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QUATERNIONS ASSOCIATED TO CURVES AND SURFACES

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Abstract This paper investigates the use of quaternions in studying space curves and surfaces in affine 3-space. First, we generate a large variety of rational space curves and rational surfaces via quaternion multiplication by taking advantage the fact that quaternions represent space rotations. Then, we prove that the curvature and the torsion of a space curve can be computed by a quaternion function that is associated to this space curve. Finally, we show that the Gaussian and the mean curvature of a surface can also be computed by a quaternion function that is associated to this surface.

1 Introduction

Quaternions were discovered by Sir William Rowan Hamilton as an extension of the complex numbers in 1843. The ring of quaternions over the real numbers is a noncommutative division algebra. An important property of quaternions is that every unit quaternion represents a rotation and this plays a special role in the study of rotations in three dimensional spaces. Quaternions are used in both theoretical and applied mathematics, especially in the areas involving calculations of three dimensional rotations, such as in three dimensional computer graphics, computer vision, animations, and aerospace applications [1], [2], [4], [5], [7].

A quaternion implementation is usually simpler, cheaper, and better behaved when compared to other alternatives, and the use of quaternions in various of applications has expanded. Many interesting research projects such as [6], [8], and [10] utilize quaternions in geometric designs. The majority of applications involve pure rotations, and the multiplications are used to represent a combination of different rotations.

A vector-valued function in a single variable $q(t) = (q_0, q_1, q_2, q_3)$ can be viewed as a quaternion function $q(t)$ or a space curve $\mathbf{q}(t) = (q_1, q_2, q_3)/q_0$; and similarly, a vector-valued function in two variables $q(s;t)$ can be viewed as a quaternion function or a surface in affine 3-space. Taking advantages of this and the fact that quaternions represent space rotations, we generate rational curves and surfaces by the quaternion multiplications of two rational space curves. The multiplication of two quaternions corresponds to the composition of rotations. Therefore, changing quaternion curves (or surfaces) can be achieved by manipulating the two space curves (or surfaces). In particular, the rotation of the curves will result in rotation of the quaternion curves (or surfaces). Hence, a large varieties of curves and surfaces can be generated using quaternion multiplications.

For a space curve, the curvature and the torsion of the curve are important subjects; and similarly, for a surface, the Gaussian and the mean curvatures have fundamental geometrical significance. Hence it is natural to ask:

1. how to relate the moving frames of a curve or a surface to a quaternion function?
2. how to compute the curvature and the torsion of a space curve using the corresponding quaternion?
3. how to compute the Gaussian and the mean curvature of a surface using the corresponding quaternion?

The goal of this paper is to answer these questions. In affine 3-space, a rotation matrix has three orthonormal columns, which may be used to represent an orthonormal moving frame for a space curve or a surface. For a space curve (or a surface), we define an associated quaternion – a quaternion is associated to this space curve (or surface) if the columns of the rotation matrix of the quaternion form the moving frames of this space curve (or surface). Using the associated quaternion as a bridge, we have affirmative and explicit answers, Theorems 3.10 and 3.14, to the last two questions.

This paper is structured as follows: Section 3 contains our main results and discussions. We start in Section 3.1 with a brief review of quaternions, and we generate rational curves and rational surfaces via quaternion product of two rational space curves in Propositions 3.4 and 3.5. In Section 3.2, we extend the Frenet - Serret frame of a space curve to a quaternion function in a single variable. We prove in Theorem 3.10 that the curvature and the torsion of a space curve can be computed by the associated quaternion. Finally, in Section 3.3, we generalize the moving frames of a surface to a quaternion function in two variables. We show in Theorem 3.14 that the Gaussian and the mean curvature of a surface can be computed by the associated quaternion. We conclude in Section 4 a short summary of our findings. Throughout the paper, we provide illustrative examples for our theorems.

2 Method

The method adopted in this paper is to associate a quaternion to a parametrizd space curve or a parametrizd surface by taking advantage the fact that quaternions represent space rotations. The key ingredient of the method is to create a quaternion whose matrix representation is exactly the orthonomal moving frames of the parametrized space curve or the parametrized surface. All the computations are verified by computer software package Mathematica Online [9].

3 Results and Discussion

3.1 Quaternion curves and Quaternion Surfaces

Quaternions

In this section, we review some basic facts about quaternions. An arbitrary quaternion has the form $q = s_q + \mathbf{v}_q$, where s_q is a scalar and $\mathbf{v}_q = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is a vector in $\mathbb{R}^3$. If $p = s_p + \mathbf{v}_p$ and $q = s_q + \mathbf{v}_q$ are two quaternions, then quaternion multiplication has the simple form

$$pq = s_p s_q - \mathbf{v}_p \cdot \mathbf{v}_q + s_p\mathbf{v}_q + s_q\mathbf{v}_p + \mathbf{v}_p \times \mathbf{v}_q.$$

Note that the cross product can be expressed in terms of quaternion multiplications:

$$\mathbf{v}_p \times \mathbf{v}_q = \frac{pq - qp}{2}. \tag{3.1}$$

The *conjugate* of q is denoted by $q^* = s_q - \mathbf{v}_q = s_q - v_1\mathbf{i} - v_2\mathbf{j} - v_3\mathbf{k}$. Notice that the product $qq^* = s_q^2 + v_1^2 + v_2^2 + v_3^2 = |q|^2$ is a scalar; $|q|$ is called the *norm* of q. If $|q| = 1$, then q is called a *unit quaternion*.

A *pure quaternion* is a quaternion q whose scalar part $s_q = 0$. A pure quaternion $\mathbf{v}_q = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is interpreted geometrically as the vector from the origin to the point located at (v_1, v_2, v_3) in $\mathbb{R}^3$. If $\mathbf{v}$ is a vector in 3-space, we let $(0, \mathbf{v})$ be the associated pure quaternion. Then the above equation implies

$$\mathbf{v} \times \mathbf{w} = \frac{(0, \mathbf{v})(0, \mathbf{w}) - (0, \mathbf{w})(0, \mathbf{v})}{2}.$$

It will be convenient to use quaternions to represent points in affine 3-space as well. The convention is that the quaternion $q = (s_q, v_1, v_2, v_3) = (q_0, q_1, q_2, q_3)$ represents the point $(\frac{q_1}{q_0}, \frac{q_2}{q_0}, \frac{q_3}{q_0})$ if $q_0 \neq 0$. In other words, we regard affine 3-space as a subset of projective 4-space (x_0, x_1, x_2, x_3) where $x_0 \neq 0$. We will distinguish between vectors in 3-space and 4-space (i.e., affine 3-space and projective 3-space), by the convention that given $x = (x_0, x_1, x_2, x_3) = (x_0, \mathbf{x})$, if $x_0 \neq 0$, then we get the point $\mathbf{x}/x_0$ in affine 3-space from the point x in projective 3-space. It is easy to see that if $(0, \mathbf{x})$ is a pure quaternion, and if q any quaternion, $q(0, \mathbf{x})q^* = (0, \mathbf{y})$ is also a pure quaternion. We denote the resulting vector $\mathbf{y} = q\mathbf{x}q^*$.

