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A SURVEY ON THE FUZZY DEGREE OF A HYPERGROUP

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Abstract This paper presents a series of results about the fuzzy degree of a hypergoup. It is considered a sequence of membership functions and of join spaces, obtained by starting with a hypergroupoid $(H, \otimes)$, see Corsini's paper [Southeast Asian Bull. Math. 27 (2003) 221-229]. The fuzzy grade is the minumum natural number i such that two consecutive associated join spaces, of the above mentioned sequence, H_i and H_{i+1} are isomorphic.

1 Introduction

Hyperstructures are a very important mathematical subject both on the theoretical point of view and for their applications to almost every topic of science. The first who worked on this matter was a French scientist, Marty [17], who introduced the notion of a hypergroup in 1934 and showed its utility in other topics: groups, algebraic functions, rational fractions. The theory has known various periods of flourishing: the 40s, then 70s and especially after the 90s the theory is studied on all continents, both theoretically and for the multitude of applications in various fields of knowledge. Several books have been written in this field, which highlight both the theoretical aspects and the applications: [10], [8], [15], [12], [13], [20].

Several classes of hypergroups are introduced and studied, such as canonical hypergroups, join spaces and complete hypergroups. Join Spaces were introduced by Prenowitz in the 40's and were utilized by him and later by him together with Jantosciak, to reconstruct several kinds of Geometry. Join spaces had already many other applications, for instance to Graphs, (Nieminen, Rosenberg, Bandelt, Mulder, Corsini), to Median Algebras (Bandelt-Hedlikova), to Hypergraphs (Corsini, Leoreanu), to Binary Relations (Chvalina, Corsini, Corsini- Leoreanu, Davvaz, De Salvo-Lo Faro, Rosenberg).

Fuzzy Sets were introduced in the 60's by Zadeh, an Iranian scientist who lived in the USA. He and others, in the following decades, found surprising applications to almost every field of science and knowledge: from enginnering to sociology, from agronomy to linguistic, from biology to computer science, from medicine to economy, from psycology to statistics and so on. They are now cultivated in all the world. Let us recall what a fuzzy set is. We know that the subsets of a universe H, can be represented as functions, the characteristic functions ϵ from H to the set $\{0, 1\}$. The notion of fuzzy subset generalizes that one of characteristic function. One considers functions μ_A from H to the closed real interval $[0, 1]$ instead of the functions ϵ. These functions, called "membership functions", express the degree of belonging of an element $x \in H$ to a subset A of H. A Fuzzy Subset of a set H is a pair (H, μ_A) where μ_A is a function $\mu_A : H \to [0, 1],\ A$ is the set $\{x \in H \mid \mu_A(x) = 1\}$.

Connections between hyperstructures and fuzzy sets have been considered by many people. In particular, the Corsini's paper [4] opened research lines studied in deep by several scientists. In this context several papers have been made in Italy, Romania, Greece, Iran, for example by Corsini, Leoreanu, Cristea, Serafimidis, Kehagias, Konstantinidou, Rosenberg. Hyperstructures endowed also with a fuzzy structure have been considered especially in Iran by Ameri, Zahedi, Davvaz and many others.

This survey is structured as follows: at the beginning basic notions in huypergroups and connections with fuzzy sets are recalled and the notion of fuzzy degree of a hypergroup is presented in particular. Then follows two examples of this notion, one connected with genetics, in particular with non-Mendelian inheritance and the other one associated with a hypergraph.

2 Hypergroups and Connections with Fuzzy Sets

An algebraic hyperstructure is a non empty set H endowed with one or some functions from $H \times H$ to the set $P^*(H)$ of non empty subsets of H. For all $(x, y) \in H^2$ one denotes by $x \circ y$ the image $f(x, y)$, where f is the function $f : H \times H \to P^*(H)$.

$x \circ y$ is called the hyperproduct of x and y and $(H; \circ)$ is called a hypergroupoid.

If $A, B \in P^*(H)$, $A \circ B$ denotes the set $\bigcup_{a \in A, b \in B} a \circ b$.

