$$

$$[a \quad 1-a] \xrightarrow{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} [1 \quad 0] \xrightarrow{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} [1 \quad 0] \xrightarrow{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} \cdots$$

$$[a \quad 1-a] \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} [0 \quad 1] \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} [0 \quad 1] \xrightarrow{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} \cdots$$

$$[a \quad 1-a] \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} [1-a \quad a] \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} [a \quad 1-a] \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \cdots$$

In this paper, our main aim is to introduce the stochastic chains which play significant role in several contexts. Though we are limited to this aim, it is interesting to look at the role of eigen values and eigen vectors of stochastic matrices in the study of stochastic chains.

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