Theorem 3.1. *(Quaternion Rotation, [3].) Let $\mathbf{v}_q = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be a pure unit quaternion and set*

$$q = \cos(\theta/2) + \sin(\theta/2)\mathbf{v}_q.$$

Then q is a unit quaternion and the map $\mathbf{x} \to q\mathbf{x}q^*$ rotates points and vectors in $\mathbb{R}^3$ by the angle θ around the line through the origin in the direction of the vector $\mathbf{v}_q$ in $\mathbb{R}^3$. Observe that both $\pm q$ give the same rotation.

If we write the unit quaternion $q = \cos(\theta/2) + \sin(\theta/2)\mathbf{v}_q = (q_0, q_1, q_2, q_3)$ where $\sum_{i=0}^{3} q_i^2 = 1$, and set $x = (0, \mathbf{x})$ with $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, then the image of the rotation in $\mathbb{R}^3$ can be represented by

$$q\mathbf{x}q^* = qxq^* = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{R}_{3\times 3} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} \text{ where } \mathbf{0} = [0,0,0] \text{ and}$$

$$\mathbf{R}_{3\times 3} = \mathbf{R}_{3\times 3}(q) = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_3q_0) & 2(q_1q_3 + q_2q_0) \\ 2(q_1q_2 + q_3q_0) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_1q_0) \\ 2(q_1q_3 - q_2q_0) & 2(q_2q_3 + q_1q_0) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \quad (3.2)$$

is the rotation matrix representing the quaternion q with orthonormal columns (and rows).

We will extend this to an action of the nonzero quaternions on projective 3-space $(x_0, \mathbf{x})$.

$$qxq^* = \begin{bmatrix} q_0^2 + q_1^2 + q_2^2 + q_3^2 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{R}_{3\times 3} \end{bmatrix} \begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix}, \text{ where } \mathbf{0} = [0,0,0], \ \mathbf{R}_{3\times 3} \text{ is given in Eq. (3.2).}$$

Remark 3.2. Any orthonormal matrix $M = \{M_{ij}\}_{i,j=\{1,2,3\}}$ is a rotation matrix representing some unit quaternion $q = (q_0, q_1, q_2, q_3)$, where

$$q_0 = \frac{\sqrt{TraceM + 1}}{2}, \quad q_1 = \frac{M_{32} - M_{23}}{4q_0}, \quad q_2 = \frac{M_{13} - M_{31}}{4q_0}, \quad q_3 = \frac{M_{21} - M_{12}}{4q_0}. \quad (3.3)$$

Remark 3.2 can be verified by observing $\mathbf{R}_{3\times 3}$ in Equation (3.2) with $d_0^2 + d_1^2 + d_2^2 + d_3^2 = 1$. We will illustrate this remark by the following example.

Example 3.3. Given an orthogonal matrix

$$M = \begin{bmatrix} -\sin t/\sqrt{2} & -\cos t & \sin t/\sqrt{2} \\ \cos t/\sqrt{2} & -\sin t & -\cos t/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}.$$

By Remark 3.2, a computation via Mathematica [9] yields a unit quaternion $q = (q_0, q_1, q_2, q_3)$ whose rotation matrix is M, where

$$q_0 = \frac{1}{2}\sqrt{\frac{(1+\sqrt{2})(1-\sin t)}{\sqrt{2}}}, \quad q_1 = \frac{\cos t}{4\sqrt{2}q_0}, \quad q_2 = \frac{\sin t - 1}{4\sqrt{2}q_0}, \quad q_3 = \frac{\cos t + \sqrt{2}\cos t}{4\sqrt{2}q_0}.$$

Generate Rational Curves and Surfaces by Quaternions

A vector valued function in a single variable $q(t) = (q_0, q_1, q_2, q_3)$ can be viewed as a quaternion function or a space curve $\mathbf{q}(t) = (q_1, q_2, q_3)/q_0$; and similarly, a vector valued function in two variables $q(s,t)$ can be viewed as a quaternion function or a surface in affine 3-space. Taking advantages of this and the fact that quaternions represent space rotations, we shall generate a large variety of rational curves and surfaces by the quaternion multiplications of two rational space curves.

We start with two rational space curves typically represented by two generically one-to-one parametrizations in protective 3-space – the director curve $d(t)$, and the radius curve $r(t)$:

$$d(t) = (d_0(t), d_1(t), d_2(t), d_3(t)) = (d_0, \mathbf{d}), \quad r(t) = (r_0(t), r_1(t), r_2(t), r_3(t)) = (r_0, \mathbf{r}), \tag{3.4}$$

where $d_\ell, r_\ell \in \mathbb{R}[t]$, $\max\{\deg(d_\ell)\} = m$, $\max\{\deg(r_\ell)\} = n$, $\ell = 0, \ldots, 3$, and $\gcd(d_0, \ldots, d_3) = \gcd(r_0, \ldots, r_3) = 1$.

Proposition 3.4. *The rational curve*

$$x(t) = (x_0(t),\, x_1(t),\, x_2(t),\, x_3(t)) = d(t)\, r(t)\, d^*(t) \tag{3.5}$$

is the homogeneous form of a rational curve in affine 3-space generated by rotating the radius $\mathbf{r}(t)/r_0(t)$ *about the director* $\mathbf{d}(t)/d_0(t)$*. The quaternion curve* $x(t)$ *has a matrix representation*

$$x(t) = \mathbf{R}_d(t)\, r(t), \text{ where } \mathbf{R}_d \text{ is the rotation matrix representing } d = (d_0, d_1, d_2, d_3)$$

$$\mathbf{R}_d = \begin{bmatrix} d_0^2 + d_1^2 + d_2^2 + d_3^2 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{R}_{3\times 3} \end{bmatrix} \tag{3.6}$$

where $\mathbf{R}_{3\times3}$ *is given in the form of Equation (3.2),* $\det(\mathbf{R}_d) = (d_0^2 + d_1^2 + d_2^2 + d_3^2)^4$*, and*

$$\mathbf{R}_d^T \mathbf{R}_d = \mathbf{R}_d \mathbf{R}_d^T = (d_0^2 + d_1^2 + d_2^2 + d_3^2)^2 \mathbf{I}_{4\times4},\ \mathbf{I}_{4\times4} \text{ is the identity matrix.}$$

Proof. We consider the curve $d(t) = (d_0(t), d_1(t), d_2(t), d_3(t))$ as a quaternion, where

$$d(t) = (d_0(t), d_1(t), d_2(t), d_3(t)) = |d| \left(\cos(\theta/2) + \sin(\theta/2) \frac{(d_1, d_2, d_3)}{\sqrt{d_1^2 + d_2^2 + d_3^2}} \right) \tag{3.7}$$

$$\cos(\theta/2) = \frac{d_0}{|d|},\ \sin(\theta/2) = \frac{\sqrt{d_1^2 + d_2^2 + d_3^2}}{|d|},\ \theta(t) = 2\tan^{-1} \frac{\sqrt{d_1^2 + d_2^2 + d_3^2}}{d_0},$$
$$|d| = \sqrt{d_0^2 + d_1^2 + d_2^2 + d_3^2}.$$

Note that $\frac{d}{|d|}(t)$ is a unit quaternion, and for each t-value, the map $r(t) \to \left(\frac{d}{|d|}(t)\right) r(t) \left(\frac{d}{|d|}(t)\right)^*$ rotates the radius $r(t)$ by the angle $\theta(t) = 2\tan^{-1} \frac{\sqrt{d_1^2+d_2^2+d_3^2}}{d_0}(t)$, where $\cos(\theta/2) = \frac{d_0(t)}{|d(t)|}$ is the scalar part of $\frac{d}{|d|}(t)$, around the line in the direction of the vector part of $\frac{d}{|d|}(t)$, that is $\frac{\langle d_1(t), d_2(t), d_3(t)\rangle}{|d(t)|}$, which is the same as the direction of $\langle d_1(t), d_2(t), d_3(t)\rangle$. Therefore we can define a rational curve by setting $x(t) = \left(\frac{d}{|d|}(t)\right) r(t) \left(\frac{d}{|d|}(t)\right)^*$. Since we work with homogeneous coordinates, for any real parameter t the denominators in $\frac{d}{|d|}(t)$ are scalars which can be ignored. Thus these rational curves are generated by the formula $x = d(t) r(t) d^*(t)$.