Definition 2.1. $(H; \circ)$ is called a *semi-hypergroup* if

$$\forall (x, y, z) \in H^3,\ (x \circ y) \circ z = x \circ (y \circ z),$$

where $(x \circ y) \circ z$ denotes the union

$$\bigcup_{u \in x \circ y} u \circ z.$$

Analoguesly,

$$x \circ (y \circ z) = \bigcup_{v \in y \circ z} x \circ v.$$

Definition 2.2. A *hypergroup* $(H; \circ)$ is a semihypergroup such that

$$\forall (a, b) \in H^2,\ \exists (x, y) \in H^2 \text{ such that}$$

$$a \in b \circ x \text{ and } a \in y \circ b$$

Definition 2.3. A join space is a commutative hypergroup $(H; \circ)$, such that the following implication is satisfied:

$$\forall (a, b, c, d) \in H^4,$$

$$a/b \cap c/d \neq \emptyset \Rightarrow a \circ d \cap b \circ c \neq \emptyset.$$

where a/b denotes the set

$$\{x \in H \mid a \in x \circ b\}.$$

Canonical hypergroups have a structure close to that of a commutative group: they are commutative, have a scalar identity e (that is $\forall x \in H, x \circ e = e \circ x = x$) every element has a unique inverse and they are reversible (that is if $x \in y \circ z$, then $z \in y^{-1} \circ x,\ y \in x \circ z^{-1}$).

Example 2.4. Let $C(n) = \{e_0, e_1, ..., e_k(n)\}$, where $k(n) = n/2$ if n is an even natural number and $k(n) = (n - 1)/2$ if n is an odd natural number. For all e_s, e_t of $C(n)$, define $e_s \circ e_t = \{e_p, e_v\}$, where $p = min\{s+t, n-(s+t)\}, v = |s-t|$. Then $(C(n), \circ)$ is a canonical hypergroup.

An important result is the next one:

Theorem 2.5. *A commutative hypergroup is canonical if and only if it is a join space with a scalar identity.*

In 1993 Corsini proved that one can associate a join space to every Fuzzy Subset of a set H , where $(\circ)$ is defined as follows: $\forall (x, y) \in H^2$,

$$x \circ y = \{z \in H \mid min\{\mu(x), \mu(y)\} \leq \mu(z) \leq max\{\mu(x), \mu(y)\}\}. \tag{1}$$

Moreover, in 2003 Corsini proved that one can associate a fuzzy subset with every hypergroupoid. Set

$$\forall u \in H,\ Q(u) = \{(x, y) \in H^2 \mid u \in x \circ y\},$$

$$q(u) = \mid Q(u) \mid,$$

$$A(u) = \sum_{(x,y) \subseteq Q(u)} 1/ \mid x \circ y \mid,$$

$$\mu(u) = A(u)/q(u). \tag{2}$$

From (1) and (2) follows clearly that every hypergroupoid (or fuzzy subset) determines a sequence of fuzzy subsets and hypergroupoids (or of hypergroupoids and fuzzy subsets), which is obtained applying consecutively (2) and (1) (or (1) and (2)). So every fuzzy subset (and every hypergroupoid) determines a sequence of join spaces and of fuzzy subsets.

The Fuzzy Grade, which is the minimum lengh of such sequences, has been calculated for several classes of hyperstructures: Corsini-Cristea for i.p.s. hypergroups (a particular case of Canonical hypergroups), and 1-hypergroups (hypergroups for which the heart is a singleton), Corsini -Leoreanu for hypergroups associated with hypergraphs. It is denoted by ∂H, where H is the starting hypergroup.

3 A Sequence of Fuzzy Sets and Hypergroups that Occour in Genetics

We analyse here the first type Epistasis of non-Mendelian inheritance, which was presented in detail in paper [16].

Consider the epistasis of dominant gene in the coat color of dogs. There are two allelomorphic pairs Aa and Bb, where A and B are dominant over a and b respectively. Summarizing this experiment, we obtain:

$$P: \ AABB \otimes aabb$$

$$F_1: \ AaBb$$

and

$$F_1 \otimes F_1: \ AaBb \otimes AaBb$$

$$F_2: White, Black, Brown.$$

A_1 denotes White, A_2 denotes Black and A_3 denotes Brown.

Set $H = \{A_1, A_2, A_3\}$. We obtain the following hypergroup:

$\otimes$	A_1	A_2	A_3
A_1	H	H	H
A_2	H	$\{A_2, A_3\}$	$\{A_2, A_3\}$
A_3	H	$\{A_2, A_3\}$	$\{A_3\}$

We generalize the above hypergroup to a hypergoup of n elements, where $n \geq 3$ in the next manner: $H = \{A_1, A_2, ..., A_n\}$ and

$\otimes$	A_1	A_2	A_3	...	A_n
A_1	H	H	H	...	H
A_2	H	$\{A_2, A_3, ..., A_n\}$	$\{A_2, A_3, ..., A_n\}$	...	$\{A_2, A_3, ..., A_n\}$
A_3	H	$\{A_2, A_3, ..., A_n\}$	$\{A_3, ..., A_n\}$	...	$\{A_3, ..., A_n\}$
...	...	...	...	...	...
A_n	H	$\{A_2, A_3, ..., A_n\}$	$\{A_3, ..., A_n\}$	...	$\{A_n\}$

We apply Corsini's construction (2) for the studied hypergroup we obtain the following calculations.

3.1 Step 1

For any $A \in H$ we have

$$\mu_1(A) = \alpha_1(A)/q_1(A).$$

If we denote $\sum_{j=1}^{n} 1/j = H_n$, then for all $1 \leq k < n$,

$$\mu_1(A_k) = (2k - H_n + H_{n-k})/[k(2n-k)].$$

Theorem 3.1. *For* $1 \leq i < j \leq n$ *we have* $\mu_1(A_i) < \mu_1(A_j)$.

Proof. Indeed, for $1 < k < n-1$ we have the equivalences $\mu_1(A_k) < \mu_1(A_{k+1})$
$\Leftrightarrow (2k - H_n + H_{n-k})/[k(2n-k)] < (2k+2-H_n+H_{n-k-1})/[(k+1)(2n-k-1)]$
$\Leftrightarrow 2/(2n-k) - T_n/k(2n-k)$
$< 2/(2n-k-1) - [T_n + 1/(n-k)]/(k+1)(2n-k-1)$, where $T_n = H_n - H_{n-k}$. Indeed, $H_n - H_{n-k-1} = T_n + 1/(n-k)$.

Hence, $\mu_1(A_k) < \mu_1(A_{k+1})$ iff
$2/(2n-k) - 2/(2n-k-1) < T_n[1/k(2n-k) - 1/(k+1)(2n-k-1)]$
$-1/[(n-k)(k+1)(2n-k-1)]$
$\Leftrightarrow -2/(2n-k)(2n-k-1) < T_n[1/k(2n-k) - 1/(k+1)(2n-k-1)]$
$-1/[(n-k)(k+1)(2n-k-1)]$
$\Leftrightarrow 1/[(n-k)(k+1)(2n-k-1)] - 2/(2n-k)(2n-k-1)$
$< T_n[1/k(2n-k) - 1/(k+1)(2n-k-1)]$
$\Leftrightarrow k(2n-2k-1)/[(n-k)(k+1)(2n-k-1)(2n-k)]$
$> T_n[1/(k+1)(2n-k-1) - 1/k(2n-k)]$, which is true since the lefthand term is positive, while the righthand term is negative.

Let us check now that $\mu(A_{n-1}) < \mu(A_n)$.
Indeed, $(2(n-1) - H_n + H_{n-(n-1)})/[(n-1)(2n-(n-1))] < (2n - H_n)/n^2$
$\Leftrightarrow (2n-1-H_n)/(n^2-1) < (2n-H_n)/n^2$
$\Leftrightarrow (-n+2)/[n(n^2-1)] < H_n/[n^2(n^2-1)]$, which is true for $n \geq 3$.
Hence we obtain the conclusion. □

We obtain a new hypergroup $(H, \circ_1)$ in which for all $A_i \in H$ we have
$A_i \circ_1 A_i = \{a_i\}$ and for all $1 \leq i < j \leq n$ we have $A_i \circ_1 A_j = A_j \circ_1 A_i = \{A_i, A_{i+1}, ..., A_j\}$. Hence,

$\circ_1$	A_1	A_2	A_3	...	A_n
A_1	A_1	$\{A_1, A_2\}$	$\{A_1, A_2, A_3\}$	...	H
A_2		A_2	$\{A_2, A_3\}$	...	$\{A_2, A_3, ..., A_n\}$
A_3			A_3	...	$\{A_3, ..., A_n\}$
...	...	...	...	...	...
A_n				...	A_n

3.2 Step 2

For any $A \in H$ we have
$$\mu_2(A) = \alpha_2(A)/q_2(A).$$

By calculations we obtain the next result:

Theorem 3.2.