In affine 3-space, the curve $x = \left(\frac{d}{|d|}\right) r \left(\frac{d}{|d|}\right)^*$ has a matrix expression $\begin{bmatrix} 1 & \mathbf{0} \\ 0 & \mathbf{R}_{3\times3} \end{bmatrix} \begin{bmatrix} 1 \\ \mathbf{r}/r_0 \end{bmatrix}$. By clearing the denominator, the homogeneous form of the quaternion curve $x = drd^* = \mathbf{R}_d\, r^T$ where the matrix $\mathbf{R}_d$ is the rotation matrix representing the quaternion d in Equation (3.6). □

Applying a similar argument as the proof of Proposition 3.4, we can generate a rational surface by two space curves with two distinct parameters, that is, we set the radius curve as $r(s)$ in stead of $r(t)$. With this manipulation, we derive the following result.

Proposition 3.5. *The rational surface*

$$x(s,t) = (x_0(s,t),\, x_1(s,t),\, x_2(s,t),\, x_3(s,t)) = d(t)\, r(s)\, d^*(t) \tag{3.8}$$

is a homogeneous form rational tensor product surface generated by rotating the radius $\mathbf{r}(s)/r_0(s)$ *about the director* $\mathbf{d}(t)/d_0(t)$*. The rational surface* $x(s,t)$ *has a matrix representation:*

$$x(s,t) \;=\; \mathbf{R}_d(t)\, r(s), \text{ where } \mathbf{R}_d \text{ is the rotation matrix representing } d \text{ in Eq. (3.6).} \tag{3.9}$$

We illustrate Propositions 3.4 and 3.5 by Example 3.6, and the resulting rational space curve and rational surface is shown in Figure 1.

Example 3.6. Given two space curve $d(t)$ and $r(t)$, where

$$\begin{aligned} d(t) &= (t^2+1, 0, t^2-1, 2t),\ r(t) = (1, t, 1, 0),\ \text{then we generate a rational curve } x(t) \text{ where} \\ x(t) &= d(t)r(t)d^*(t) = \mathbf{R}_d(t)\, r(t) \\ &= \begin{bmatrix} 2(t^2+1)^2 & 0 & 0 & 0 \\ 0 & 0 & -4t(t^2+1) & -2t^4+2 \\ 0 & 4t(t^2+1) & 2(t^2-1)^2 & 4t(t^2-1) \\ 0 & -2(t^4-1) & 4t(t^2-1) & 8t^2 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} (t^2+1)^2 \\ -2t(t^2+1) \\ 3t^4+1 \\ -t(t^2-1)^2 \end{bmatrix}. \end{aligned}$$

In Euclidean 3-space, the curve $d(t)$ is a circle $y^2 + z^2 = 1$ (in red), the radius $r(t)$ is the intersection of the planes $y = 1$ and $z = 0$ (in green), and the curve $x(t) = d(t)r(t)d^*(t)$ (in blue) is generated by rotating the line $r(t)$ about the director circle $d(t)$ which is illustrated on the left of the Figure 1.

Figure 1. A rational curve (left) and a rational surface (right) generated by rotating a line about a circle.

Using the same two curves, and let s be the parameter for the curve r, then

$$\begin{aligned} d(t) &= (t^2+1, 0, t^2-1, 2t),\ r(s) = (1, s, 1, 0),\ \text{we generate a rational surface } x(s,t) \text{ where} \\ x(s,t) &= d(t)r(s)d^*(t) = \mathbf{R}_d(t)\, r(s) \\ &= \begin{bmatrix} 2(t^2+1)^2 & 0 & 0 & 0 \\ 0 & 0 & -4t(t^2+1) & -2t^4+2 \\ 0 & 4t(t^2+1) & 2(t^2-1)^2 & 4t(t^2-1) \\ 0 & -2(t^4-1) & 4t(t^2-1) & 8t^2 \end{bmatrix} \begin{bmatrix} 1 \\ s \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2(t^2+1)^2 \\ -4t(t^2+1) \\ 2(2t^3s+2st+t^4-2t^2+1) \\ -2t^4s+2s+4t^3-4t \end{bmatrix}. \end{aligned}$$

In Euclidean 3-space, the surface $x = d(t)r(s)d^*(t)$ is generated by rotating the line $r(s)$ (in green) about the circle $d(t)$ (in red), and the surface $x(s,t)$ looks like a double fortune cookie illustrated on the right of Figure 1.

3.2 Compute The Curvature and The Torsion of Space Curves via Quaternion

Quaternion Differentials

Suppose $q(t) = (q_0(t), q_1(t), q_2(t), q_3(t))$ is a quaternion function in t. Then the derivative of the quaternion $q(t)$ with respect to t is denoted as $q'(t)$, and

$$q'(t) = (q_0'(t), q_1'(t), q_2'(t), q_3'(t)).$$

If $p(t)$ is another quaternion function in t, then the product rule for the derivative of the quaternion product is

$$(p(t)q(t))' = p'(t)q(t) + p(t)q'(t).$$

Suppose $q(t) = (q_0, \mathbf{q})$ is a unit quaternion, then $qq^* = 1$ and $(qq^*)' = q'q^* + q(q^*)' = 0$. Since

$$\begin{aligned} (qq^*)' &= q'q^* + q(q^*)' \\ &= [q_0'q_0 + \mathbf{q}' \cdot \mathbf{q} - q_0'\mathbf{q} + q_0\mathbf{q}' + \mathbf{q}' \times \mathbf{q}] + [q_0q_0' + \mathbf{q} \cdot \mathbf{q}' - q_0\mathbf{q}' + q_0'\mathbf{q} + \mathbf{q} \times \mathbf{q}'] \\ &= 2(q_0'q_0 + \mathbf{q}' \cdot \mathbf{q}), \end{aligned}$$

we must have that $q_0'q_0 + \mathbf{q}' \cdot \mathbf{q} = 0$.