(i) *For* $n = 2s$ *we have:*
$$\mu_2(A_1) = \mu_2(A_{2s}),\ \mu_2(A_2) = \mu_2(A_{2s-1}), ..., \mu_2(A_s) = \mu_2(A_{s+1})$$
and $\mu_2(A_1) > \mu_2(A_2) > ... > \mu_2(A_s)$.

(ii) *For* $n = 2s+1$ *we have:*
$$\mu_2(A_1) = \mu_2(A_{2s+1}),\ \mu_2(A_2) = \mu_2(A_{2s}), ..., \mu_2(A_s) = \mu_2(A_{s+2})$$
and $\mu_2(A_1) > \mu_2(A_2) > ... > \mu_2(A_s) > \mu_2(A_{s+1})$.

Now, taking account of $(**)$ we obtain a new hypergroup $(H, \circ_2)$ in which for all $A_i \in H$ we have $A_i \circ_2 A_i = \{A_i, A_{n+1-i}\}$. Moreover, for all $1 \leq i < j \leq n$ we have $A_i \circ_2 A_j = A_j \circ_2 A_i = \{A_i, A_{n+1-i}, A_{i+1}, A_{n-i}, ..., A_j, A_{n+1-j}\}$.

Denote $\hat{A}_i = \{A_i, A_{n+1-i}\}$. Hence

$$A_i \circ_2 A_i = \hat{A}_i, \;\; A_i \circ_2 A_j = A_j \circ_2 A_i = \hat{A}_i \cup \hat{A_{i+1}} \cup ... \cup \hat{A}_j.$$

In what follows we analyze separately the cases n even and n odd.

3.3 Step 3

Case $n = 2s$.

$\circ_2$	A_1	A_2	...	A_s	A_{s+1}	...	A_{2s-1}	A_{2s}
A_1	$\hat{A}_1$	$\hat{A}_1 \cup \hat{A}_2$	...	$\hat{A}_1 \cup ... \cup \hat{A}_s$	$\hat{A}_1 \cup ... \cup \hat{A}_s$	...	$\hat{A}_1 \cup \hat{A}_2$	$\hat{A}_1$
A_2		$\hat{A}_2$	...	$\hat{A}_2 \cup ... \cup \hat{A}_s$	$\hat{A}_2 \cup ... \cup \hat{A}_s$	...	$\hat{A}_2$	$\hat{A}_1 \cup \hat{A}_2$
...	...	...	...	...	...	...	...	...
A_s				$\hat{A}_s$	$\hat{A}_s$	...	...	$\hat{A}_1 \cup ... \cup \hat{A}_s$
A_{s+1}				$\hat{A}_s$	$\hat{A}_s$	...	...	$\hat{A}_1 \cup ... \cup \hat{A}_s$
...	...	...	...	...	...	...	...	...
A_{2s}	$\hat{A}_1$	$\hat{A}_1 \cup \hat{A}_2$		$\hat{A}_1 \cup ... \cup \hat{A}_s$	$\hat{A}_1 \cup ... \cup \hat{A}_s$		$\hat{A}_1 \cup \hat{A}_2$	$\hat{A}_1$

We have

$$\forall A \in H, \; \mu_3(A) = \alpha_3(A)/q_3(A).$$

If s is even then $\mu_3(A_1) = \mu_3(A_s), \; \mu_3(A_2) = \mu_3(A_{s-1}), ..., \mu_3(A_{s/2}) = \mu_3(A_{1+s/2})$.
If s is odd then $\mu_3(A_1) = \mu_3(A_s), \; \mu_3(A_2) = \mu_3(A_{s-1}), ..., \mu_3(A_{(s-1)/2}) = \mu_3(A_{(s+3)/2})$.

Case $n = 2s + 1$.

Clearly, if $n = 2s + 1$ then $\hat{A_{s+1}} = \{A_{s+1}\}$. Notice that this is the only class containig only one element.

$\circ_2$	A_1	A_2	...	A_s	A_{s+1}	A_{s+2}	...	A_{2s+1}
A_1	$\hat{A}_1$	$\hat{A}_1 \cup \hat{A}_2$	...	$\hat{A}_1 \cup ... \cup \hat{A}_s$	$\hat{A}_1 \cup ... \cup \hat{A_{s+1}}$	$\hat{A}_1 \cup ... \cup \hat{A}_s$	...	$\hat{A}_1$
A_2		$\hat{A}_2$	...	$\hat{A}_2 \cup ... \cup \hat{A}_s$	$\hat{A}_2 \cup ... \cup \hat{A_{s+1}}$	$\hat{A}_2 \cup ... \cup \hat{A}_s$	...	$\hat{A}_1 \cup \hat{A}_2$
...	...	...	...	...	...	...	...	...
A_s				$\hat{A}_s$	$\hat{A}_s \cup \hat{A_{s+1}}$	$\hat{A}_s$	...	$\hat{A}_1 \cup ... \cup \hat{A}_s$
A_{s+1}					$\hat{A_{s+1}}$	$\hat{A}_s \cup \hat{A}_{s+1}$	...	$\hat{A}_1 \cup ... \cup \hat{A}_s$
...	...	...	...	...	...	...	...	...
A_{2s+1}								$\hat{A}_1$

The column and the line corresponding to element A_{s+1} represent the axes of symmetry of the table.

Again $\forall A \in H, \; \mu_3(A) = \alpha_3(A)/q_3(A)$ and we have

$$\mu_3(A_{s+1}) > \mu_3(A_s) > ... > \mu_3(A_2) > \mu_3(A_1).$$

We check that for $n \in \{5, 7\}$ the fuzzy degree of $(H, \otimes)$ is 3.

As a conclusion,

(i) if $n = 6$ then the fuzzy degree of $(H, \otimes)$ is 3. (ii) if $n = 8$ then the fuzzy degree of $(H, \otimes)$ is 4. Notice that for $n = 2s$ and s is odd, then the classes $[A_i]$ have 4 elements for all $i \in \{1, ..., (s-1)/2\}$ and $[A_{(s+1)/2}]$ has 2 elements in the table of $\circ_3$, while if s is even all classes $[A_i]$ have 4 elements for all $1 \leq i \leq s$.

(iii) If $n = 5$ then the fuzzy degree of $(H, \otimes)$ is 3.

(iv) If $n = 7$ then the fuzzy degree of $(H, \otimes)$ is 3.

4 A sequence of fuzzy sets and join spaces determined by a hypergraph

Here the context is the following one: we consider a certain hypergraph, that is a non empty set H, endowed with a family of non empty subsets, whose union is H. Then we associate a hyperoperation with this hypergraph and we construct the sequence defined by Corsini. This sequence is analyzed for several particular cases and finally we obtain the general formulae for all the fuzzy sets of this sequence, in the general case. These results are presented in detail in [9].

Set $E_1 = \{1, 2\}$, $E_2 = \{2, 3, 4\}$ and $f_2 = 2$. For all $s \geq 3$, define

$$f_s = f_2 + 2 + 3 + ... + s - 1 \text{ and } E_s = \{f_s, f_s + 1, ..., f_s + s\}.$$

We have $E_s = \dfrac{s^2 - s}{2} + 1$. It follows

$$E_s = \left\{ \frac{s^2 - s}{2} + 1, \frac{s^2 - s}{2} + 2, ..., \frac{s^2 - s}{2} + s + 1 \right\}.$$

Let $H = \bigcup\limits_{1 \leq i \leq s} E_i$ and for all $x \in H$, define $x \circ x = \bigcup\limits_{x \in E_i} E_i$.

For all $(x, y) \in H^2$, define $x \circ y = x \circ x \cup y \circ y$.