Again, $(qq^*)' = q'q^* + q(q^*)' = 0$ yields $q'q^* = -q(q^*)'$. Right multiplying by q, or left multiplying by q^*, to both sides of $q'q^* = -q(q^*)'$, and setting $q_0'q_0 + \mathbf{q}' \cdot \mathbf{q} = 0$ yield

$$\begin{aligned} q' &= -q[(q^*)'q] = -q[q_0'q_0 + \mathbf{q}' \cdot \mathbf{q} + q_0'\mathbf{q} - q_0\mathbf{q}' - \mathbf{q}' \times \mathbf{q})] \\ &= q[q_0\mathbf{q}' - q_0'\mathbf{q} - \mathbf{q} \times \mathbf{q}'] = qk; \qquad (3.10) \\ (q^*)' &= -[q^*q']q^* = -[q_0q_0' + \mathbf{q} \cdot \mathbf{q}' + q_0\mathbf{q}' - q_0'\mathbf{q} - \mathbf{q} \times \mathbf{q}']q^* \\ &= -[q_0\mathbf{q}' - q_0'\mathbf{q} - \mathbf{q} \times \mathbf{q}']q^* = -kq^*; \qquad (3.11) \\ &\text{where } k = (0, \mathbf{k}),\ \mathbf{k} = q_0\mathbf{q}' - q_0'\mathbf{q} - \mathbf{q} \times \mathbf{q}'. \qquad (3.12) \end{aligned}$$

Quaternion and Frenet-Serret Frame

We start this section with a brief review of Frenet-Serret formulas for space curves. Let $\mathbf{x}(t)$ be a curve in Euclidean space, representing the position vector of the particle as a function of time. Then the arc length $s(t) = \int_0^t |\mathbf{x}'(\sigma)| d\sigma$ measures the length that the particle has moved along the curve in time t. If $\mathbf{x}'(t) \neq \mathbf{0}$, then $s(t)$ is a strictly monotonically increasing function, and it is possible to solve for t as a function of s, and thus to write $\mathbf{x}(s) = \mathbf{x}(t(s))$. The curve is thus parametrized its arc length.

If a curve $\mathbf{x}(s)$ with $\mathbf{x}'(s) \neq \mathbf{0}$ is parameterized by its arc length, then the unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$, and the unit binormal vector $\mathbf{B}$ are defined as

$$\mathbf{T} = \frac{d\mathbf{x}}{ds}, \quad \mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{|\frac{d\mathbf{T}}{ds}|}, \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

The curvature that measures the failure of a curve to be a straight line, and the torsion that measures the failure of a curve to be planar are defined by

$$\kappa = |\frac{d\mathbf{T}}{ds}|, \quad \tau = |\frac{d\mathbf{B}}{ds}|,$$

Theorem 3.7. *(Frenet-Serret Formula) Let $\mathbf{x}(t) : \mathbb{R} \to \mathbb{R}^3$ be a regular space curve, not necessarily parametrized by arc length. Then the curvature, the torsion, and the Frenet-Serret formula can be computed by:*

$$\kappa = \frac{|\mathbf{x}'(t) \times \mathbf{x}''(t)|}{|\mathbf{x}'(t)|^3}, \quad \tau = -\frac{(\mathbf{x}'(t) \times \mathbf{x}''(t)) \cdot \mathbf{x}'''(t)}{|\mathbf{x}'(t) \times \mathbf{x}''(t)|^2}, \quad \begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = |\mathbf{x}'(t)| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

For a quaternion function $q(t) = (q_0, q_1, q_2, q_3)$ such that $|q|^2 = qq^* = \sum_{i=0}^3 q_i^2 = 1$, let $\mathbf{R}_{3\times 3}$ and $\mathbf{R}_q$ be the rotation matrix representing q in the forms of Equations (3.2) and (3.6). Let $\mathbf{e}_i$ be the standard basis in $\mathbb{R}^3$, and quaternion $e_i = (0, \mathbf{e}_i)$ for $i = 1, 2, 3$. It is easy to see that for each $i = 1, 2, 3$, the quaternion product

$$q\mathbf{e}_i q^* = qe_iq^* = \mathbf{R}_q e_i^T = (0, \mathbf{R}_{3\times 3}\mathbf{e}_i) = \mathbf{R}_{3\times 3}\mathbf{e}_i = \text{ i-th column of the matrix } \mathbf{R}_{3\times 3}$$

results a pure quaternion whose vector part corresponds to the three orthonormal columns of the matrix $\mathbf{R}_{3\times 3}$.

Definition 3.8. A unit quaternion function $q(t)$ is *associated* to a space curve $\mathbf{y}(t)$ if the columns of the rotation matrix $\mathbf{R}_{3\times 3}$ of q are the unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$, and the unit binormal vector $\mathbf{B}$ of the curve $\mathbf{y}(t)$.

Remark 3.9. Example 3.3 computes a unit quaternion q that is associated to some space curve $\mathbf{y}(t)$ although the parametrization of $\mathbf{y}(t)$ is unknown. In fact, by taking the anti-derivative, we know that this unit quaternion is associated to a helix curve $\mathbf{y}(t) = (\cos t, \sin t, t)$.

If a unit quaternion q is associated to some parametric curve $\mathbf{y}(t) : \mathbb{R} \to \mathbb{R}^3$, that is, the columns of the rotation matrix $\mathbf{R}_{3\times 3}$ of q are the unit tangent vector $\mathbf{T}$, the unit normal vector $\mathbf{N}$, and the unit binormal vector $\mathbf{B}$ of $\mathbf{y}(t)$, then one can write $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ in terms of matrix multiplications, and express $\mathbf{T}', \mathbf{N}', \mathbf{B}'$ in terms of q' as:

$$
\begin{aligned}
\mathbf{T} &= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 \\ 2(q_1q_2 + q_3q_0) \\ 2(q_1q_3 - q_2q_0) \end{bmatrix} = \begin{bmatrix} q_0 & q_1 & -q_2 & -q_3 \\ q_3 & q_2 & q_1 & q_0 \\ -q_2 & q_3 & -q_0 & q_1 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = Aq, \\
\mathbf{T}' &= \begin{bmatrix} 2(q_0q_0' + q_1q_1' - q_2q_2' - q_3q_3') \\ 2(q_1'q_2 + q_1q_2' + q_3'q_0 + q_3q_0') \\ 2(q_1'q_3 + q_1q_3' - q_2'q_0 - q_2q_0') \end{bmatrix} = 2Aq', \\
\mathbf{N} &= \begin{bmatrix} 2(q_1q_2 - q_3q_0) \\ q_0^2 - q_1^2 + q_2^2 - q_3^2 \\ 2(q_2q_3 + q_1q_0) \end{bmatrix} = \begin{bmatrix} -q_3 & q_2 & q_1 & -q_0 \\ q_0 & -q_1 & q_2 & -q_3 \\ q_1 & q_0 & q_3 & q_2 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = Bq, \\
\mathbf{N}' &= \begin{bmatrix} 2(q_1'q_2 + q_1q_2' - q_3'q_0 - q_3q_0') \\ 2(q_0q_0' - q_1q_1' + q_2q_2' - q_3q_3') \\ 2(q_2'q_3 + q_2q_3' + q_1'q_0 + q_1q_0') \end{bmatrix} = 2Bq', \\
\mathbf{B} &= \begin{bmatrix} 2(q_1q_3 + q_2q_0) \\ 2(q_2q_3 - q_1q_0) \\ q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} = \begin{bmatrix} q_2 & q_3 & q_0 & q_1 \\ -q_1 & -q_0 & q_3 & q_2 \\ q_0 & -q_1 & -q_2 & q_3 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = Cq, \\
\mathbf{B}' &= \begin{bmatrix} 2(q_1'q_3 + q_1q_3' + q_2'q_0 + q_2q_0') \\ 2(q_2'q_3 + q_2q_3' - q_1'q_0 - q_1q_0') \\ 2(q_0q_0' - q_1q_1' - q_2q_2' + q_3q_3') \end{bmatrix} = 2Cq'.
\end{aligned}
$$

Applying Theorem 3.7, we have

$$
\begin{bmatrix} 2Aq' \\ 2Bq' \\ 2Cq' \end{bmatrix} = \begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = |\mathbf{y}'(t)| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} = |\mathbf{y}'(t)| \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} Aq \\ Bq \\ Cq \end{bmatrix}. \tag{3.13}
$$

Next, we shall express q' in terms of the product of a matrix with entries in κ, τ and the vector q. To do so, let A_i, B_i, C_i be the i-th row of matrix A, B, C. We observe that Equation (3.13) yields that

$$
\begin{aligned}
2(A_1 + C_3)q' &= |\mathbf{y}'(t)|(\kappa B_1 - \tau B_3)q, \text{ and} \\
2(A_1 + C_3)q' &= 4[q_0, 0, -q_2, 0]q' = 4(q_0q_0' - q_2q_2') \\
|\mathbf{y}'(t)|(\kappa B_1 - \tau B_3)q &= |\mathbf{y}'(t)|[-\kappa q_3 - \tau q_1, \kappa q_2 - \tau q_0, \kappa q_1 - \tau q_3, -\kappa q_0 - \tau q_2]q \\
&= 2|\mathbf{y}'(t)|[q_0(-\kappa q_3 - \tau q_1) + q_2(\kappa q_1 - \tau q_3)].
\end{aligned}
$$

To equate $2(A_1 + C_3)q' = |\mathbf{y}'(t)|(\kappa B_1 - \tau B_3)q$, comparing the coefficients of q_0 and q_2 of both sides implies

$$\begin{bmatrix} q_0' \\ q_2' \end{bmatrix} = \frac{|\mathbf{y}'(t)|}{2} \begin{bmatrix} -\kappa q_3 - \tau q_1 \\ -(\kappa q_1 - \tau q_3) \end{bmatrix} = \frac{|\mathbf{y}'(t)|}{2} \begin{bmatrix} 0 & -\tau & 0 & -\kappa \\ 0 & -\kappa & 0 & \tau \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}. \tag{3.14}$$

Similarly, Equation (3.13) yields that

$$\begin{aligned} 2(A_3 + C_1)q' &= |\mathbf{y}'(t)|(\kappa B_3 - \tau B_1)q, \text{ and} \\ 2(A_3 + C_1)q' &= 4[0, q_3, 0, q_1]q' = 4(q_3 q_1' + q_1 q_3') \\ |\mathbf{y}'(t)|(\kappa B_3 - \tau B_1)q &= |\mathbf{y}'(t)|[\kappa q_1 + \tau q_3, \kappa q_0 - \tau q_2, \kappa q_3 - \tau q_1, \kappa q_2 + \tau q_0]q \\ &= 2|\mathbf{y}'(t)|[q_3(\kappa q_2 + \tau q_0) + q_1(\kappa q_0 - \tau q_2)]. \end{aligned}$$

Again, to equate $2(A_3 + C_1)q' = |\mathbf{y}'(t)|(\kappa B_3 - \tau B_1)q$, comparing the coefficients of q_3 and q_1 of both sides implies

$$\begin{bmatrix} q_1' \\ q_3' \end{bmatrix} = \frac{|\mathbf{y}'(t)|}{2} \begin{bmatrix} \kappa q_2 + \tau q_0 \\ \kappa q_0 - \tau q_2 \end{bmatrix} = \frac{|\mathbf{y}'(t)|}{2} \begin{bmatrix} \tau & 0 & \kappa & 0 \\ \kappa & 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}. \tag{3.15}$$

Finally, combining Equations (3.14) and (3.15), we derive $q'(t)$ in terms of the torsion and curvature of the curve $\mathbf{y}(t)$ as the following

$$q'(t) = \begin{bmatrix} q_0' \\ q_1' \\ q_2' \\ q_3' \end{bmatrix} = \frac{|\mathbf{y}'(t)|}{2} \begin{bmatrix} 0 & -\tau & 0 & -\kappa \\ \tau & 0 & \kappa & 0 \\ 0 & -\kappa & 0 & \tau \\ \kappa & 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{|\mathbf{y}'(t)|}{2} \begin{bmatrix} 0 & -\tau & 0 & -\kappa \\ \tau & 0 & \kappa & 0 \\ 0 & -\kappa & 0 & \tau \\ \kappa & 0 & -\tau & 0 \end{bmatrix} q(t). \tag{3.16}$$

Compute Curvature and Torsion via Quaternion

In this section, we determine the curvature and the torsion of a space curve $\mathbf{y}(t)$ based on a unit quaternion associated to $\mathbf{y}(t)$.

Theorem 3.10. *Let $\mathbf{y}(t) \in \mathbb{R}^3$ be a space curve such that $|\mathbf{y}'(t)| \neq 0$. Assume $q(t)$ is a unit quaternion that is associated to $\mathbf{y}(t)$. Then the curvature κ and the torsion τ of the curve $\mathbf{y}(t)$ are given as*

$$\begin{bmatrix} \tau \\ \kappa \end{bmatrix} = \frac{2 \begin{bmatrix} q_0 - q_1 & q_3 - q_2 \\ q_2 - q_3 & q_0 - q_1 \end{bmatrix} \begin{bmatrix} q_0' + q_1' \\ q_2' + q_3' \end{bmatrix}}{|\mathbf{y}'(t)|[1 - 2(q_0 q_1 + q_2 q_3)]}, \quad \forall t \notin \{t \mid 1 - 2(q_0 q_1 + q_2 q_3)(t) = 0\}.$$

Proof. By Equation (3.16)

$$\begin{bmatrix} q_0' \\ q_1' \\ q_2' \\ q_3' \end{bmatrix} = \frac{|\mathbf{y}'(t)|}{2} \begin{bmatrix} 0 & -\tau & 0 & -\kappa \\ \tau & 0 & \kappa & 0 \\ 0 & -\kappa & 0 & \tau \\ \kappa & 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{|\mathbf{y}'(t)|}{2} \begin{bmatrix} -\tau q_1 - \kappa q_3 \\ \tau q_0 + \kappa q_2 \\ -\kappa q_1 + \tau q_3 \\ \kappa q_0 - \tau q_2 \end{bmatrix} \implies$$

$$\begin{bmatrix} q_0' + q_1' \\ q_2' + q_3' \end{bmatrix} = \frac{|\mathbf{y}'(t)|}{2} \begin{bmatrix} (q_0 - q_1)\tau + (q_2 - q_3)\kappa \\ (q_3 - q_2)\tau + (q_0 - q_1)\kappa \end{bmatrix} = \frac{|\mathbf{y}'(t)|}{2} \begin{bmatrix} q_0 - q_1 & q_2 - q_3 \\ q_3 - q_2 & q_0 - q_1 \end{bmatrix} \begin{bmatrix} \tau \\ \kappa \end{bmatrix}.$$

Therefore, for all t such that $\det \begin{bmatrix} q_0 - q_1 & q_2 - q_3 \\ q_3 - q_2 & q_0 - q_1 \end{bmatrix} = 1 - 2(q_0q_1 + q_2q_3) \neq 0$, we have

$$\begin{bmatrix} \tau \\ \kappa \end{bmatrix} = \frac{2}{|\mathbf{y}'(t)|} \begin{bmatrix} q_0 - q_1 & q_2 - q_3 \\ q_3 - q_2 & q_0 - q_1 \end{bmatrix}^{-1} \begin{bmatrix} q_0' + q_1' \\ q_2' + q_3' \end{bmatrix} = \frac{2 \begin{bmatrix} q_0 - q_1 & q_3 - q_2 \\ q_2 - q_3 & q_0 - q_1 \end{bmatrix} \begin{bmatrix} q_0' + q_1' \\ q_2' + q_3' \end{bmatrix}}{|\mathbf{y}'(t)|[1 - 2(q_0q_1 + q_2q_3)]}.$$

□

We will use the following example to illustrate Theorem 3.10.