4.1 The case $s = 2$

We have $H = E_1 \cup E_2 = \{1, 2, 3, 4\}$, where $E_1 = \{1, 2\}$, $E_2 = \{2, 3, 4\}$. We get the following hypergroupoid:

$\circ$	1	2	3	4
1	1,2	H	H	H
2	H	H	H	H
3	H	H	2,3,4	2,3,4
4	H	H	2,3,4	2,3,4

We obtain

$$\widetilde{\mu}_1(2) > \widetilde{\mu}_1(3) = \widetilde{\mu}_1(4) > \widetilde{\mu}_1(1),$$

whence we obtain the following join space $(H, \circ_{\widetilde{\mu}_1})$:

$\circ_{\widetilde{\mu}_1}$	1	3	4	2
1	1	1,3,4	1,3,4	H
3	1,3,4	3,4	3,4	2,3,4
4	1,3,4	3,4	3,4	2,3,4
2	H	2,3,4	2,3,4	2

Now, by (2) we obtain a new fuzzy set $\widetilde{\mu}_2$ and we have $\widetilde{\mu}_2(1) = \widetilde{\mu}_2(2) > \widetilde{\mu}_2(3) = \widetilde{\mu}_2(4)$. Hence we obtain the following join space $(H, \circ_{\widetilde{\mu}_2})$:

$\circ_{\widetilde{\mu}_2}$	1	2	3	4
1	1,2	1,2	H	H
2	1,2	1,2	H	H
3	H	H	3,4	3,4
4	H	H	3,4	3,4

Now by (2) we get a new fuzzy set $\widetilde{\mu}_3$ for which
$\widetilde{\mu}_3(1) = \widetilde{\mu}_3(2) = \widetilde{\mu}_3(3) = \widetilde{\mu}_3(4) = 0.3333$ and so the join space $(H, \circ_{\widetilde{\mu}_3})$ is the total hypergroup. Hence the grade of this sequence, which is the length of it, is 3.

Therefore, we obtain the next theorem.

Theorem 4.1. *For s=2 the grade of the sequence defined by* (2) *is 3.*

4.2 The case $s = 3$

We have $H = \{1, 2, 3, 4, 5, 6, 7\}$, where $E_1 = \{1, 2\}$, $E_2 = \{2, 3, 4\}$ and $E_3 = \{4, 5, 6, 7\}$. Then $(H, \circ)$ is the following commutative hypergroupoid:

$\circ$	1	2	3	4	5	6	7
1	1,2	1,2,3,4	1,2,3,4	H	H-{3}	H-{3}	H-{3}
2		1,2,3,4	1,2,3,4	H	H	H	H
3			2,3,4	H-{1}	H-{1}	H-{1}	H-{1}
4				H-{1}	H-{1}	H-{1}	H-{1}
5					4,5,6,7	4,5,6,7	4,5,6,7
6						4,5,6,7	4,5,6,7
7							4,5,6,7

We have

$$(*) \qquad \widetilde{\mu}_1(1) > \widetilde{\mu}_1(4) > \widetilde{\mu}_1(2) > \widetilde{\mu}_1(3) > \widetilde{\mu}_1(5) = \widetilde{\mu}_1(6) = \widetilde{\mu}_1(7).$$

Using (1), we obtain the following join space $(H, \circ_{\widetilde{\mu}_1})$:

$\circ_{\widetilde{\mu}_1}$	1	2	3	4	5	6	7
1	1	1,2,4	1,2,3,4	1,4	H	H	H
2		2	2,3	2,4	H-{1,4}	H-{1,4}	H-{1,4}
3			3	2,3,4	3,5,6,7	3,5,6,7	3,5,6,7
4				4	H-{1}	H-{1}	H-{1}
5					5,6,7	5,6,7	5,6,7
6						5,6,7	5,6,7
7							5,6,7

We obtain

$$\widetilde{\mu}_2(1) > \widetilde{\mu}_2(4) > \widetilde{\mu}_2(2) > \widetilde{\mu}_2(3) > \widetilde{\mu}_2(5) = \widetilde{\mu}_2(6) = \widetilde{\mu}_2(7).$$

Since $\widetilde{\mu}_1$ satisfies $(*)$, it follows that the join space $(H, \circ_{\widetilde{\mu}_2})$ associated with $\widetilde{\mu}_2$ coincides with the join space $(H, \circ_{\widetilde{\mu}_1})$ associated with $\widetilde{\mu}_1$.

Hence, we have obtained the next result:

Theorem 4.2. *For s=3 the grade of the sequence defined by* (2) *is 1.*

4.3 The case $s = 4$

We have $H = \{n \in \mathbb{N}^* \mid 1 \leq n \leq 11\}$ and $E_1 = \{1,2\}$, $E_2 = \{2,3,4\}$, $E_3 = \{4,5,6,7\}$, $E_4 = \{7,8,9,10,11\}$.