Example 3.11. We continue with Example 3.3 where a unit quaternion $q = (q_0, q_1, q_2, q_3)$ is associated to a space curve $\mathbf{y}(t)$. If $|\mathbf{y}'(t)| = \sqrt{2}$, then by Theorem 3.10, a direct computation via Mathematica Online [9] gives the curvature and torsion of the curve $\mathbf{y}(t)$:

$$\begin{bmatrix} \tau \\ \kappa \end{bmatrix} = \frac{2}{\sqrt{2}[(q_0 - q_1)^2 + (q_2 - q_3)^2]} \begin{bmatrix} q_0 - q_1 & q_3 - q_2 \\ q_2 - q_3 & q_0 - q_1 \end{bmatrix} \begin{bmatrix} q_0' + q_1' \\ q_2' + q_3' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Remark 3.12. A vector valued function $x(t) = (x_0, x_1, x_2, x_3)$ can be viewed as a curve in affine 3-space, or as a quaternion. If x is considered as a space curve, then it is parametrized as $\mathbf{x}(t) = (x_1, x_2, x_3)/x_0$ in affine 3-space. The curvature and the torsion of the curve $\mathbf{x}(t)$ can be computed directly by the derivatives of $\mathbf{x}(t)$. If viewed as a quaternion, $x(t)$ is associated to some curve $\mathbf{y}(t)$ in affine 3-space. Theorem 3.10 provides a mean to calculate the curvature and the torsion of such curve $\mathbf{y}(t)$ given only $|\mathbf{y}'(t)|$.

3.3 Compute the Gaussian and the Mean Curvature of Surfaces via Quaternion

Quaternion and Gaussian and Mean curvatures

We start this section by recalling the Gaussian curvature and the mean curvature of a surface in $\mathbb{R}^3$. If $\mathbf{x}(s,t) : U \to \mathbb{R}^3$ is a regular patch, $|\mathbf{x}_s \times \mathbf{x}_t| \neq 0$, and $\mathbf{N} = \frac{\mathbf{x}_s \times \mathbf{x}_t}{|\mathbf{x}_s \times \mathbf{x}_t|}$ is the normal vector of the surface, then the Gaussian curvature K and the mean curvature H are given by

$$K = \frac{LM - N^2}{EG - F^2}, \quad H = \frac{EN - 2FM + GL}{2(EG - F^2)},$$

where

$$E = \mathbf{x}_s \cdot \mathbf{x}_s, \quad F = \mathbf{x}_s \cdot \mathbf{x}_t, \quad G = \mathbf{x}_t \cdot \mathbf{x}_t; \quad L = \mathbf{x}_{uu} \cdot \mathbf{N}, \quad M = \mathbf{x}_{uv} \cdot \mathbf{N}, \quad N = \mathbf{x}_{vv} \cdot \mathbf{N}$$

are the coefficients of the first fundamental form $E\,du^2 + 2F\,du\,dv + G\,dv^2$, and the second fundamental form $L\,du^2 + 2M\,du\,dv + N\,dv^2$.

If $\mathbf{x}_s = \mathbf{T}_1$ and $\mathbf{x}_t = \mathbf{T}_2$ are orthonormal basis for the tangent space to the surface $\mathbf{x}(s,t)$ in affine 3-space, then

$$\begin{aligned}
\begin{bmatrix} \mathbf{N}_s & \mathbf{N}_t \end{bmatrix} &= \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \end{bmatrix} D, \text{ and} \\
\mathbf{N}_s \times \mathbf{N}_t &= (a_{11}\mathbf{T}_1 + a_{21}\mathbf{T}_2) \times (a_{21}\mathbf{T}_1 + a_{22}\mathbf{T}_2) = a_{11}a_{22}\mathbf{T}_1 \times \mathbf{T}_2 + a_{21}a_{12}\mathbf{T}_2 \times \mathbf{T}_1 \\
&= \det(D)\mathbf{T}_1 \times \mathbf{T}_2 = K\mathbf{N}, \\
\mathbf{N}_s \times \mathbf{T}_2 + \mathbf{T}_1 \times \mathbf{N}_t &= (a_{11}\mathbf{T}_1 + a_{21}\mathbf{T}_2) \times \mathbf{T}_2 + \mathbf{T}_1 \times (a_{21}\mathbf{T}_1 + a_{22}\mathbf{T}_2) \\
&= \text{Trace}(D)\mathbf{T}_1 \times \mathbf{T}_2 = 2H\mathbf{N}.
\end{aligned}$$

In the event that $\mathbf{x}_s$ and $\mathbf{x}_t$ do not form an orthonormal basis for the tangent space of the surface, then the general formula is that

$$\mathbf{N}_s \times \mathbf{N}_t = K|\mathbf{x}_s \times \mathbf{x}_t|\mathbf{N}, \quad \text{and} \quad \mathbf{N}_s \times \mathbf{T}_2 + \mathbf{T}_1 \times \mathbf{N}_t = 2H|\mathbf{x}_s \times \mathbf{x}_t|\mathbf{N}. \tag{3.17}$$

We observe that in the frame of $\mathbf{T}_1, \mathbf{T}_2, \mathbf{N}$, the Gaussian curvature is the scalar projection of the vector $\mathbf{N}_s \times \mathbf{N}_t$ on $\mathbf{N}$; and the mean curvature is half of the scalar projection of the vector $\mathbf{N}_s \times \mathbf{T}_2 + \mathbf{T}_1 \times \mathbf{N}_t$ on $\mathbf{N}$.

Now, we shall see how to use a unit quaternion to compute the Gaussian and the mean curvatures of a surface. To do so, we first introduce a definition below.

Definition 3.13. Let $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{N}$ be a set of orthonormal moving frame of the surface $\mathbf{x}(s,t)$, that is, $\mathbf{T}_1$, $\mathbf{T}_2$ form an orthonormal basis for the tangent space to a surface $\mathbf{x}(s,t)$, and $\mathbf{N}$ a unit normal to the surface $\mathbf{x}(s,t)$. A unit quaternion $q(s,t)$ is *associated* to a surface $\mathbf{x}(s,t)$ if the columns of the rotation matrix $\mathbf{R}_{3\times 3}$ of q are the orthonormal moving frames $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{N}$ of the surface $\mathbf{x}(s,t)$.