Then
$$\widetilde{\mu}_1(1) > \widetilde{\mu}_1(2) > \widetilde{\mu}_1(3) > \widetilde{\mu}_1(4) > \widetilde{\mu}_1(7) > \widetilde{\mu}_1(5)$$
$$= \widetilde{\mu}_1(6) > \widetilde{\mu}_1(8) = \widetilde{\mu}_1(9) = \widetilde{\mu}_1(10) = \widetilde{\mu}_1(11).$$
In a similar way we obtain

$$\widetilde{\mu}_2(1) > \widetilde{\mu}_2(2) > \widetilde{\mu}_2(3) > \widetilde{\mu}_2(4) > \widetilde{\mu}_2(7) > \widetilde{\mu}_2(5) = \widetilde{\mu}_2(6) > \widetilde{\mu}_2(8) = \widetilde{\mu}_2(9) = \widetilde{\mu}_2(10) = \widetilde{\mu}_2(11).$$

Hence, according to (1), the fuzzy sets $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ determine the same join space.

Therefore we obtain the next result:

Theorem 4.3. *For s=4 the grade of the sequence defined by* (2) *is 1.*

4.4 The case $s = 5$

We have $H = \{n \in \mathbb{N}^* \mid 1 \leq n \leq 16\}$ and $E_1 = \{1,2\}$, $E_2 = \{2,3,4\}$, $E_3 = \{4,5,6,7\}$, $E_4 = \{7,8,9,10,11\}$, $E_5 = \{11,12,13,14,15,16\}$.

Then
$$\widetilde{\mu}_1(1) > \widetilde{\mu}_1(2) > \widetilde{\mu}_1(3) > \widetilde{\mu}_1(4) > \widetilde{\mu}_1(5) = \widetilde{\mu}_1(6) > \widetilde{\mu}_1(7) > \widetilde{\mu}_1(8) > \widetilde{\mu}_1(9) = \widetilde{\mu}_1(10) > \widetilde{\mu}_1(12)$$
$$= \widetilde{\mu}_1(13) = \widetilde{\mu}_1(14) = \widetilde{\mu}_1(15) = \widetilde{\mu}_1(16).$$
In a similar way it follows that

$$\widetilde{\mu}_2(1) > \widetilde{\mu}_2(2) > \widetilde{\mu}_2(3) > \widetilde{\mu}_2(4) > \widetilde{\mu}_2(5) = \widetilde{\mu}_2(6) > \widetilde{\mu}_2(7) > \widetilde{\mu}_2(11) > \widetilde{\mu}_2(8) = \widetilde{\mu}_2(9) = \widetilde{\mu}_2(10)$$
$$> \widetilde{\mu}_2(12) = \widetilde{\mu}_2(13) = \widetilde{\mu}_2(14) = \widetilde{\mu}_2(15) = \widetilde{\mu}_2(16).$$
Therefore, $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ determine the same join space, according to (1).

We have obtained the next result:

Theorem 4.4. *For $s = 5$ the grade of the sequence defined by* (2) *is* 1.

Before generalizing, we present here a conclusion for the previous particular cases:

1°. For $s = 2$, we have:
$\widetilde{\mu}_1(2) > \widetilde{\mu}_1(3) = \widetilde{\mu}_1(4) > \widetilde{\mu}_1(1)$,
$\widetilde{\mu}_2(1) = \widetilde{\mu}_2(2) > \widetilde{\mu}_2(3) = \widetilde{\mu}_2(4)$ and
$\widetilde{\mu}_3(1) = \widetilde{\mu}_3(2) = \widetilde{\mu}_3(3) = \widetilde{\mu}_3(4)$, whence $\partial H = 3$.

2°. For $s = 3$, we have:
$\widetilde{\mu}_i(1) > \widetilde{\mu}_i(4) > \widetilde{\mu}_i(2) > \widetilde{\mu}_i(3) > \widetilde{\mu}_i(5) = \widetilde{\mu}_i(6) = \widetilde{\mu}_i(7)$, for $i \in \{1,2\}$, whence $\partial H = 1$.