It is easy to observe that if $q(s,t)$ is a unit quaternion associated to a surface $\mathbf{x}(s,t)$, then $\mathbf{T}_1, \mathbf{T}_2, \mathbf{N}$ are the orthonormal frame formed by the columns of the matrix $\mathbf{R}_{3\times 3}$ of q. In terms of quaternion multiplication, the unit vectors

$$\begin{aligned}\mathbf{T}_i &= (0,\mathbf{T}_i) = (0,\mathbf{R}_{3\times 3}\mathbf{e}_i) = q(0,\mathbf{e}_i)q^* = q\mathbf{e}_i q^*, \text{ for } i = 1,2,\\ \mathbf{N} &= (0,\mathbf{N}) = (0,\mathbf{R}_{3\times 3}\mathbf{e}_3) = q(0,\mathbf{e}_3)q^* = q\mathbf{e}_3 q^*.\end{aligned}$$

By Equations (3.10) and (3.12)

$$\begin{aligned} q_s &= \frac{\partial q}{\partial s} = q(0,\mathbf{a}) \text{ where } \mathbf{a} = q_0\mathbf{q}_s - q_{0s}\mathbf{q} - \mathbf{q}\times\mathbf{q}_s,\\ q_t &= \frac{\partial q}{\partial t} = q(0,\mathbf{b}) \text{ where } \mathbf{b} = q_0\mathbf{q}_t - q_{0t}\mathbf{q} - \mathbf{q}\times\mathbf{q}_t. \end{aligned} \tag{3.18}$$

Thus, taking partial derivative of $\mathbf{N} = q\mathbf{e}_3 q^*$, replacing q_s^*, q_t^* by the expressions given in Equation (3.11), and using the relationship between the cross product of the vectors and quaternions as in Equation (3.1) yield

$$\begin{aligned} \mathbf{N}_s &= q_s\mathbf{e}_3 q^* + q\mathbf{e}_3 q_s^* = q(0,\mathbf{a})(0,\mathbf{e}_3)q^* - q(0,\mathbf{e}_3)(0,\mathbf{a})q^*\\ &= q[(0,\mathbf{a})(0,\mathbf{e}_3) - (0,\mathbf{e}_3)(0,\mathbf{a})]q* = 2q(\mathbf{a}\times\mathbf{e}_3)q^*;\\ \mathbf{N}_t &= q_t\mathbf{e}_3 q^* + q\mathbf{e}_3 q_t^* = q(0,\mathbf{b})(0,\mathbf{e}_3)q^* - q(0,\mathbf{e}_3)(0,\mathbf{b})q^*\\ &= q[(0,\mathbf{b})(0,\mathbf{e}_3) - (0,\mathbf{e}_3)(0,\mathbf{b})]q^* = 2q(\mathbf{b}\times\mathbf{e}_3)q^*;\\ \mathbf{N}_s\times\mathbf{N}_t &= 4q[(\mathbf{a}\times\mathbf{b}\cdot\mathbf{e}_3)\mathbf{e}_3]q^* = 4(\mathbf{a}\times\mathbf{b}\cdot\mathbf{e}_3)(q\mathbf{e}_3q^*) = 4(\mathbf{a}\times\mathbf{b})_3\mathbf{N}. \end{aligned}$$

Hence Equation (3.17) implies

$$K = \frac{4(\mathbf{a}\times\mathbf{b})_3}{|\mathbf{x}_s\times\mathbf{x}_t|}. \tag{3.19}$$

Furthermore,

$$\begin{aligned} & \mathbf{N}_s\times\mathbf{T}_2 + \mathbf{T}_1\times\mathbf{N}_t\\ =\ & \frac{(0,\mathbf{N}_s)(0,\mathbf{T}_2) - (0,\mathbf{T}_2)(0,\mathbf{N}_s)}{2} + \frac{(0,\mathbf{T}_1)(0,\mathbf{N}_t) - (0,\mathbf{N}_t)(0,\mathbf{T}_1)}{2}\\ =\ & [q(\mathbf{a}\times\mathbf{e}_3)q^*q(\mathbf{e}_2)q^* - q(\mathbf{e}_2)q^*q(\mathbf{a}\times\mathbf{e}_3)q^*] + [q(\mathbf{e}_1)q^*q(\mathbf{b}\times\mathbf{e}_3)q^* - q(\mathbf{b}\times\mathbf{e}_3)q^*q(\mathbf{e}_1)q^*]\\ =\ & q[(0,\mathbf{a}\times\mathbf{e}_3)(0,\mathbf{e}_2) - (0,\mathbf{e}_2)(0,\mathbf{a}\times\mathbf{e}_3)]q^* + q[(0,\mathbf{e}_1)(0,\mathbf{b}\times\mathbf{e}_3) - (0,\mathbf{b}\times\mathbf{e}_3)(0,\mathbf{e}_1)]q^*\\ =\ & 2q(\mathbf{a}\times\mathbf{e}_3)\times\mathbf{e}_2q^* + 2q\mathbf{e}_1\times(\mathbf{b}\times\mathbf{e}_3)q^* = 2q[(\mathbf{a}\times\mathbf{e}_3)\times\mathbf{e}_2 + \mathbf{e}_1\times(\mathbf{b}\times\mathbf{e}_3)]q^*\\ =\ & 2q[(a_2-b_1)\mathbf{e}_3]q^* = 2(a_2-b_1)(q\mathbf{e}_3q^*) = 2(a_2-b_1)\mathbf{N}. \end{aligned}$$

Hence Equation (3.17) implies

$$H = \frac{a_2 - b_1}{|\mathbf{x}_s\times\mathbf{x}_t|}. \tag{3.20}$$

Compute the Gaussian and the Mean curvature via Quaternion

In this section, we compute the Gaussian curvature and the mean curvature of a surface $\mathbf{x}(s,t)$ using the unit quaternion associated to $\mathbf{x}(s,t)$.

Theorem 3.14. *Assume $q(s,t)$ is a unit quaternion associated to a surface $\mathbf{x}(s,t)$ with $|\mathbf{x}_s \times \mathbf{x}_t| \neq 0$. Let $\mathbf{R}_q$ be the rotation matrix representing q of the form in Equation (3.6). Then the Gaussian and the mean curvatures of the surface $\mathbf{x}(s,t)$ are*

$$K = \frac{\mathbf{R}_q^{-1}(q_s q_t^*) \cdot (0,0,0,-4)}{|\mathbf{x}_s \times \mathbf{x}_t|}, \qquad H = \frac{\mathbf{R}_q^{-1}(qe_1 q_s^* + qe_2 q_t^*) \cdot (0,0,0,-1)}{|\mathbf{x}_s \times \mathbf{x}_t|}.$$

Proof. First, we observe that

$$\begin{aligned}
& q_s q_t^* = q(0,\mathbf{a})(0,-\mathbf{b})q^* \text{ by Eq. (3.18) and Eq. (3.11)} \\
= \;& -q(0,\mathbf{a})(0,\mathbf{b})q^* = -q(-\mathbf{a}\cdot\mathbf{b}, \mathbf{a}\times\mathbf{b})q^* \\
= \;& -q[-\mathbf{a}\cdot\mathbf{b}, (\mathbf{a}\times\mathbf{b})_1\mathbf{e}_1 + (\mathbf{a}\times\mathbf{b})_2\mathbf{e}_2 + (\mathbf{a}\times\mathbf{b})_3\mathbf{e}_3]q^* \\
= \;& (\mathbf{a}\cdot\mathbf{b}, -(\mathbf{a}\times\mathbf{b})_1\mathbf{T}_1 - (\mathbf{a}\times\mathbf{b})_2\mathbf{T}_2 - (\mathbf{a}\times\mathbf{b})_3\mathbf{N}) \\
= \;& \mathbf{R}_q\,(\mathbf{a}\cdot\mathbf{b}, -(\mathbf{a}\times\mathbf{b})_1, -(\mathbf{a}\times\mathbf{b})_2, -(\mathbf{a}\times\mathbf{b})_3)^T \\
\Longrightarrow \;& K = \frac{\mathbf{R}_q^{-1}(q_s q_t^*) \cdot (0,0,0,-4)}{|\mathbf{x}_s \times \mathbf{x}_t|} \text{ since } K = \frac{4(\mathbf{a}\times\mathbf{b})_3}{|\mathbf{x}_s \times \mathbf{x}_t|} \text{ by Eq. (3.19).}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& qe_1 q_s^* + qe_2 q_t^* = q(0,\mathbf{e}_1)(0,-\mathbf{a})q^* + q(0,\mathbf{e}_2)(0,-\mathbf{b})q^* \text{ by Eq. (3.18) and Eq. (3.11)} \\
= \;& -q[(0,\mathbf{e}_1)(0,\mathbf{a}) + (0,\mathbf{e}_2)(0,\mathbf{b})]q^* = -q[(-\mathbf{e}_1\cdot\mathbf{a} + \mathbf{e}_1\times\mathbf{a}) + (-\mathbf{e}_2\cdot\mathbf{b} + \mathbf{e}_2\times\mathbf{b})]q^* \\
= \;& -q[(-a_1, 0, -a_3, a_2) + (-b_2, b_3, 0, -b_1)]q^* = -q[(-a_1 - b_2, b_3, -a_3, a_2 - b_1)]q^* \\
= \;& -q[(-a_1 - b_2, b_3\mathbf{e}_1 - a_3\mathbf{e}_2 + (a_2 - b_1)\mathbf{e}_3)]q^* \\
= \;& ((a_1 + b_2), -b_3\mathbf{T}_1 + a_3\mathbf{T}_2 - (a_2 - b_1)\mathbf{N}) \\
= \;& \mathbf{R}_q\,((a_1 + b_2), -b_3, a_3, -(a_2 - b_1))^T \\
\Longrightarrow \;& H = \frac{\mathbf{R}_q^{-1}(qe_1 q_s^* + qe_2 q_t^*) \cdot (0,0,0,-1)}{|\mathbf{x}_s \times \mathbf{x}_t|} \text{ since } H = \frac{a_2 - b_1}{|\mathbf{x}_s \times \mathbf{x}_t|} \text{ by Eq. (3.20).}
\end{aligned}$$

□

Remark 3.15. A vector valued function $x(s,t) = (x_0, x_1, x_2, x_3)$ can be viewed as a surface in affine 3-space or as a quaternion. If x is considered as a surface in affine 3-space, then $\mathbf{x}(s,t) = (x_1, x_2, x_3)/x_0$. The Gaussian and the mean curvature of the surface $\mathbf{x}(s,t)$ can be computed directly by the first and second fundamental forms. If viewed as a quaternion, then $x(s,t)$ is associated to some parametrized surface $\mathbf{y}(s,t) : \mathbb{R}^2 \to \mathbb{R}^3$. Theorem 3.14 provides a mean to calculate the Gaussian and the mean curvature of such surface $\mathbf{y}(s,t)$ given only $|\mathbf{y}_s \times \mathbf{y}_t|$.

We will use the following example to illustrate Theorem 3.14 and Remark 3.15.

Example 3.16. Suppose that an orthonormal moving frames for a surface $\mathbf{x}(s,t)$ are given by the columns of the matrix:

$$M = \begin{bmatrix} \frac{t\sin s}{\sqrt{1+t^2}} & \cos s & \frac{\sin s}{\sqrt{1+t^2}} \\ -\frac{t\cos s}{\sqrt{1+t^2}} & \sin s & \frac{-\cos s}{\sqrt{1+t^2}} \\ -\frac{1}{\sqrt{1+t^2}} & 0 & \frac{t}{\sqrt{1+t^2}} \end{bmatrix}, \quad \text{and} \quad |\mathbf{x}_s \times \mathbf{x}_t| = \sqrt{1+t^2}.$$

By Equation (3.3), a computation via Mathematica Online [9] yields a unit quaternion q that is associated to the surface $\mathbf{x}(s,t)$ where q is

$$\left(\frac{AB}{2}, \frac{\cos s}{2AB\sqrt{1+t^2}}, \frac{A}{2B\sqrt{1+t^2}}, \frac{\cos s}{2A\sqrt{1+t^2}}\right), \quad \text{where } A = \sqrt{1+\sin s}, B = \sqrt{t + \sqrt{1+t^2}}.$$

In addition, computing the Gaussian and the mean curvature via Mathematica Online [9] by Theorem 3.14 yield

$$K = \frac{1}{1+t^2}, \quad H = 0.$$

Note, in this example, the surface parametrization is unknown. In fact, by solving partial differential equations, we obtain a parametrization of this surface, which is a Helicoid $\mathbf{x}(s,t) = (t\cos s, t\sin s, s)$. Computation of the Gaussian and the mean curvatures by the fundamental forms of the parametrization give exactly the same result.

4 Conclusion

This paper investigates the use of quaternions in studying space curves and surfaces in affine 3-space. First, we generate a large variety of rational space curves and rational surfaces via quaternion multiplication. Then, we prove that the curvature and the torsion of a space curve can be computed by the quaternion associated to the curve; similarly, the Gaussian and the mean curvature of a surface can be computed by the quaternion associated to the surface.

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References

[1] P. J. Besl and N. D. McKay, *A method for registration of 3-D shapes*, IEEE Transactions on pattern analysis and machine intelligence, 14(2):239-256 (1992).

[2] O. D. Faugeras and M. Hebert, *The representation, recognition, and locating of 3-D objects*, International Journal of Robotics Research, 5(3):27-52 (1986).

[3] R. Goldman, *Rethinking Quaternions: Theory and Computation*, Synthesis Lectures on Computer Graphics and Animation, ed. Brian A. Barsky, No. 13. San Rafael: Morgan & Claypool Publishers (2010).

[4] B. K. P. Horn, *Closed-form solution of absolute orientation using unit quaternions*, Journal of Optical Society of America A, 4(4):629-642 (1987).

[5] J. B. Kuipers, *Quaternions and Rotation Sequences*, Princeton University Press (1999).

[6] J. E. Mebius, *A matrix-based proof of the quaternion representation theorem for four-dimensional rotations* (2005). http://arxiv.org/abs/math/0501249.

[7] J. T. Schwartz and M. Sharir, *Identification of partially obscured objects in two and three dimensions by matching noisy characteristic curves*, International Journal of Robotics Research, 6(2):29-44 (1987).

[8] K. Shoemake, *Quaternions and* 4×4 *matrices*, J. Arvo (Ed.), Graphics Gems II, Academic Press, New York, 351-354 (1991).

[9] Wolfram Research, Inc. (www.wolfram.com), Mathematica Online, Champaign, IL (2021).

[10] S. Zube, *A circle representation using complex and quaternion numbers*, Lithuanian Mathematical Journal, 46, 246-255 (2006).

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