3°. For $s = 4$, we have:
$\widetilde{\mu}_i(1) > \widetilde{\mu}_i(2) > \widetilde{\mu}_i(3) > \widetilde{\mu}_i(4) > \widetilde{\mu}_i(7) > \widetilde{\mu}_i(5) = \widetilde{\mu}_i(6) > \widetilde{\mu}_i(8) = \widetilde{\mu}_i(9) = \widetilde{\mu}_i(10) = \widetilde{\mu}_i(11)$, for $i \in \{1,2\}$, whence $\partial H = 1$.

4°. For $s = 5$, we have:
$\widetilde{\mu}_i(1) > \widetilde{\mu}_i(2) > \widetilde{\mu}_i(3) > \widetilde{\mu}_i(4) > \widetilde{\mu}_i(5) = \widetilde{\mu}_i(6) > \widetilde{\mu}_i(7) > \widetilde{\mu}_i(11) > \widetilde{\mu}_i(8) = \widetilde{\mu}_i(9) = \widetilde{\mu}_i(10) > \widetilde{\mu}_i(12) = \widetilde{\mu}_i(13) = \widetilde{\mu}_i(14) = \widetilde{\mu}_i(15) = \widetilde{\mu}_i(16)$, for $i \in \{1,2\}$, whence $\partial H = 1$.

4.5 The general case

Let $n = f_s + s$, where $s \geq 2$. Notice that $f_i + i = \dfrac{i^2+i+2}{2} = f_{i+1}$, hence $E_i \cap E_{i+1} = \{f_i + i\}$, for all $1 \leq i \leq s$. Therefore, $H = \{k \in \mathbb{N}^* \mid 1 \leq k \leq n\}$.

Suppose

$\widetilde{\mu}_1(x_1) > \widetilde{\mu}_1(x_2) > ... > \widetilde{\mu}_1(x_m) > \widetilde{\mu}_1(x_{m+1}) = \widetilde{\mu}_1(x_{m+2}) = ... = \widetilde{\mu}_1(x_{m+k_1}) > \widetilde{\mu}_1(x_{m+k_1+1})$
$= \widetilde{\mu}_1(x_{m+k_1+2}) = ... = \widetilde{\mu}_1(x_{m+k_1+k_2}) > ... > \widetilde{\mu}_1(x_{m+k_1+k_2+...+k_t+1})$
$= \widetilde{\mu}_1(x_{m+k_1+k_2+...+k_t+2}) = ... = \widetilde{\mu}_1(x_{m+k_1+k_2+...+k_{t+1}})$,
where $m + k_1 + k_2 + ... + k_{t+1} = n$ and $\{x_i \mid 1 \leq i \leq n\} = \{i \in \mathbb{N}^* \mid 1 \leq i \leq n\}$.

Notice that for $s = 2$, we have $m = 1$, $k_1 = 2$, $k_2 = 1$; for $s = 3$, we have $m = 4$, $k_1 = 3$; for $s = 4$, we have $m = 5$, $k_1 = 2$, $k_2 = 4$, while for $s = 5$ we have $m = 4$, $k_1 = 2$, $k_2 = k_3 = 1$, $k_4 = 3$ and $k_5 = 5$.

We have

$$\widetilde{\mu}_2(x_{m+1}) = ... = \widetilde{\mu}_2(x_{m+k_1}), \widetilde{\mu}_2(x_{m+k_1+1}) = ... = \widetilde{\mu}_2(x_{m+k_1+k_2}), ...,$$

$$\widetilde{\mu}_2(x_{m+k+1+...+k_t+1}) = ... = \widetilde{\mu}_2(x_{m+k_1+...+k_{t+1}}).$$

For $s \in \{3, 4, 5\}$, we have checked that if $\widetilde{\mu}_1(x_i) > \widetilde{\mu}_1(x_j)$, then $\widetilde{\mu}_2(x_i) > \widetilde{\mu}_2(x_j)$. This does not happen for $s = 2$. Indeed, we have $\widetilde{\mu}_1(2) > \widetilde{\mu}_1(1)$ and $\widetilde{\mu}_2(2) = \widetilde{\mu}_2(1)$. We also have $\widetilde{\mu}_1(4) > \widetilde{\mu}_1(1)$ and $\widetilde{\mu}_2(4) < \widetilde{\mu}_2(1)$.

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